Chromaticity Considerations for Nonlinear Integrable Lattices

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2014 Advanced Accelerator Concepts Workshop

Outline

- *Integrable Optics for the Intensity Frontier*
- *Conditions for integrability*
- *The Off-Momentum Problem*
- *Dispersion*
- *Chromaticity*
- *Future work*

Fermilab to Advance the Intensity Frontier

- *Study of neutrino physics & rare processes*
  - *Requires large beam currents and high brightness (PIP-II parameters):*
    - *CW proton beam*
    - *60-120 GeV*
    - *1.2 MW*
  - *Intensity-dependent effects limit designs*
    - *beam halo*
    - *Laslett tune shift*

- *Future accelerators must mitigate these effects*
  - *Most collective effects arise from single betatron tunes*
  - *Highly nonlinear lattices will decohere collective resonances*
  - *Requires:*
    - *large tune spreads*
    - *bounded, (integrable?) dynamics*
    - *large dynamic aperture*

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Integrable Optics for the Intensity Frontier

- *Integrable Optics Test Accelerator (IOTA)*
  - *Study nonlinear integrable optics of Danilov & Nagaitsev*[1]
  - *Under construction @ FermiLab ASTA (http://asta.fnal.gov)*
  - *Implements practical lattice with*
    - *tune spreads of order unity*
    - *bounded, integrable motion*
    - *current plan to study single-particle dynamics*
  - *Other work underway (talks past and future)*
    - *Collective effects with intense beams (D. Bruhwiler earlier)*
    - *Construction of the elliptic potential magnet (F. O'Shea next)*
    - *Experimental program @ IOTA (S. Nagaitsev Thurs. 9:30)*
    - *Octupole lattices (S. Antipov Thurs. 11:20)*
    - *Nonlinear optics @ UMER (K. Ruisard Thurs. 11:40)*

[1] *V. Danilov & S. Nagaitsev, "Nonlinear accelerator lattices with one and two analytic invariants", PR ST-AB* ***13****, 084002 (2010).*

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Conditions for Integrability

- *Bertrand-Darboux equation*
  - *Hamiltonians with 2nd invariants quadratic in momentum satisfy:*

$$xy\left(\partial_x^2 U - \partial_y^2 U\right) + \left(y^2 - x^2 + c^2\right)\partial_{xy} U + 3y\partial_x U - 3x\partial_y U = 0$$

    - *differential equation is* ***linear***
    - *any superposition of potentials that satisfy this differential equation will have a 2nd invariant and be integrable*
  - *Other auxiliary conditions for accelerators:*
    - *matched beta functions in the drifts with these nonlinear elements*
    - *equal vertical and horizontal linear tunes*

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The Off-Momentum Problem

- *Off-momentum particles couple motion to energy*
  - *Linear lattice chromaticity:*
    - *energy-dependent tune could cross nonlinear resonance*
    - *no loss of integrability (assuming linear RF bucket/coasting beam)*
  - *Linear lattice dispersion:*
    - *large dispersion can cause large beam size*
  - *Potential problems for elliptic potential*
    - *unequal tunes violates the Bertrand-Darboux equation*
    - *dispersion violates the equal beta function requirement*

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Single-turn Map

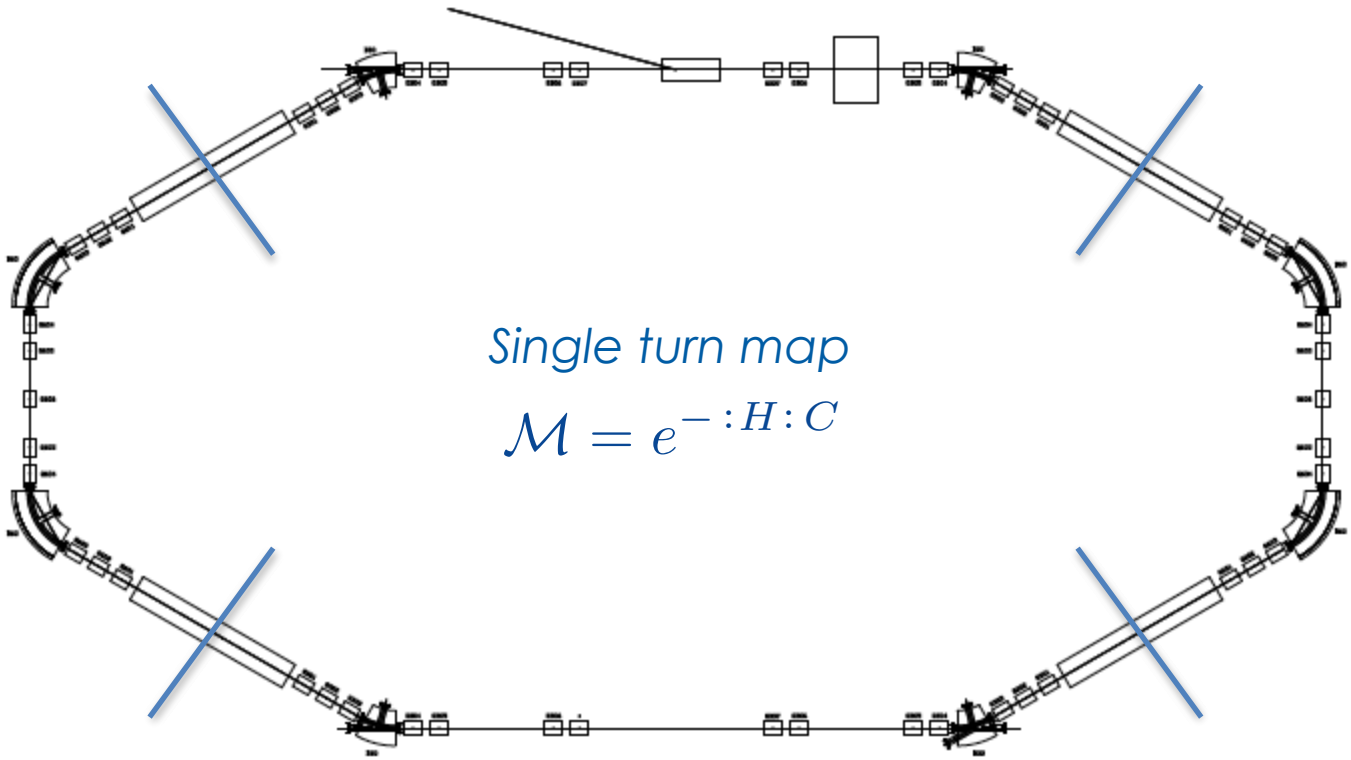

Figure from S. Nagaitsev, "IOTA Physics Goals" (2012)

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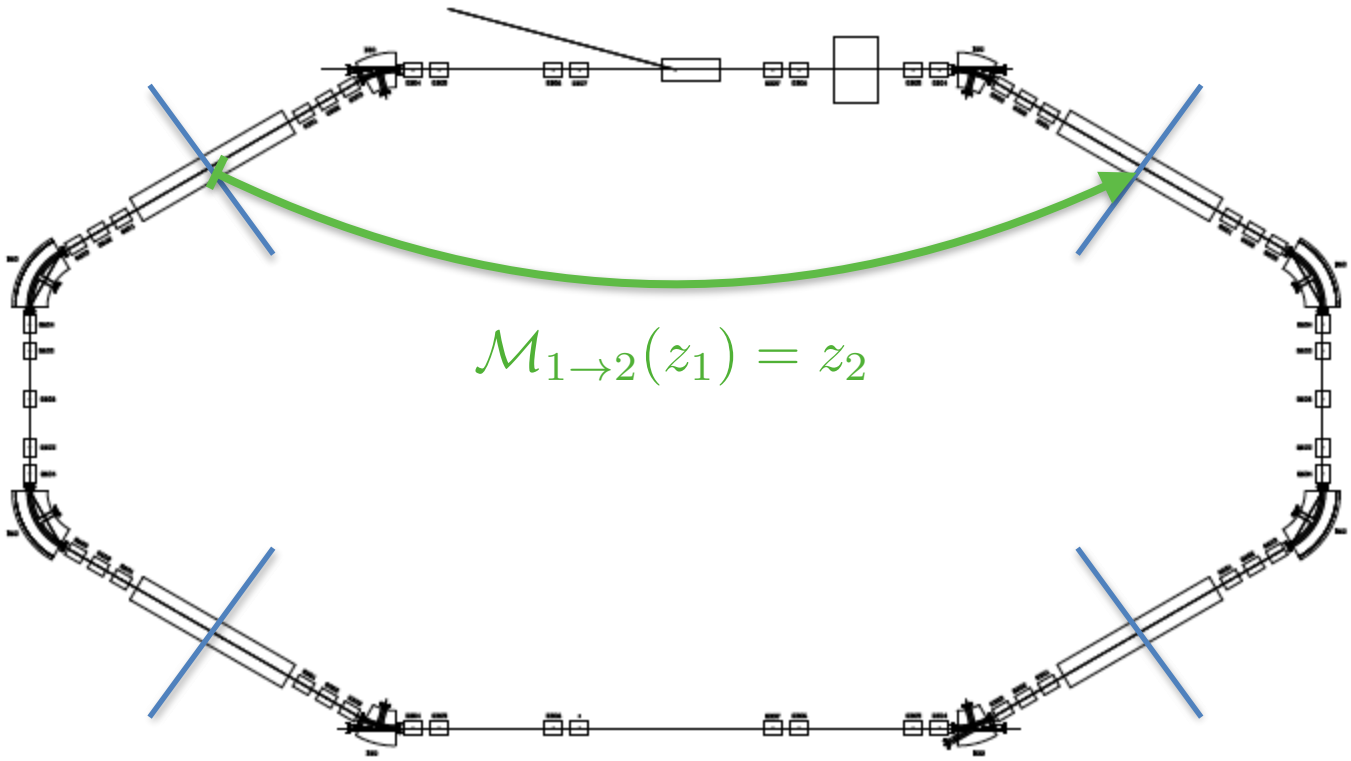
Single-turn Map



Single-turn Map

- *The Hamiltonian H is an invariant*
  - *Determines single particle dynamics*
    - *related to a general concept of emittance & matching*
    - *reduces to Courant-Snyder invariants for linear lattices*
    - *other invariants associated with integrable lattices*
  - *Contains all the information — dispersion, chromaticity…*
- *Normalized coordinates:*

$$e^{-:H:C} = \mathcal{A}\, e^{-:\overline{H}:}\, \mathcal{A}^{-1}$$

$$e^{-:H:C} = \left(\prod_{i=0}^{N/2} \exp\left\{ -:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y):\Delta s \right\} \right) \times \ldots$$

$$e^{-:h_0:} \left(\prod_{i=N/2}^{N} \exp\left\{ -:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y):\Delta s \right\} \right)$$

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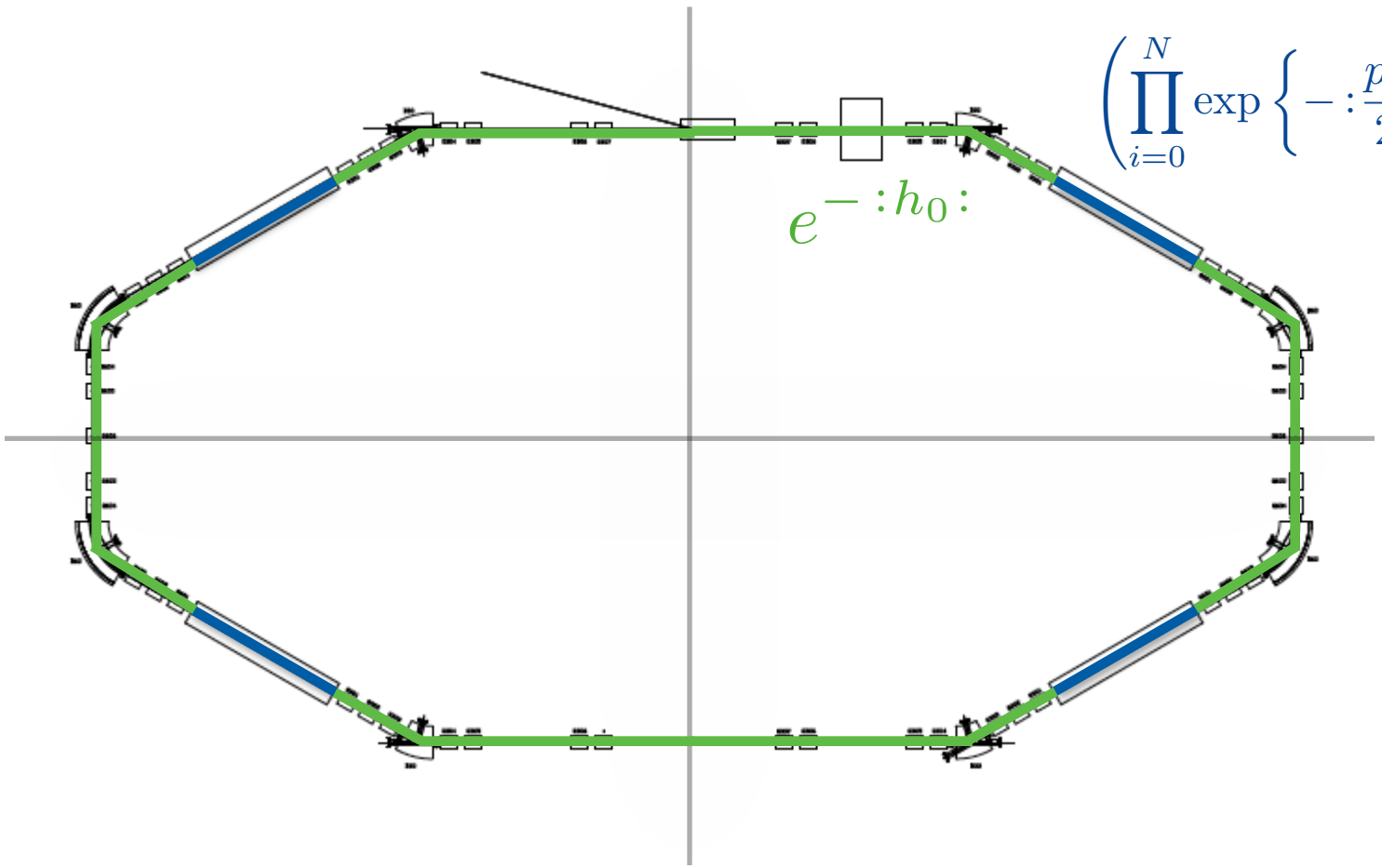
Single-turn Map



$$\left(\prod_{i=0}^{N} \exp\left\{-:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y):\Delta s\right\}\right)$$

Dispersion

- *After much algebra…*

$$\overline{H} = \frac{\mu_0}{2}\left[(1 - C_x\delta)\left(\overline{p}_x^2 + \overline{x}^2\right) + (1 - C_y\delta)\left(\overline{p}_y^2 + \overline{y}^2\right) + \ldots\right.$$
$$\left.(1-\delta)t\int_0^{\ell_{\mathrm{drift}}} \mathcal{V}\left(\overline{x} - \delta(\eta(s')/\sqrt{\beta_x(s')}), \overline{y}\right) ds'\right] + \mathrm{h.o.t.}$$

- *treating dispersion as a perturbation…*
  - *obtain $H_0 + \delta H_1$ with H_0 integrable and*

$$H_1 \approx \left(\delta(1-\delta)t\int_0^{\ell_{\mathrm{drift}}} \frac{\eta(s')}{\sqrt{\beta_x(s')}} ds'\right) \partial_{\overline{x}}\mathcal{V}(\overline{x}, \overline{y})$$

  - *Conclusions:*
    - *dispersion-free drifts for the magnets are ideal*
    - *failing that, minimize the integrated dispersion*

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Chromaticity

$$\overline{H} = \frac{\mu_0}{2}\left[(1 - C_x\delta)\left(\overline{p}_x^2 + \overline{x}^2\right) + (1 - C_y\delta)\left(\overline{p}_y^2 + \overline{y}^2\right) + \ldots\right.$$

$$\left.(1-\delta)t\int_0^{\ell_{\text{drift}}} \mathcal{V}\left(\overline{x} - \delta(\eta(s')/\sqrt{\beta_x(s')}), \overline{y}\right) ds'\right] + \text{h.o.t.}$$

- *Bertrand-Darboux equation needs isotropic linear term*
  - *Creates non-integrable perturbation H_2*

$$C_0 = \frac{1}{2}(C_x + C_y)$$

$$\Delta_C = \frac{1}{2}(C_x - C_y)$$

$$H_2 = \delta\ \Delta_C\left(\overline{p}_y^2 - \overline{p}_x^2 + \overline{y}^2 - \overline{x}^2\right)$$

  - *Conclusions:*
    - *defocussing quadratic perturbation due to differing chromaticities*
    - *already have large tune spreads — no need to remove all the chromaticity*
    - *correct to make $C_x = C_y$*

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Conclusions

- *Off-momentum dynamics breaks integrability*
  - *Dispersion violates the equal-beta-function requirement*
  - *Chromaticity introduces quadrupole perturbation which breaks the integrability*
- *Design to ameliorate the problem*
  - *Tune the chromaticities to be equal — don't have to cancel entirely!*
    - *conventional chromatic sextupole families suffice*
    - *can use weaker sextupoles, which saves more dynamic aperture*
  - *Use dispersion-free drifts for the nonlinear magnets*
  - *More study required to be sure…*

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Thank you for your attention

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Digression on Lie Operators

- *Lie operators from Poisson brackets*

$$\dot{z} = -\{H, z\} \mapsto \dot{z} = -:H:z \qquad\qquad z(t) = e^{-:H:t} z(0)$$

  - *Advantages*
    - *can multiply maps, cannot multiply Hamiltonians*
    - *maps make coördinate transformations into similarity transformations*

  - *Disadvantages*
    - *a lot of formalism to get to the physics*
    - *difficult to work with time-varying Hamiltonians*

- *Key Identities*

  - *BHC Identity* $e^{:C:} = e^{:A:} e^{:B:} \qquad C = A + B + \frac{1}{2}:A:B + \frac{1}{12}(:A:^2 B + :B:^2 A) + \ldots$

  - *Similarity transformation* $e^{:A:} e^{:B:} e^{-:A:} = \exp\left(:e^{:A:} B:\right)$

When are sextupoles optically transparent?

- *Lie operator approach*

$$\mathcal{M} = e^{-S_n :z^n:} e^{- :h_2:} e^{-S_n :z^n:}$$

$$e^{- :h_2:} = \mathcal{A}^{-1} \quad \underbrace{\mathcal{R}(\theta)}_{\text{pure rotation}} \quad \mathcal{A}$$

$$\mathcal{M} = e^{- :h_2:} \exp(-S_n :e^{:h_2:} z^n:) \exp(-S_n :z^n:)$$

$$\mathcal{M} = \mathcal{A}^{-1} \mathcal{R} \exp(-S_n :\mathcal{R}(\overline{z})^n:) \exp(-S_n :\overline{z}^n:)$$

$$\mathcal{R} \propto -1, \theta = (2n+1)\pi \implies \exp(-S_n :\mathcal{R}(\overline{z})^n:) \exp(-S_n :\overline{z}^n:) \mathcal{A} = 1$$

- *Off-momentum particles do not cancel exactly because θ is energy-dependent. This is the basis of chromaticity correction.*

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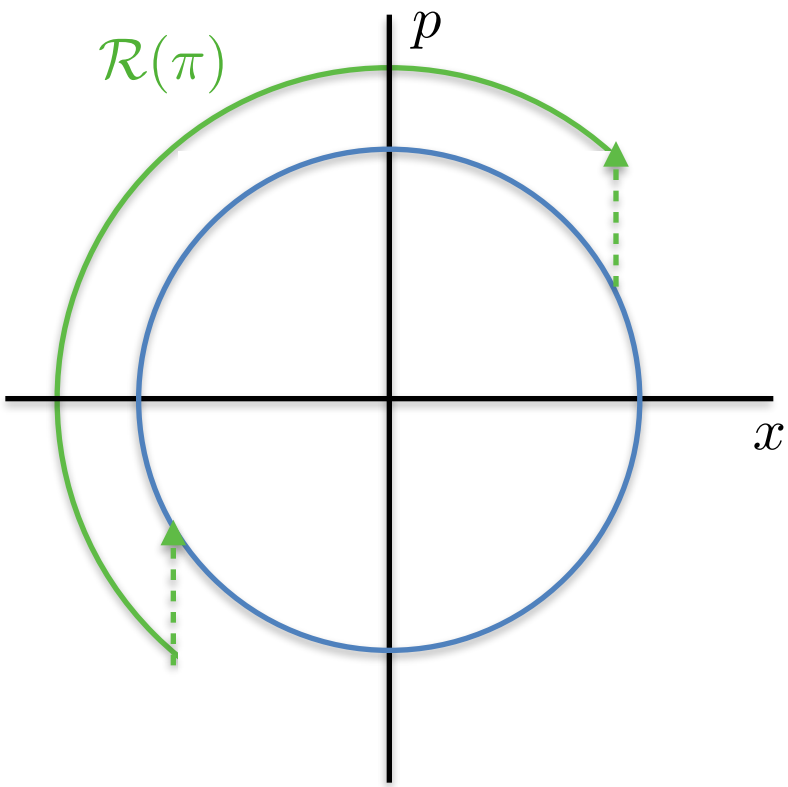
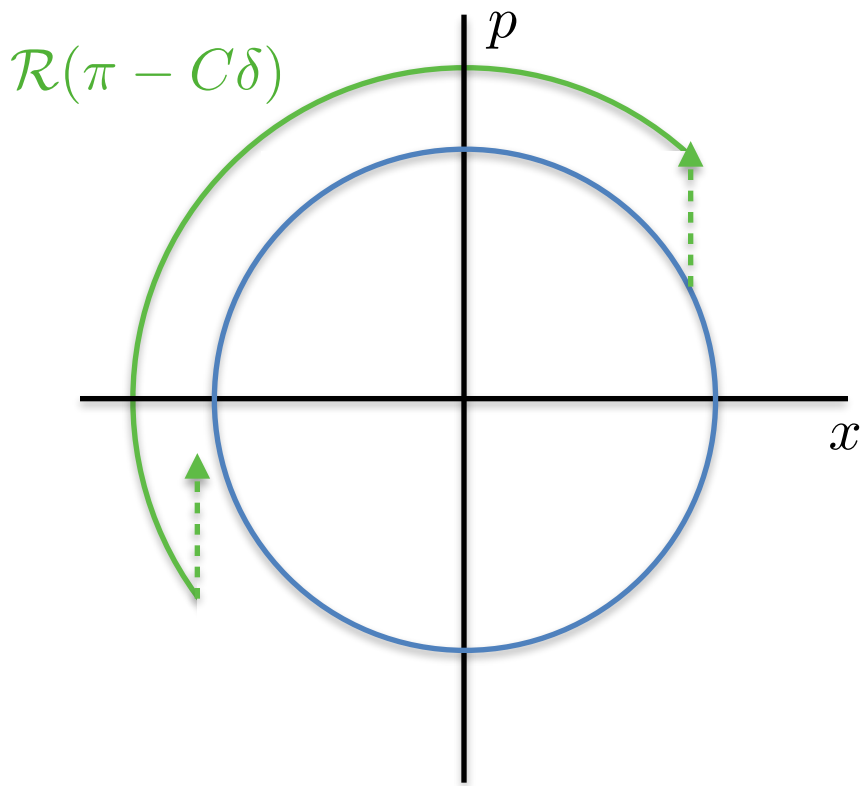
When are sextupoles optically transparent?

- *Pictorial approach (design momentum)*



When are sextupoles optically transparent?

- *Pictorial approach (off-momentum)*



The horror…

$$\mathcal{M} = \left(\prod_{i=0}^{N/2} \exp\left\{-:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y):\Delta s\right\}\right) e^{-:h_0:} \left(\prod_{i=N/2}^{N} \exp\left\{-:\frac{p^2}{2} + t_i \mathcal{V}_i(x,y):\Delta s\right\}\right)$$

$$= \left(\prod_{i=0}^{N/2} \exp\left\{-t_i :e^{-(i+1/2):p^2/2:\Delta s}\mathcal{V}_i(x,y):\Delta s\right\}\right) \circ$$

$$\underbrace{e^{-:p^2/2:\ell/2} e^{-:h_0:} e^{-:p^2/2:\ell/2}}_{e^{-:h_2:}} \circ$$

$$\left(\prod_{i=N/2}^{N} \exp\left\{-t_i :e^{(i+1/2):p^2/2:\Delta s}\mathcal{V}_i(x,y):\Delta s\right\}\right)$$

… the horror

$$e^{-:h_2:} = \mathcal{A} e^{-:\overline{h_2}:} \mathcal{A}^{-1}$$

Normalized coördinates

$$\mathcal{A}^{-1} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 & 0 & 0 & 0 & -\eta/\sqrt{\beta_x} \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 & 0 & -\alpha_x\eta+\beta_x\eta'/\sqrt{\beta_x} \\ 0 & 0 & 1/\sqrt{\beta_y} & 0 & 0 & 0 \\ 0 & 0 & \alpha_x/\sqrt{\beta_y} & \sqrt{\beta_y} & 0 & 0 \\ \eta' & \eta & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Courant-Snyder Parameterization

$$\mathcal{M} = \left(\prod_{i=0}^{N/2} \exp\left\{ -t_i :e^{-(i+1/2):p^2/2:\Delta s} \mathcal{V}_i(x,y) : \Delta s \right\} \right) \circ$$
$$\left(\mathcal{A} e^{-:\overline{h_2}:} \mathcal{A}^{-1} \right) \circ$$
$$\left(\prod_{i=N/2}^{N} \exp\left\{ -t_i :e^{(i+1/2):p^2/2:\Delta s} \mathcal{V}_i(x,y) : \Delta s \right\} \right)$$

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$$\mathcal{A}^{-1}\exp\left\{-t_i :e^{(i+1/2):p^2:2}\mathcal{V}_i(x,y):\Delta s\right\} =$$

$$\mathcal{A}^{-1}\exp\left\{-t_i :e^{(i+1/2):p^2:2}\mathcal{V}_i(x,y):\Delta s\right\}\mathcal{A}\mathcal{A}^{-1} =$$

$$\underbrace{\mathcal{A}^{-1}e^{-(i+1/2):p^2/2:\Delta s}}_{\mathcal{A}_i^{-1}}\exp\left\{-t_i :\mathcal{V}_i(x,y):\Delta s\right\}\underbrace{e^{(i+1/2):p^2/2:\Delta s}\mathcal{A}}_{\mathcal{A}_i}$$

The Danilov-Nagaitsev potential normalizing trick as follows:

$$A_0^{(i)} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 & 0 & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} & 0 & 0 \\ 0 & 0 & 1/\sqrt{\beta_y} & 0 \\ 0 & 0 & \alpha_x/\sqrt{\beta_y} & \sqrt{\beta_y} \end{pmatrix}$$

$$\mathcal{V}_i(x,y) = \mathcal{V}_i\left(A_0^{(i)}(x,y)\right)$$

$$\mathcal{A}^{-1} e^{-(i+1/2):p^2/2:\Delta s} \exp\left\{-t_i :\mathcal{V}_i(x,y): \Delta s\right\} e^{(i+1/2):p^2/2:\Delta s} \mathcal{A} =$$

$$\exp\left\{-t :\mathcal{V}\left(\overline{x} - \delta\frac{\eta}{\sqrt{\beta_x}}, \overline{y}\right): \Delta s\right\}$$

Final transfer map in normalized coordinates

$$\left(\prod_{i=0}^{N/2} \exp\left\{-:\frac{p^2}{2} + t_i\mathcal{V}_i(x,y): \Delta s\right\}\right) e^{-:h_0:} \left(\prod_{i=N/2}^{N} \exp\left\{-:\frac{p^2}{2} + t_i\mathcal{V}_i(x,y): \Delta s\right\}\right) =$$

$$\mathcal{A}\ \exp\left\{\sum_i -(1-\delta)t :\mathcal{V}\left(\overline{x} - \delta\frac{\eta_i}{\sqrt{\beta_i}}, \overline{y}\right):\right\} e^{-:\overline{h_2}:} \exp\left\{\sum_i -(1-\delta)t :\mathcal{V}\left(\overline{x} - \delta\frac{\eta_i}{\sqrt{\beta_i}}, \overline{y}\right):\right\}\ \mathcal{A}^{-1}$$

$$\overline{h}_2 = \frac{\mu_0}{2}\left[(1 - C_x\delta)\left(\overline{p}_x^2 + \overline{x}^2\right) + (1 - C_y\delta)\left(\overline{p}_y^2 + \overline{y}^2\right)\right]$$