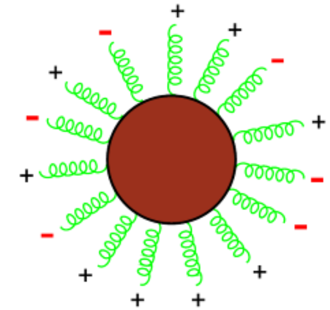

QCD and Monte Carlo
1. Introduction to QCD

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Fermilab-CERN Hadron Collider Physics Summer School

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Why bother ?

Understanding/testing the QCD theory of SM
in a new kinematic range never explored before (LHC)
- at many different scales (HERA, TeVatron, LHC)
- in a variety of initial states: ep, ppbar, pp

Our level of understanding and modeling of the QCD interactions has direct impact on the potential we have for precision measurements and discovery

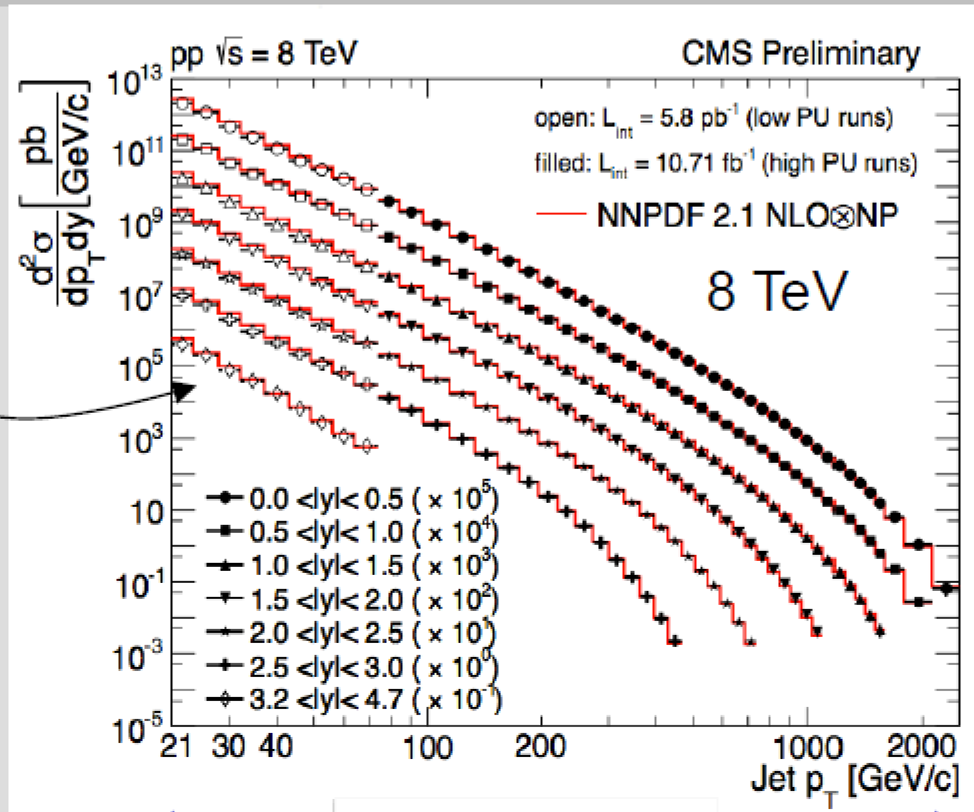
Largest syst. For Higgs cross-section is from σ(ggF): 7% scales / 7% PDFs

W/Z+jet is often one of the largest background to top-quark, SUSY, Higgs and exotic searches.

Slide taken from Chiara Roda, *Probing QCD & Hadron Physics*, ICHEP 2014

STDM-2013-11 CMS-PAS-FSQ-12-031 CMS-PAS-SMP-12-012

Inclusive cross-section @ 8 TeV



Low pile-up data to extend to the low p_T range down to 20 GeV and |y|<4.7

11 orders of magnitude

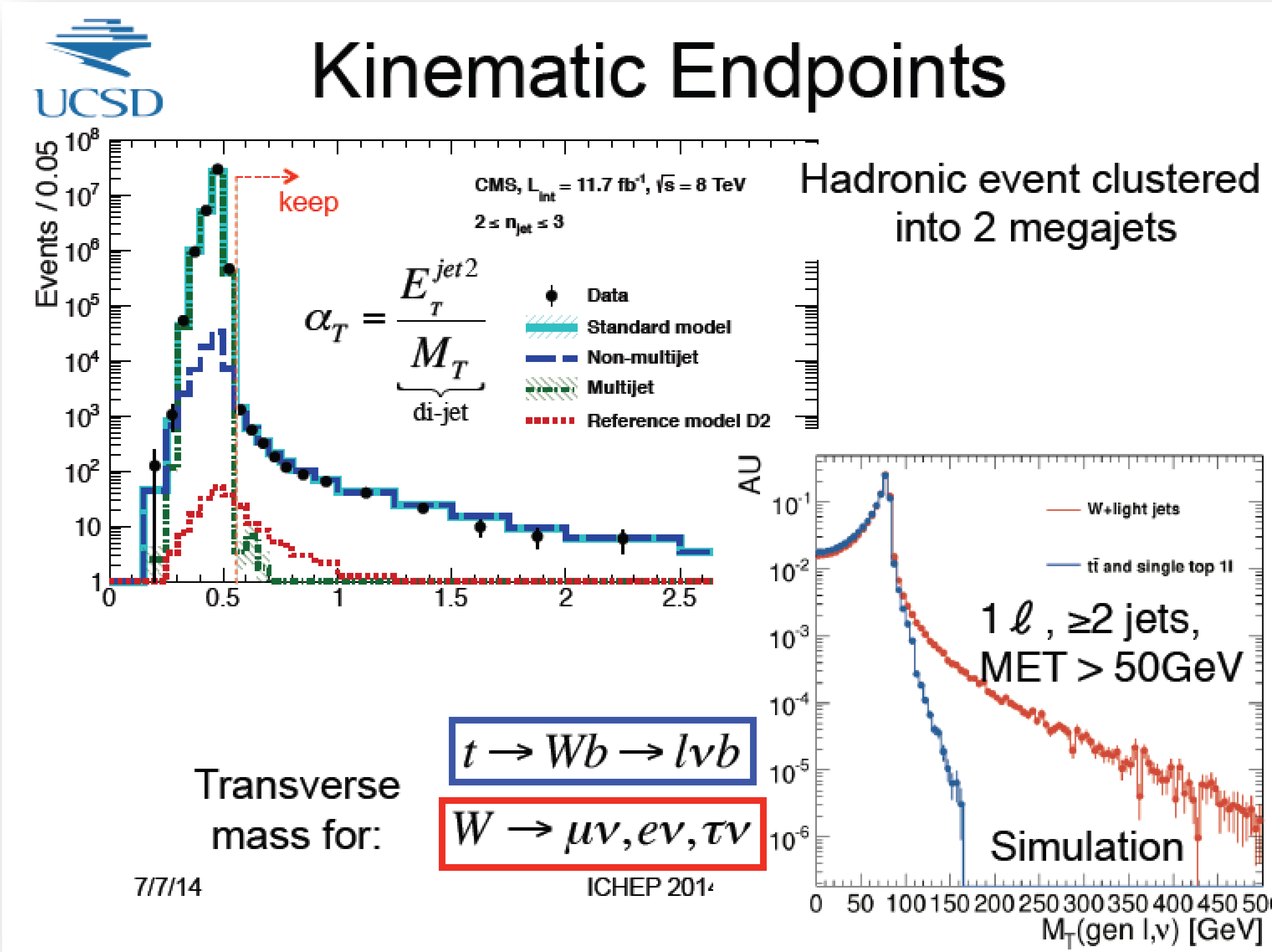
20 Gev – 2 TeV

LHC data allows pQCD tests in a new kinematic regime – extended in p_T and y

Covers 11 orders of magnitude / two jet sizes

Reference prediction: NLOJET + NNPDF2.1 but other PDF tested

Slide taken from Chiara Roda, *Probing QCD & Hadron Physics*, ICHEP 2014



Slide taken from Frank Wurthwein, *Beyond The Standard Model*, ICHEP 2014

WHY BOTHER?

- Because QCD is a fundamental part of nature
- We might come to a new era of precision QCD
- Because the main working tools at a hadron machine are Jets, Missing ET, Leptons, …
- And a possible new physics signal will show up on top of lots of *old physics* processes
- And… because it is beautiful!!

Useful References

- R.K. Ellis, W.J. Stirling and B.R. Webber, *QCD and Collider Physics*
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- S. Bethke, G. Dissertori and G.P. Salam, *Quantum Chromodynamics,* (Particle Data Group), 2012
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- Z. Bern, S. Dittmaier, L. Dixon et al., *The NLO Multileg Working Group: Summary Report*, arXiv:0803.0494

QCD AT THE LHC

THE QCD LAGRANGIAN

QED, Gauge invariance, SU(N), Feynman Rules

RUNNING COUPLING

RGE, QCD beta function, Running alphas, Measurements, Scales in QCD

FACTORIZATION OF QCD AMPLITUDES

Collinear limit, Splitting Funcs, Gral Soft/Coll Relations, An Application

Local Gauge Invariance

Start by writing the classical Fermion Lagrangian:

$$\mathcal{L}_{fermion} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi = \bar{\psi}(i\not{\partial} - m)\psi \tag{1}$$

with $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Notice how the global invariance $\psi \to e^{i\theta}\psi$ of this Lagragian can be made local ($\theta \to \theta(x)$) by replacing ∂_μ with the covariant derivative:

$$\not{D}_\mu = \partial_\mu + ieA_\mu \tag{2}$$

where A_μ is a new field that transforms as:

$$A_\mu(x) \to A_\mu(x) + \frac{i}{e}(\partial e^{i\theta(x)})e^{-i\theta(x)} \tag{3}$$

For this new field we introduce a kinematic term with the use of the *field strength tensor* $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

The QED Lagrangian

We arrive then at the classical Lagrangian

$$\mathcal{L}_{classical} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \quad (4)$$

Notice that a mass term $m^2 A^\mu A_\mu$ for the vector field is not allowed by gauge invariance!

We want to extract the Feynman rules for the quantum theory. In finding the propagator of the vector field we encounter the problem of solving:

$$\Delta_{\mu\nu}(p) i \left[p^2 g^{\nu\sigma} - p^\nu p^\sigma\right] = \delta_\mu^\sigma \quad (5)$$

Which actually have no solution. This is a consequence of the redundancy of gauge invariant terms in the action of our theory. We then add a gauge fixing term to the Lagrangian:

$$\mathcal{L}_{gauge-fix} = \begin{cases} -\frac{1}{2\lambda}(\partial^\mu A_\mu)^2 & \text{covariant gauge} \\ -\frac{1}{2\lambda}(n^\mu A_\mu)^2 & \text{axial gauge} \end{cases}$$

The Photon Propagator

In the covariant gauge, we arrive at the equation for the photon propagator:

$$\Delta_{\mu\nu}(p) i \left[p^2 g^{\nu\sigma} - \left(1 - \frac{1}{\lambda} \right) p^\nu p^\sigma \right] = \delta_\mu^\sigma \tag{6}$$

which returns the propagator:

$$\Delta_{\mu\nu}(p) = \frac{i}{p^2} \left(-g_{\mu\nu} + (1 - \lambda) \frac{p_\mu p_\nu}{p^2} \right) \tag{7}$$

λ is a free parameter, the gauge parameter, and physical quantities should not depend on it. Picking a λ we *fix* a gauge. Common choices are:

$$\lambda = 1 \qquad \text{the Feynman gauge} \tag{8}$$

$$\lambda \to 0 \qquad \text{the Landau gauge} \tag{9}$$

Axial Gauge

Similarly we can find the propagator of the photon in an axial gauge $n^\mu A_\mu = 0$:

$$\Delta_{\mu\nu}(p) = \frac{i}{p^2}\left(-g_{\mu\nu} + \frac{p_\mu n_\nu + n_\mu p_\nu}{n \cdot p} - \frac{n^2 + \lambda p^2}{(n \cdot p)^2} p_\mu p_\nu\right) \tag{10}$$

Although more complicated, axial gauges have the nice properties that in axial gauges photons have two polarization states transverse to their momentum $\rightarrow$ *physical gauge* (see more later).

A common choice of gauge parameters λ and n is the so called *lightcone gauge*, for which $\lambda \rightarrow 0$ and $n^2 = 0$.

Color $SU(3)$

The Group $SU(N)$

- Group of unitary $N \times N$ matrices with determinant 1
- An element of the group $M \in SU(N)$ close to the identity can be written with the relation $M = 1 + i\epsilon G$, as long as G is Hermitian and traceless
- A basis t^A for Hermitian traceless $N \times N$ matrices have $N^2 - 1$ elements $(A = 1, \cdots, N^2 - 1)$
- They form a Lie Algebra $(su(N))$ with $[t^A, t^B] = if^{ABC} t^C$, where f^{ABC} are called the *structure constants* of the group
- A general $K \in SU(N)$ can be expressed as $K = \exp(i\theta^A t^A)$

Color $SU(3)$

The particular case of $SU(3)$ is of special interest, as it is the *gauge group* that builds QCD. Commonly we write $t^A = \lambda^A/2$, with λ^A the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- t^A defines the dimension 3 *fundamental* representation of $SU(3)$ (quarks)
- Constructing the matrices $(T^A)_{BC} = -if^{ABC}$ one obtains another set of matrices that obey the Lie Algebra
- T^A defines the dimension $3^2 - 1 = 8$ *adjoint* representation of $SU(3)$ (gluons)

QCD Lagrangian

Let's now write the Lagrangian for QCD

$$\mathcal{L}_{QCD} = \mathcal{L}_{Yang-Mills} + \mathcal{L}_{fermions} + \mathcal{L}_{gauge-fix} + \mathcal{L}_{ghost} \qquad (11)$$

Where we have:

$$\mathcal{L}_{Yang-Mills} = -\frac{1}{4} F^A_{\mu\nu} F^{A\mu\nu} \ , \quad F^A_{\mu\nu} = \partial_\mu \mathcal{A}^A_\nu - \partial_\nu \mathcal{A}^A_\mu - g_s f^{ABC} \mathcal{A}^B_\mu \mathcal{A}^C_\nu$$

$$\mathcal{L}_{fermions} = \sum_{flavours} \bar{q}_i \left(i \not{D}_{ij} + m_q \delta_{ij} \right) q_j, \quad D_{\mu ij} = \delta_{ij} \partial_\mu + i g_s (t^A \mathcal{A}^A_\mu)_{ij} \qquad (12)$$

$i, j = 1, 2, 3$ and $A = 1, \cdots, 8$. These pieces of the Lagrangian are invariant under the local $SU(3)$ gauge transformations:

$$q_i(x) \to q'_i(x) = \left(e^{i\theta^A(x) t^A} \right)_{ij} q_j(x)$$

$$t^A \mathcal{A}^A_\mu(x) = t \cdot \mathcal{A}_\mu(x) \to t \cdot \mathcal{A}'_\mu = e^{it\cdot\theta} t \cdot \mathcal{A}_\mu e^{-it\cdot\theta} + \frac{i}{g_s} \left(\partial_\mu e^{it\cdot\theta} \right) e^{-it\cdot\theta} \qquad (13)$$

QCD Lagrangian

We want to quantize the theory, and then the need form the *gauge fixing* and *ghost* terms.

$$\mathcal{L}_{gauge-fix} = \begin{cases} -\frac{1}{2\lambda}(\partial^\mu \mathcal{A}_\mu^A)^2 & \text{covariant gauge} \\ -\frac{1}{2\lambda}(n^\mu \mathcal{A}_\mu^A)^2 & \text{axial gauge} \end{cases} \quad (14)$$

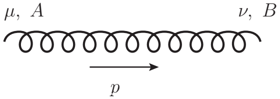
If like in QED we add only gauge fixing terms, we would find that unphysical degrees of freedom propagate for gluons. For that reason one introduces a complex scalar field η, with Fermi statistics:

$$\mathcal{L}_{ghost} = \begin{cases} \partial^\mu (\eta^A)^\dagger (D_\mu^{AB} \eta^B) & \text{covariant gauge} \\ -(\eta^A)^\dagger n_\mu (\partial^\mu \eta^A) & \text{axial gauge} \end{cases} \quad (15)$$

- In covariant gauges we need to include Feynman rules for the ghost field
- In axial gauges the ghost do not couple to the gluons, and so only physical d.o.f propagate $\rightarrow$ *physical gauges* (although not so physical...)

Gluon Propagator and Polarizations

Finally we encounter the gluon propagator:



$$\longrightarrow \qquad \Delta^{AB}_{\mu\nu}(p) = \frac{i\delta^{AB}}{p^2} d_{\mu\nu}(p)$$

where the tensor $d_{\mu\nu}(p)$ is connected to the sum over vector polarizations:

$$d_{\mu\nu}(p) = \sum_{\text{polarizations}} \epsilon^*_\mu \epsilon_\nu$$

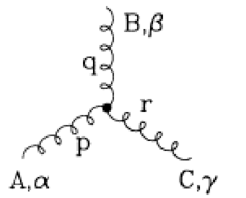
$$= \begin{cases} -g_{\mu\nu} + (1-\lambda)\dfrac{p_\mu p_\nu}{p^2} & \text{convariant gauge} \\ -g_{\mu\nu} + \dfrac{p_\mu n_\nu + p_\nu n_\mu}{p \cdot n} & \text{lightcone gauge} \end{cases}$$

QCD Feynman Rules

$$\delta^{AB}\,\frac{i}{(p^2+i\epsilon)}$$ ghost propagator

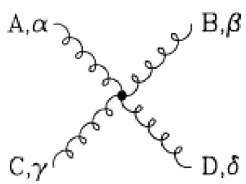
$$\delta^{ab}\,\frac{i}{(\not p-m+i\epsilon)_{ji}}$$ fermion propagator

Other Propagators



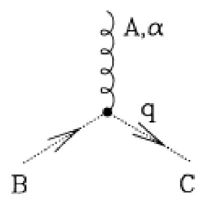
$$-g\, f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

(all momenta incoming, $p+q+r = 0$)

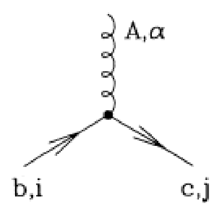


$$-ig^2\, f^{XAC}f^{XBD}\,[g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\, f^{XAD}f^{XBC}\,[g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\, f^{XAB}f^{XCD}\,[g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$

Gluon Self Interactions



$$g\, f^{ABC}q^{\alpha}$$ gluon-ghost vertex

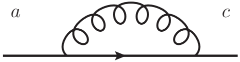


$$-ig\,(t^A)_{cb}\,(\gamma^{\alpha})_{ji}$$ gluon-quark vertex

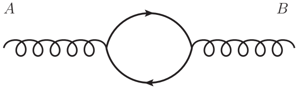
ffV vertices

From Ellis, Stirling and Webber

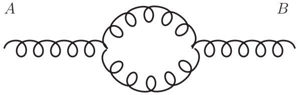
Few Color Identities



$$\longrightarrow \qquad t^A_{ab} t^A_{bc} = C_F \delta_{ac} \tag{16}$$



$$\longrightarrow \qquad \mathrm{Tr}(t^A t^B) = T_R \delta^{AB} \tag{17}$$



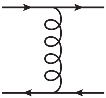
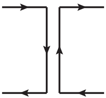
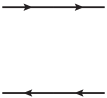
$$\longrightarrow \qquad \mathrm{Tr}(T^A T^B) = C_A \delta_{AB} \tag{18}$$

- $C_F = \frac{N^2-1}{2N} \overset{N=3}{=} \frac{4}{3}$
- $T_R = \frac{1}{2}$
- $\sum_{CD} f^{CDA} f^{CDB} = N\delta^{AB} \to C_A = N \overset{N=3}{=} 3$

Few Color Identities

The Fierz Identity

$$t^A_{ab} t^A_{cd} \quad = \quad \frac{1}{2}\delta_{ad}\delta_{cb} \quad - \frac{1}{2N}\delta_{ab}\delta_{cd} \tag{19}$$

And finally an useful identity for writing amplitudes in a *color ordered* way:

$$[t^A, t^B] = i f^{ABC} t^C \;\Rightarrow\; f^{ABC} = -2i\mathrm{Tr}([t^A, t^B] t^C) \tag{20}$$

QCD AT THE LHC

THE QCD LAGRANGIAN

QED, Gauge invariance, SU(N), Feynman Rules

RUNNING COUPLING

RGE, QCD beta function, Running alphas, Measurements, Scales in QCD

FACTORIZATION OF QCD AMPLITUDES

Collinear limit, Splitting Funcs, Gral Soft/Coll Relations, An Application

Coupling Constant α_s

In $\mathcal{L}_{QCD}$ we introduced the parameter g_s. The strong coupling constant is defined by $\alpha_s = g_s^2/(4\pi)$.

- A dimensionless observable R only depending on a single large energy scale Q is computed by perturbations as a series in α_s
- Although we would expect R to be a constant, renormalization introduces a second scale μ_r, which makes R generally depending on Q^2/μ_r^2
- But μ_r is an unphysical scale, then if having R depending on Q^2/μ_r^2 and α_s we find the *renormalization group equation*:

$$\mu_r^2 \frac{\partial}{\partial \mu_r^2} R(Q^2/\mu_r^2, \alpha_s) = \left[\mu_r^2 \frac{\partial}{\partial \mu_r^2} + \mu_r^2 \frac{\partial \alpha_s}{\partial \mu_r^2} \frac{\partial}{\partial \alpha_s} \right] R = 0 \tag{21}$$

$$\tau = \log\left(\frac{Q^2}{\mu_r^2}\right), \ \beta(\alpha_s) = \mu_r^2 \frac{\partial \alpha_s}{\partial \mu_r^2} \ \rightarrow \ \left[-\frac{\partial}{\partial \tau} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right] R = 0 \tag{22}$$

The Running α_s

Defining $\alpha_s(\mu_r^2) = \alpha_s$ and writing

$$t = \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)} \rightarrow \frac{\partial \alpha_s(Q^2)}{\partial t} = \beta(\alpha_s(Q^2))$$

the RGE is shown to be solved by $R(1, \alpha_s(Q^2))$!

- Dependence in the scale Q^2 in R comes from renormalization
- As long as α_s small, we can compute R perturbatively and then the β function

$$\beta(\alpha_s) = -\alpha_s^2(\beta_0 + \beta_1 \alpha_s + \cdots) \tag{23}$$

It is found

$$\beta_0 = \frac{1}{12\pi}(11C_A - 4T_R n_f) = \frac{1}{12\pi}(11N - 2n_f)$$

and so $\beta(\alpha_s) < 0$, that is $\alpha_s(Q^2)$ decreases for growing Q^2!

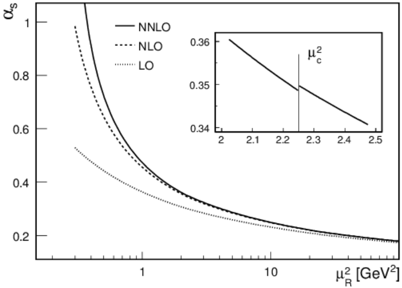
The Running α_s

Keeping only the term β_0, we find the *leading log* expression:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\beta_0 \log(\frac{Q^2}{\mu^2})} \qquad (24)$$

Compare with the analogous QED results:

$$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \log(\frac{Q^2}{m_e^2})} \qquad (25)$$



Asymptotic freedom and IR slavery

In Eq. (24) a special scale Λ ($\sim$ 300 MeV) at which the coupling diverges $\rightarrow$ *dimensional transmutation*

α_s Measurements

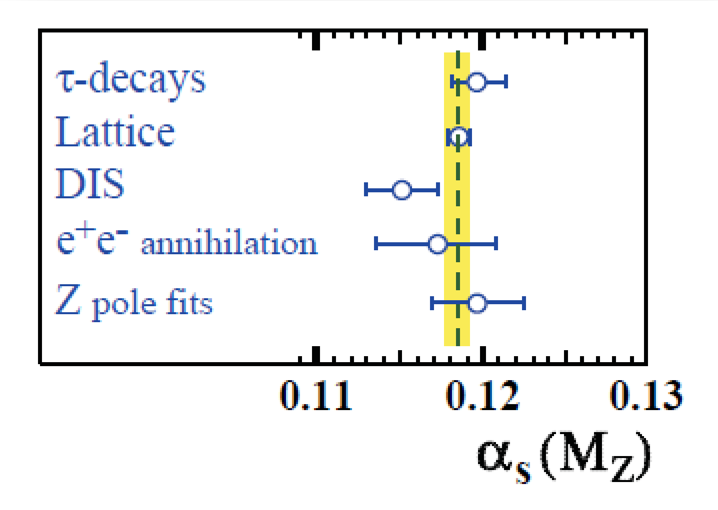
- The Strong Coupling Constant is in itself not a physical observable
- It enters the perturbative expression of experimentally measurable observables
- For example it is studied in jet production cross sections, hadron and τ lepton decays, event shapes, etc
- Consequently, determinations of α_s depend on the availability of precise predictions for the related observables
- Finally, it is customary to relate measurements at different scales through running to that of the value of $\alpha_s(M_Z^2)$

The world average value of the strong coupling constant is:

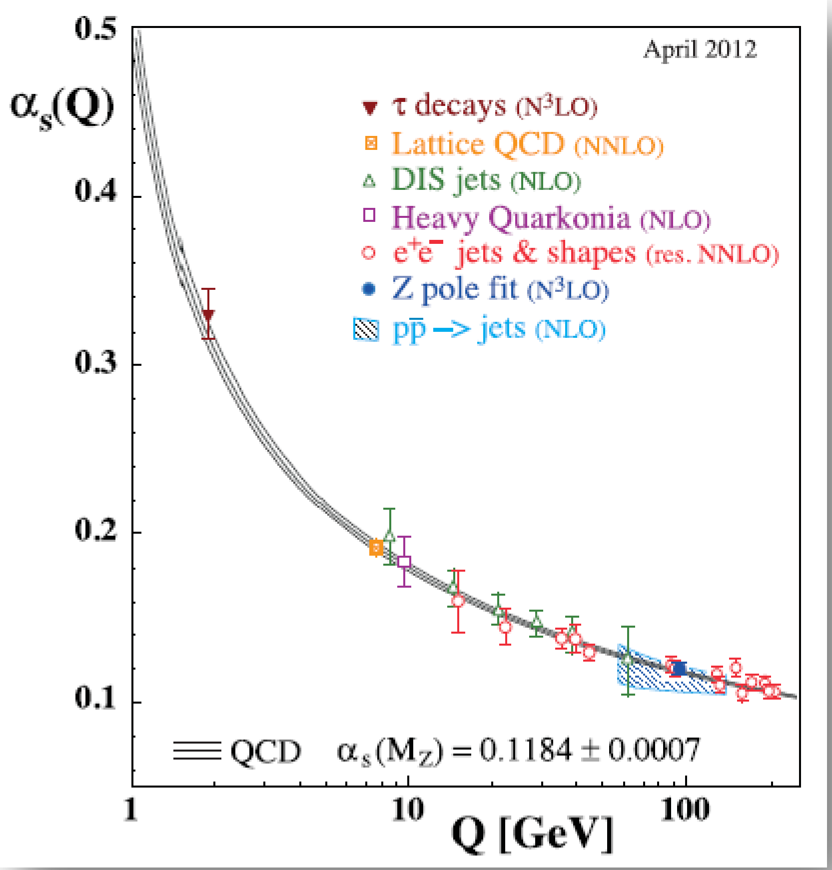
$$\alpha_s(M_Z^2) = 0.1185 \pm 0.0006$$

As presented on the *Review on Particle Physics* by the PDG in 2013

α_s Measurements



$$\alpha_s(M_Z^2) = \quad 0.1185 \pm 0.0006$$



Some of the most precise data on α_s comes from hadronic τ decay, results from the lattice, structure functions in DIS, hadron production at lepton colliders

Notice the different scales at which these observables are measured! The running is evident in the data!

The problem with unphysical scales

- Physical Observable R computed as a perturbative series in α_s
- Although α_s depends on μ_r, *in principle R* should not depend on the unphysical μ_r
- In practice the perturbative series of R is truncated, and computed at *Fixed Order*
- If we keep only the first term on the perturbative series we call it the LO (*leading order*) approximation, two terms NLO (*next-to-leading order*), three NNLO (*next-to-next-to-leading order*), and so on…
- The truncated theoretical observables (R_{LO}, R_{NLO}, R_{NNLO},…) acquire then a dependence on μ_r
- Such spurious dependence will decrease for Higher Order Calculations
- Actually (for high enough order) the spurious dependence will be of the order of the higher order terms not included (as they will cancel such dependence!)
- Then, the unphysical scale dependence CAN be used as a PROXY of the theoretical uncertainty of the perturbative calculation (WITH CARE, of course…)

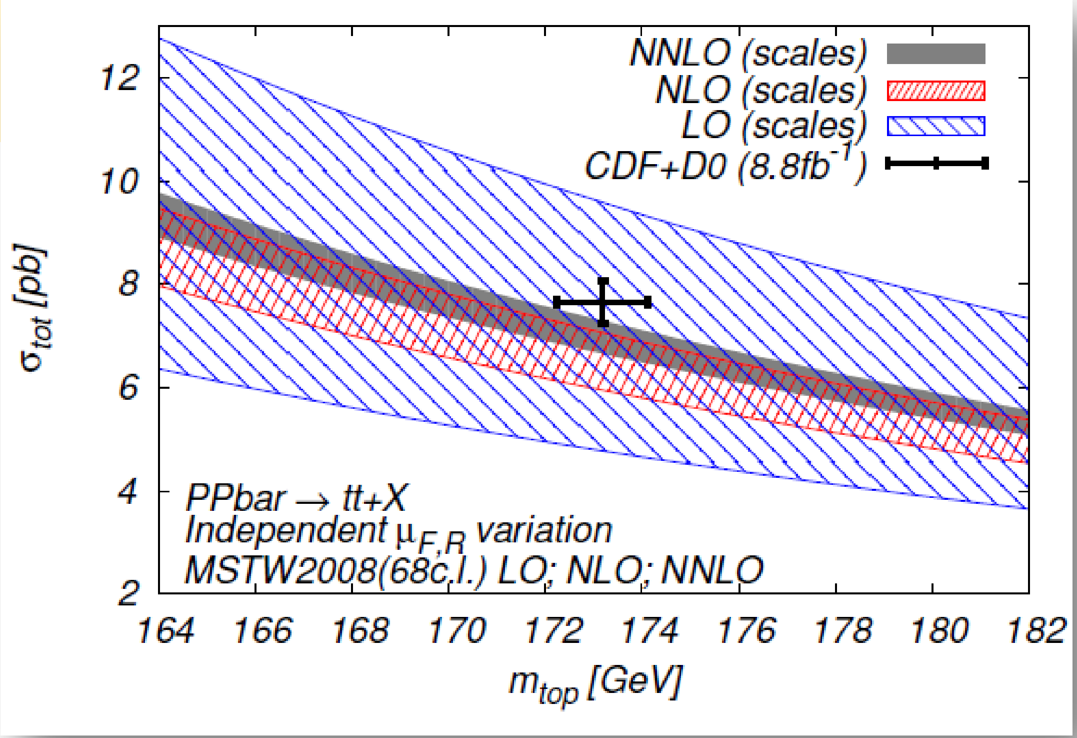
Similar considerations can be made for another unphysical scale that appears in calculations for hadron colliders, called the factorization scale μ_f
(***We will come back to this!***)

Top pair production example

- R as a perturbative series in α_s
- R should in principle not depend on μ_r
- *Fixed Order* $R \rightarrow R_{LO}, R_{NLO}, R_{NNLO},\ldots$
- They acquire a spurious dependence on μ_r
- Dependence decrease *LO* → *NLO* → *NNLO*…
- Unphysical scale dependence CAN be used as a PROXY of the theoretical uncertainty!
- WITH CARE!!!

The convergence of the perturbative series for this observable is clear. Notice the *scale band* overlap. Similar features are found for many observables!

Czakon, Fiedler, Mitov, Rojo arXiv:1305.3892



CARE! Special features might appear

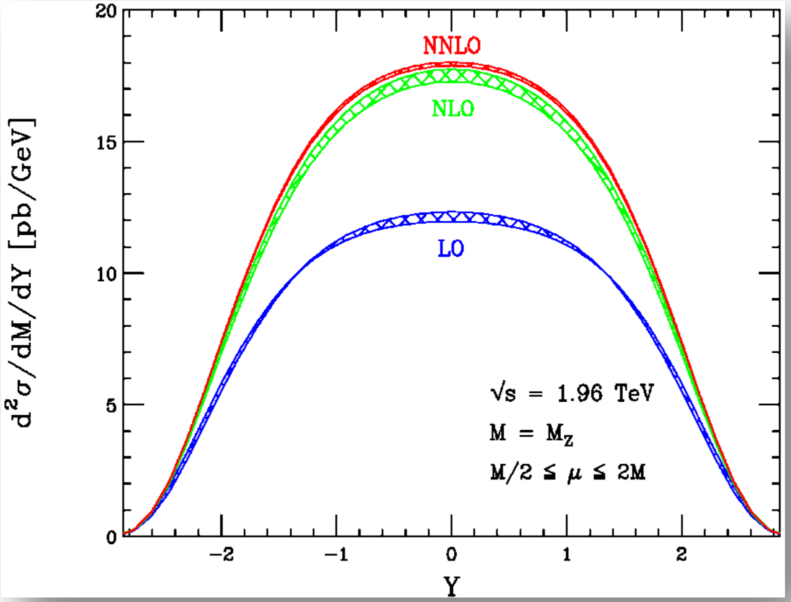
Compare for example Drell-Yan Production at NNLO with Top pair Production at NNLO

Certain processes might present special features that even seem to question the pertinence of perturbation theory, due for example to the presence of large *K-factors* (i.e. large (N)NLO/LO ratios)

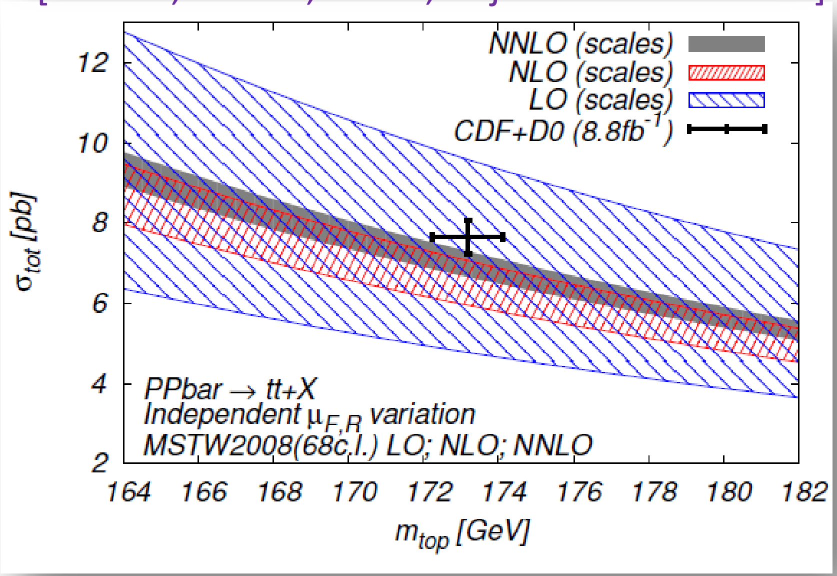
But this is in well understood:

- Not so small α_s at scales of relevance
- Opening of new (gluon) initiated subprocesses at higher orders
- Release of kinematical constrains in quantum corrections

[Anastasiou, Dixon, Melnikov, Petriello hep-ph/0312266]



[Czakon, Fiedler, Mitov, Rojo arXiv:1305.3892]

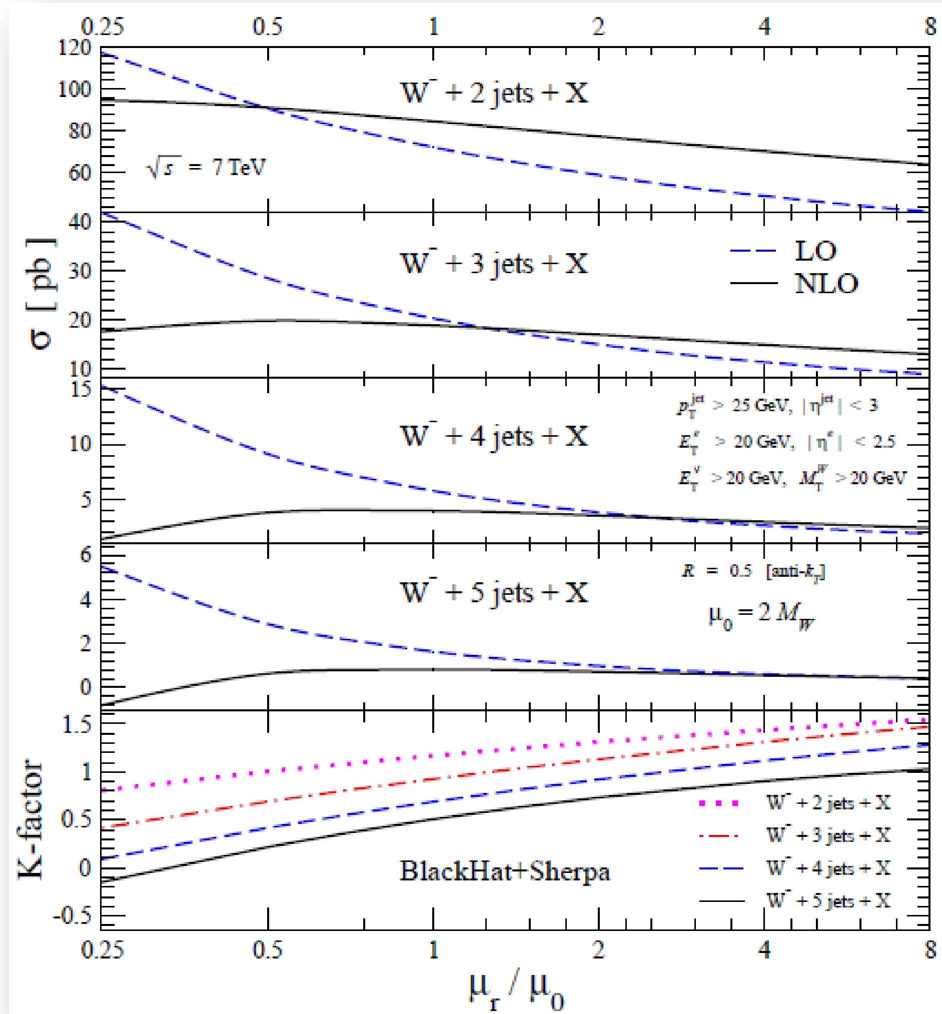


NLO the first level for quantitative predictions

[Bern, Dixon, FFC, Hoeche, Ita, Kosower, Maitre, Ozeren arXiv:1304.1253]

W+ *n* Jet Production

- LO unphysical scale dependence is large
- It grows with jet multiplicity
- Even more, shapes of distributions modified by quantum corrections
- NLO scale uncertainty more stable over multiplicity of jets
- NLO gives first quantitative prediction for observables
- Precision QCD (down to few percent uncertaity) needs NNLO!

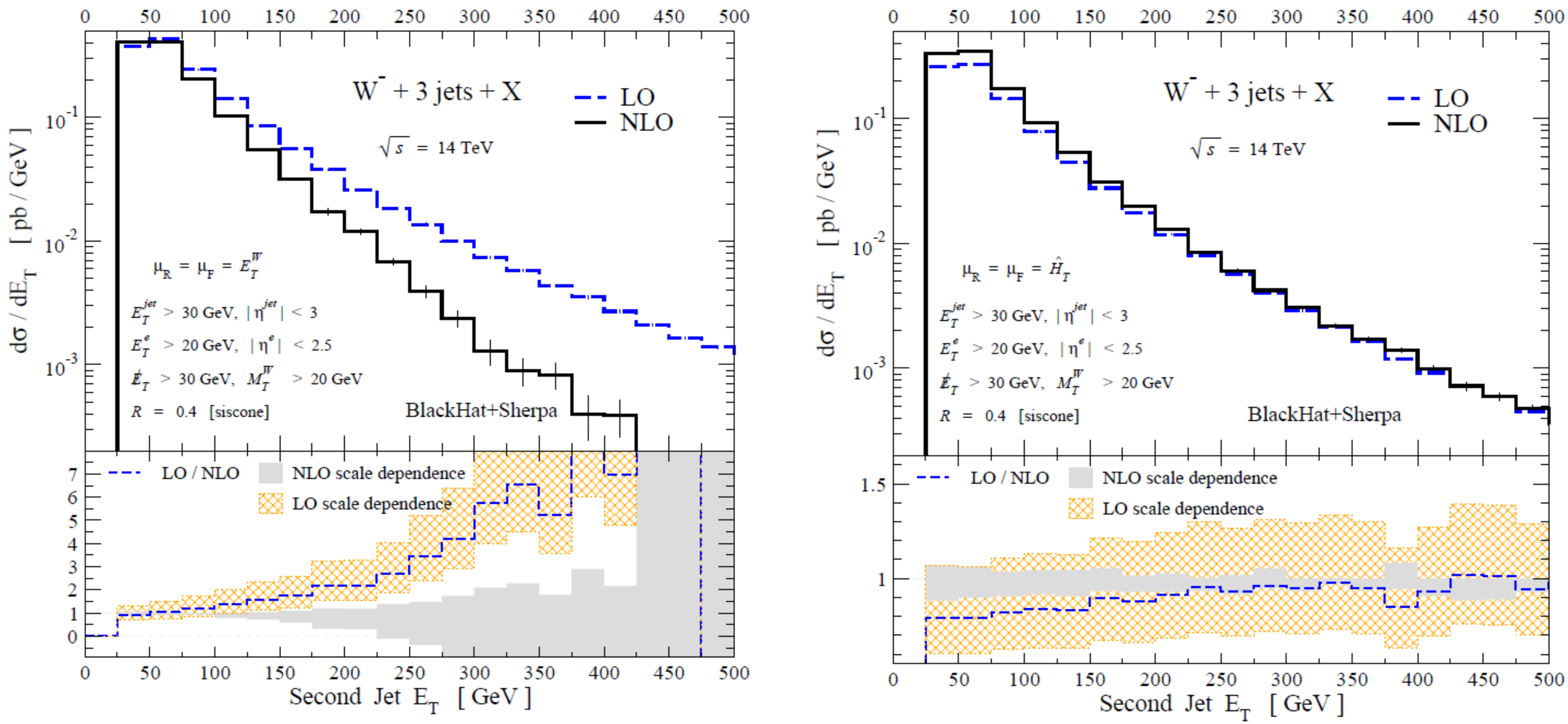


Dynamical Scales

At the LHC one samples large kinematical ranges!

[Berger, Bern, Dixon, FFC, Forde, et al arXiv:0907.1984]

$\mu = E_T^W$ $\qquad$ $\mu = \hat{H}_T$



Fixed scales are not proper! What to choose for fixed order calculations?

- LO/NLO ratio sensible.
- NLO guides scale choices

QCD AT THE LHC

THE QCD LAGRANGIAN

QED, Gauge invariance, SU(N), Feynman Rules

RUNNING COUPLING

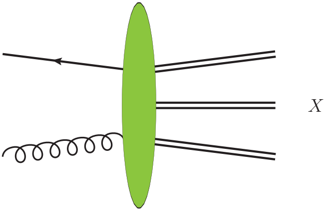
RGE, QCD beta function, Running alphas, Measurements, Scales in QCD

FACTORIZATION OF QCD AMPLITUDES

Collinear limit, Splitting Funcs, Gral Soft/Coll Relations, An Application

Producing X via a $\bar{q}g$ channel

Suppose you are studying some production channels of your preferred signal X

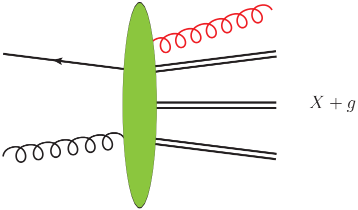


Start for computing the born level cross section, and then ask

how can I get extra radiation on on top of X?

Start with adding a gluon!

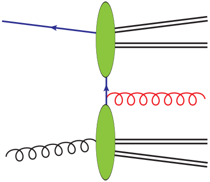
- $\mathcal{O}(\alpha_s)$ corrections to your signal
- Part of the *real* NLO corrections



Extra gluon emission $\bar{q}g \to X + g$

Pay attention to the diagrams in which the extra gluon couples to the external $\bar{q}$ line:

$\mathcal{A}_{\bar{q}g\to g+X}$ =



$$+ \underbrace{\cdots}_{\text{Other diagrams with } g \text{ not coupling to } \bar{q} \text{ line}} = \sum_i D_i + \cdots$$

In the square of the amplitude we then find:

$$|\mathcal{A}_{\bar{q}g\to g+X}|^2 = \sum_i |D_i|^2 + \sum_{i\neq j} D_i^\dagger D_j + \cdots \qquad (26)$$

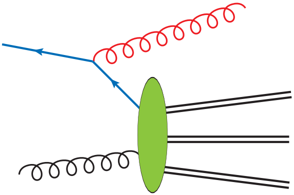
Notice that the propagator leading to the vertex that couples g and $\bar{q}$ in diagram D_j leads to a term like (we set $m_{\bar{q}} = 0$ for now!):

$$\frac{1}{(-p_{\bar{q}} + p_g + p_{X_j'})^2}$$

And so in Eq. 26 we find a potential divergent terms of the form $1/(2p_{\bar{q}} \cdot p_g)^2$!

Exploring Singularities of QCD Tree Amplitudes

These (most) singular terms come in $|\mathcal{A}_{\bar{q}g\to g+X}|^2$ from the square of the set of diagrams (let's call them D_1):



So let's explore in detail D_1 contributions!

First:

$$D_1 = g_s t^a \bar{v}(p_{\bar{q}})\gamma_\mu \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2}\ \tilde{\mathcal{A}}_{\bar{q}g\to X}\ \epsilon^{\mu*} \qquad (27)$$

In the matrix element square, we need to deal with the sum over polarizations of the g. We introduce a light-like vector n^μ with $n \cdot q \neq 0$ and write:

$$\sum_{\text{polarizations}} \epsilon^{\mu*}\epsilon^{\nu} = -g^{\mu\nu} + \frac{p_g^\mu n^\nu + p_g^\nu n^\mu}{p_g \cdot n} \qquad (28)$$

Exploring Singularities of QCD Tree Amplitudes

And then, in a sum over initial and final states degrees of freedom, we find:

$$\begin{aligned}
\sum |D_1|^2 &= g_s^2 C_F \\
&\quad \mathrm{Tr}\left\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \left[\gamma_\nu \not{p}_{\bar{q}} \gamma_\mu\right] \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \tilde{\mathcal{A}}_{\bar{q}g\to X} \right\} \\
&\qquad \left(-g^{\mu\nu} + \frac{p_g^\mu n^\nu + p_g^\nu n^\mu}{p_g \cdot n} \right) \\
&= g_s^2 C_F \\
&\quad \mathrm{Tr}\left\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \left[-\gamma^\mu \not{p}_{\bar{q}} \gamma_\mu + \frac{\not{n}\not{p}_{\bar{q}}\not{p}_g + \not{p}_g \not{p}_{\bar{q}} \not{n}}{n \cdot p_g} \right] \right. \\
&\quad \left. \frac{\not{p}_g - \not{p}_{\bar{q}}}{(p_g - p_{\bar{q}})^2} \tilde{\mathcal{A}}_{\bar{q}g\to X} \right\} \qquad (29)
\end{aligned}$$

Employing identities for Dirac's γ matrices (like $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$, etc) we obtain the compact expresion:

Exploring Singularities of QCD Tree Amplitudes

$$\begin{aligned}\sum |D_1|^2 &= g_s^2 C_F \frac{2}{(2p_{\bar{q}} \cdot p_g)^2 (n \cdot p_g)} \,\mathrm{Tr}\Big\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} (\not{p}_g - \not{p}_{\bar{q}}) \\ &\quad \Big[(n \cdot p_{\bar{q}}) \not{p}_g + (p_{\bar{q}} \cdot p_g) \not{n} \Big] (\not{p}_g - \not{p}_{\bar{q}}) \tilde{\mathcal{A}}_{\bar{q}g\to X} \Big\} \\ &= g_s^2 C_F \frac{2}{(2p_{\bar{q}} \cdot p_g)(n \cdot p_g)} \,\mathrm{Tr}\Big\{ \tilde{\mathcal{A}}^{\dagger}_{\bar{q}g\to X} \\ \Big[(n \cdot p_{\bar{q}}) \not{p}_{\bar{q}} &+ n \cdot (p_g - p_{\bar{q}}) (\not{p}_g - \not{p}_{\bar{q}}) + (p_{\bar{q}} \cdot p_g) \not{n} \Big] \tilde{\mathcal{A}}_{\bar{q}g\to X} \Big\} \end{aligned} \tag{30}$$

Here it comes the crucial step!

If we explore the regions were our diagrams diverge (i.e. were $(2p_{\bar{q}} \cdot p_g) \to 0$), this occurs either because g is *soft* or because g turns *collinear* to $\bar{q}$!

Collinear Singularities in QCD

Characterize the collinear region with the help of the *Sudakov parameterization* ($k_\perp$ is a space-live vector $\perp$ to both p_g and $p_{\bar{q}}$):

$$p_g = (1-z)p_{\bar{q}} + \beta n^\mu - k_\perp^\mu \qquad (31)$$

where picking $\beta = -k_\perp^2/(2(1-z)(n\cdot p_{\bar{q}}))$ ensures $p_g^2 = 0$.

We are going to let $k_\perp$ go to zero, and with it have a measure of how collinear is our configuration! We get:

$$\sum |D_1|^2 = g_s^2 C_F \frac{2}{(2p_{\bar{q}}\cdot p_g)(n\cdot p_g)} \mathrm{Tr}\Big\{ \tilde{\mathcal{A}}^\dagger_{\bar{q}g\to X} \left[\frac{(n\cdot p_g)}{(1-z)}\not{p}_{\bar{q}} - \frac{(p_g\cdot n)z}{(1-z)}(\not{p}_g - \not{p}_{\bar{q}}) - \frac{k_\perp^2}{2(p_g\cdot n)}\not{n}\right]\tilde{\mathcal{A}}_{\bar{q}g\to X}\Big\} \qquad (32)$$

Collinear Singularities in QCD

Now, with the use of the simple identity:

$$\not{p}_{\bar{q}} = \frac{1}{z}\left(-(\not{p}_g - \not{p}_{\bar{q}}) - \not{k}_\perp - \frac{k_\perp^2}{2(1-z)(n\cdot p_{\bar{q}})}\not{n}\right)$$

we find:

$$\sum |D_1|^2 = 2g_s^2 C_F \frac{-1}{k_\perp^2} \,\mathrm{Tr}\Big\{\tilde{\mathcal{A}}^\dagger_{\bar{q}g\to X} \Big[(-\frac{1}{z} - z)(\not{p}_g - \not{p}_{\bar{q}}) + \mathcal{O}(k_\perp^2)\Big]\tilde{\mathcal{A}}_{\bar{q}g\to X}\Big\} \quad (33)$$

And notice that in the collinear limit ($k_\perp^2$ going to zero), the singular piece approximates the full amplitude square:

$$\sum |\mathcal{A}_{\bar{q}g\to g+X}|^2 \stackrel{k_\perp^2\to 0}{\approx} \sum |D_1|^2 \quad (34)$$

Collinear Singularities in QCD

And then we encounter an interesting relation!

$$\sum |\mathcal{A}_{\bar{q}g\to g+X}|^2 \overset{k_\perp^2\to 0}{\approx} 2g_s^2 C_F \frac{-1}{k_\perp^2}\frac{1+z^2}{z}\mathrm{Tr}\left\{\mathcal{A}^\dagger_{\bar{q}g\to X}(\not{p}_g - \not{p}_{\bar{q}})\mathcal{A}_{\bar{q}g\to X}\right\}$$

$$= 2g_s^2 C_F \left(-\frac{1}{k_\perp^2}\right)\frac{1+z^2}{z}\sum |\mathcal{A}_{\bar{q}g\to X}|^2 \quad (35)$$

Now suppose that you are interested in the behavior of the differential cross section around the collinear limit. Notice that you can factorize the Lorentz Invariant Phase-Space of the collinear gluon like:

$$\frac{d^3p_g}{(2\pi)^3}\frac{1}{2E_g} \overset{k_\perp^2\to 0}{\approx} \frac{1}{16\pi^2}\frac{dz}{(1-z)}d(-k_\perp^2)\frac{d\phi}{2\pi} = \frac{1}{16\pi^2}\frac{dz}{(1-z)}d(-k_\perp^2) \quad (36)$$

Where in the last step we implicitly integrate the azimuthal angle.

Collinear Factorization in QCD

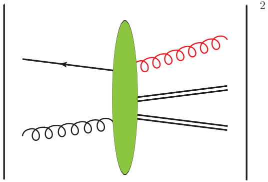
We arrive to this important *collinear* relation:

$$d\hat{\sigma}_{\bar{q}g\to g+X} \overset{k_\perp^2\to 0}{\approx} \frac{dk_\perp^2}{k_\perp^2}\frac{dz}{z}\frac{\alpha_s}{2\pi}\underbrace{\frac{1+z^2}{1-z}}_{\tilde{\mathrm{P}}_{\mathrm{qq}}(\mathrm{z})} \; d\hat{\sigma}_{\bar{q}g\to X} \tag{37}$$

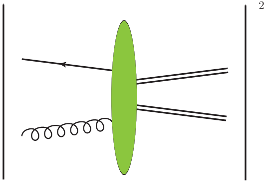
- The function $\tilde{P}_{qq}(z)$ is associated to the so called Altarelli-Parisi splitting function for a q to turn into a collinear q (and a g).
- Notice that as written, $\tilde{P}_{qq}(z)$ has a divergence for $z \to 1$, which is actually associated with a *soft* divergence.
- This is commonly regulated in order to avoid double counting when soft divergences are treated separately.

Collinear Factorization in QCD

We have found a picture of the factorization of our process $\bar{q}g \to g + X$ when the g goes collinear with the $\bar{q}$ like:



$$\overset{k_\perp^2 \to 0}{\propto} \tilde{P}_{qq}(z)$$



Comments

- If g goes collinear with the initial state gluon we find a similar result. Also for any other colored parton in the final state an associated relation is found.
- In such cases corresponding *Splitting functions appear*.
- Notice that integration over $dk_\perp^2/k_\perp^2$ is divergent, so there is need of a regularization procedure!

Mass regularization of Collinear Divergences

Consider a collinear splitting $g \to q'\bar{q}'$, and suppose the quarks q' have a mass $m > 0$. In such situation one finds that, up to powers of m^2, the singular transverse integral changes according to:

$$\frac{d|k_\perp^2|}{|k_\perp^2|} \xrightarrow{m>0} \frac{d|k_\perp^2|}{|k_\perp^2| + m^2} \tag{38}$$

Which then allows to integrate down to $k_\perp^2 = 0$, returning a $\log(Q^2/m^2)$ (Q^2 some large scale).

- The divergence is now explicit in the log of the (small) mass.
- Although a useful regularization procedure for collinear divergences with quark masses, we can't do the proper with gluon masses (as we would explicitly break gauge invariance).
- If the quark mass is of relevance for your studies (e.g. certain b quark studies) large logarithms might be present!
- Soft divergences are not regularized by m.

The $d = 4 - 2\epsilon$ Trick

A way to regularize divergences in gauge theories is the procedure called *Dimensional Regularization*. Preservation of gauge invariance, regularization of both soft and collinear divergences (and also UV!), extraction of divergences as poles in a Laurent series, are some of the properties that makes it a standard in perturbative calculation in gauge theories!

A simple idea...

$$\int d^3 r \frac{1}{|\vec{r}|^3} \to \int_{r_1}^{r_2} |\vec{r}|^2 d|\vec{r}| \frac{1}{|\vec{r}|^3} \to \log\left(\frac{r_2}{r_1}\right) \xrightarrow{r_1 \to 0} \infty$$

$$\Downarrow$$

$$\int d^{3-2\epsilon} r \frac{1}{|\vec{r}|^3} \to \int_{r_1=0}^{r_2} |\vec{r}|^{2-2\epsilon} d|\vec{r}| \frac{1}{|\vec{r}|^3} \xrightarrow{\epsilon<0} -\frac{1}{\epsilon} r_2^{|\epsilon|}$$

Volume Integrals in d Dimensions

But how to get a grasp of continuous dimensions?
(Most of the time) *Just don't!*

Recursive $(d-1)$ Solid Angle Calculation

- $d = 2 \Rightarrow \int d\Omega_1 = \int d\phi = 2\pi$, polar coordinates in $\boldsymbol{R}^2$
- $d = 3 \Rightarrow \int d\Omega_2 = \int d\phi \sin(\theta) d\theta = 4\pi$, spherical coord in $\boldsymbol{R}^3$
- $d = 4 \Rightarrow \int d\Omega_3 = \int d\phi \sin(\theta') d\theta' \sin^2(\theta) d\theta = 2\pi^2$
- $d \Rightarrow \int d\Omega_{d-1} = \int d\Omega_{d-2} \sin^{d-2}(\theta) d\theta = 2\pi^{d/2}/\Gamma(d/2)$

- The space dimension is then a parameter in your calculation and amplitudes become a Laurent series in ϵ
- By the KLN theorem, ϵ poles will cancel off phys. observables
- To keep integral dimensions correctly, one introduces a dimensionful parameter μ, the *regularization scale* (which gets identified with μ_r and μ_f), $d^4p \to \mu^{2\epsilon} d^{d=4-2\epsilon} p$

Spitting Functions in Dimensional Regularization

We can then go ahead and revisit our collinear factorization in d dimensions. We would find a similar picture, with the leading order, d dimensional, massless, unregulated, averaged over polarizations Splitting functions $\hat{P}_{ij}(z)$ for the spitting process $i \to jk$:

Altarelli-Parisi Splitting Functions

- $\hat{P}_{qq}(z) = C_F \left(\frac{1+z^2}{1-z} - (1-z)\epsilon \right)$
- $\hat{P}_{qg}(z) = C_F \left(\frac{1+(1-z)^2}{z} - (z)\epsilon \right)$
- $\hat{P}_{gq}(z) = T_R \left(1 - \frac{2z(1-z)}{1-\epsilon} \right)$
- $\hat{P}_{gg}(z) = C_A \left(\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right)$

QCD General Factorization in Soft and Collinear Limits

Some of the most important properties for tree level QCD amplitudes are indeed their factorizing behavior when soft and collinear limits are taken. We are ready to enunciate these relations (and you can prove them before the discussion session!)

- For a process like $a(p_a) + b(p_b) \to i_1(p_1) + \cdots + i_n(p_n)$ we write the QCD tree level amplitude like $\mathcal{A}(\{c_a, s_a, p_a\}, \{c_b, s_b, p_b\}; \{c_1, s_1, p_1\}, \cdots, \{c_n, s_n, p_n\}) \equiv \mathcal{A}_{2,n}$
- Construct a ket $|a, b; 1, \cdots, n\rangle_{2,n}$ in color and spin space such that the coefficient of a given element in color and spin space $|\{c_a, s_a\}, \{c_b, s_b\}; \{c_1, s_1\}, \cdots, \{c_n, s_n\}\rangle$ would be this amplitude
- With this notation you get the relation:

$$\sum_{\text{colors,spins}} |\mathcal{A}_{2,n}|^2 = {}_{2,n}\langle a, b; 1, \cdots, n|a, b; 1, \cdots, n\rangle_{2,n}$$

Collinear Limits

Consider the final state splitting $(ij) \to ij$. Employing the Sudakov parameterization:

$$p_i^\mu = zp^\mu + k_\perp^\mu - \frac{k_\perp^2}{2zp\cdot n} n^\mu, \quad p_j^\mu = (1-z)p^\mu - k_\perp^\mu - \frac{k_\perp^2}{2(1-z)p\cdot n} n^\mu$$

We can then generalize our previous collinear relation to:

$$_{2,n+1}\langle a, b; 1, \cdots, n+1 | a, b; 1, \cdots, n+1\rangle_{2,n+1} \xrightarrow{k_\perp^2 \to 0}$$

$$\frac{4\pi\mu^{2\epsilon}\alpha_s}{p_i \cdot p_j} \, _{2,n}\left\langle a, b; \underbrace{1, \cdots, n+1}_{i,\ j \text{ replaced by} (ij)} \middle| \hat{P}_{(ij),i}(z, k_\perp, \epsilon) \middle| a, b; \underbrace{1, \cdots, n+1}_{i,\ j \text{ replaced by} (ij)} \right\rangle_{2,n}$$

Here $\hat{P}_{(ij),i}(z, k_\perp, \epsilon)$ can in general be polarization dependent (*spin correlations!*). If the splitting parton was in the initial state, we reproduce our previous result (with the extra $1/z$ factor).

Soft Limits in QCD

Soft divergences appear when a final state gluon momenta goes to zero. Let's introduce a dimensionless parameter λ to parameterize the soft limit:

$$p_j^\mu = \lambda q^\mu$$

Then, in the limit $\lambda \to 0$ it is found:

$$_{2,n+1}\langle a,b;1,\cdots,n+1|a,b;1,\cdots,n+1\rangle_{2,n+1} \longrightarrow$$

$$-\frac{8\pi\mu^{2\epsilon}\alpha_s}{\lambda^2}\sum_i \frac{1}{p_i\cdot q}\sum_{k\neq i}\frac{p_k\cdot\pi}{(p_i+p_k)\cdot q}$$

$$_{2,n}\left\langle a,b;\underbrace{1,\cdots,n+1}_{j\ \text{removed}}\middle|\mathbf{C}_{ki}\middle|a,b;\underbrace{1,\cdots,n+1}_{j\ \text{removed}}\right\rangle_{2,n}$$

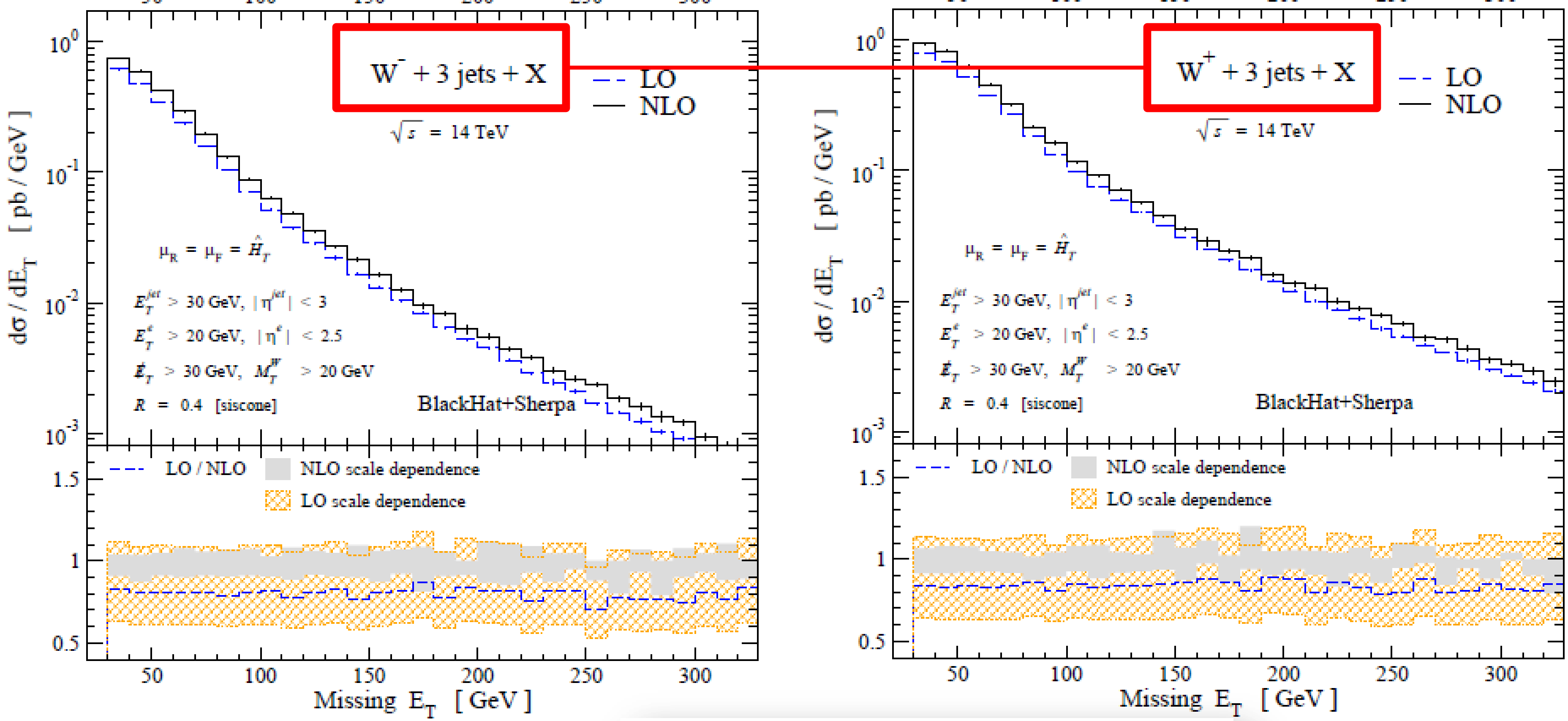
The last amplitude is a color correlated amplitude, in which the operator $\mathbf{C}_{ki}$ represents an insertion of the color degrees of freedom of a gluon between the partons k on the left and i on the right.

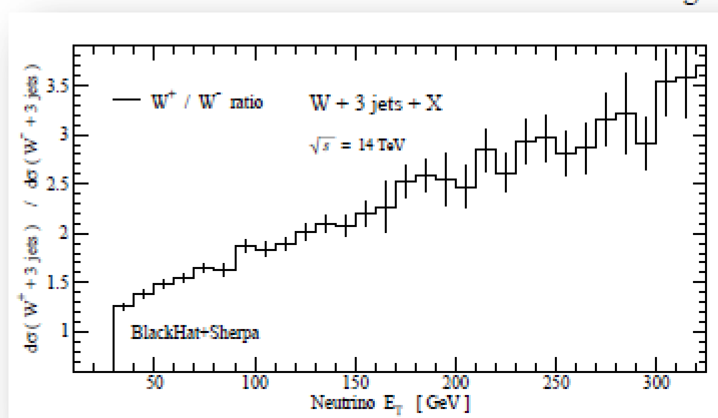
IR Limits in QCD Processes

- After two partons go collinear, square of QCD amplitudes factorize into a lower point amplitudes times a divergent term and a Splitting function. Spin correlations remain.
- If a final state gluon goes soft, square of QCD amplitudes produce a divergent term times a color correlated amplitude.
- These divergences are commonly regulated using dimensional regularization.
- In the same spirit of what we studied, multi-particle divergences appear in QCD amplitudes. Later in this set of lectures we will employ them to further our understanding of gauge theory amplitudes!

LARGE PT W POLARIZATION IN FACTORIZATION

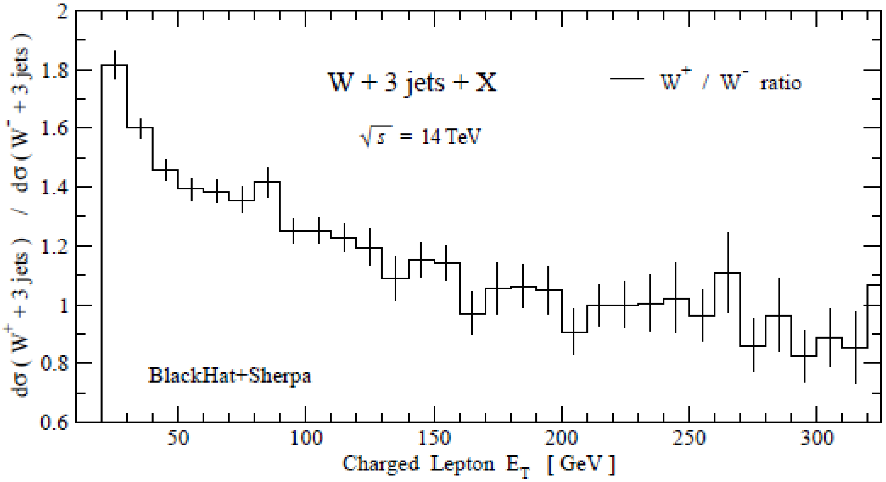
Finding W Polarization In An Odd Place

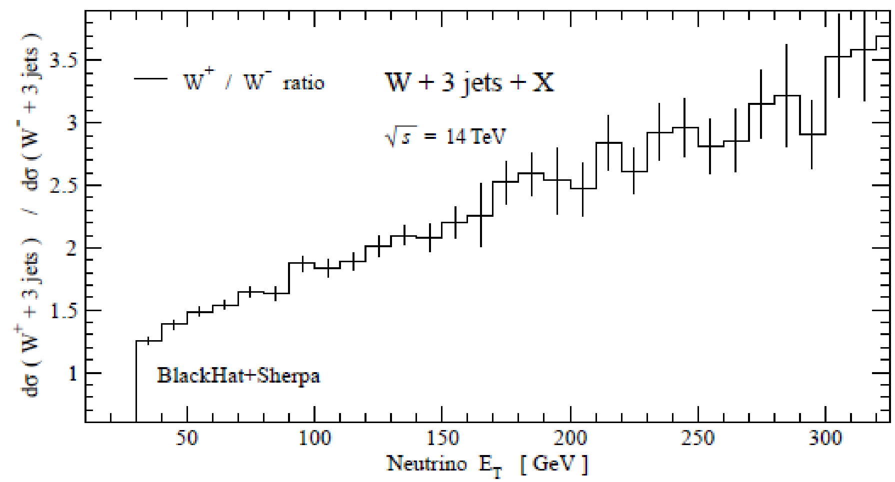




Leptonic E_T in W + 3 jets at LHC

[Bern, Dixon, FFC, Hoeche, Ita, et al. arXiv:0907.1984, arXiv:1103.5445]





– W^+/W^- transverse lepton ratios trace a remarkably large left-handed W polarization at large $p_T(W)$

– independent of number of jets

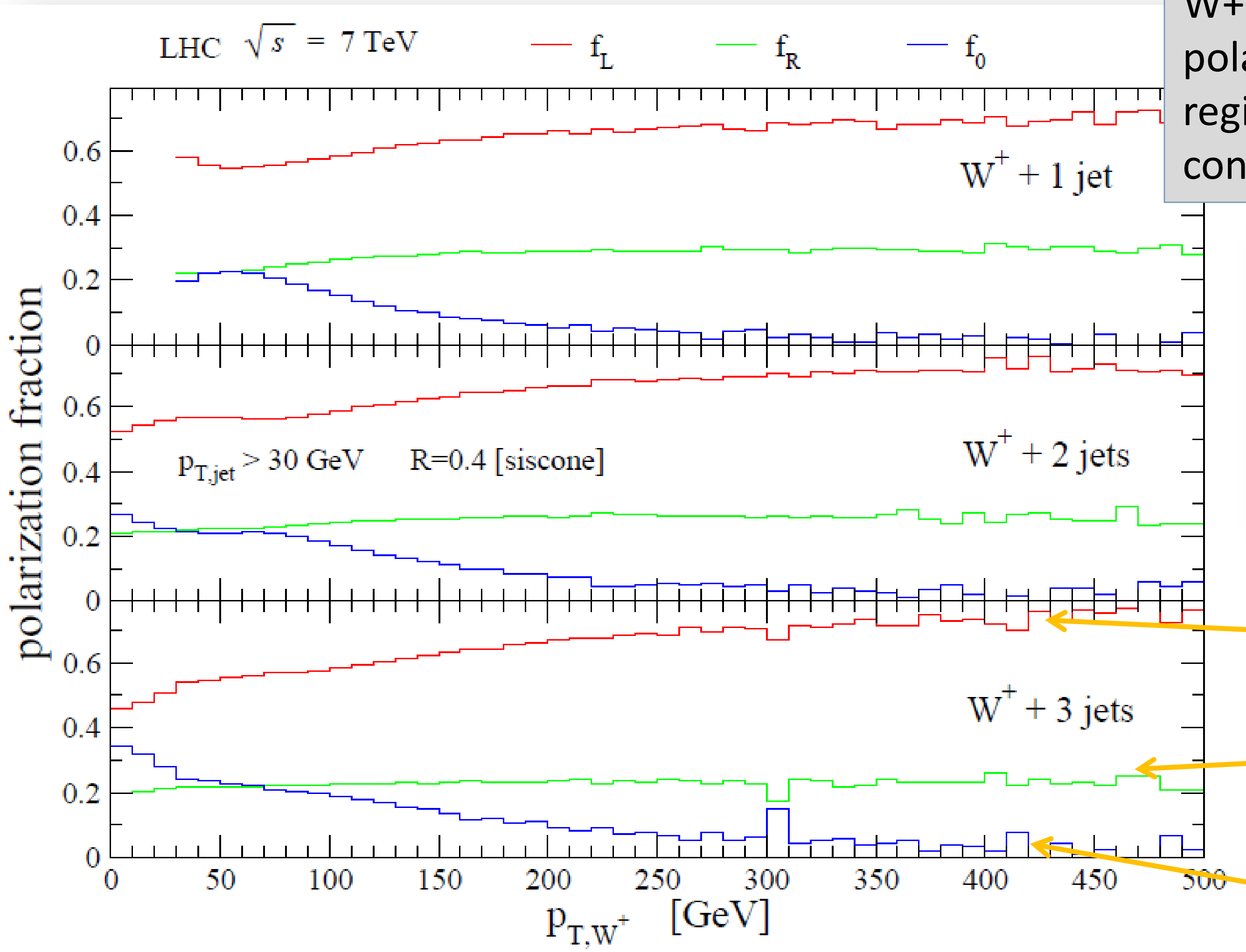
– stable under QCD corrections

– will be useful to separate W + n jets from top, maybe also from new physics

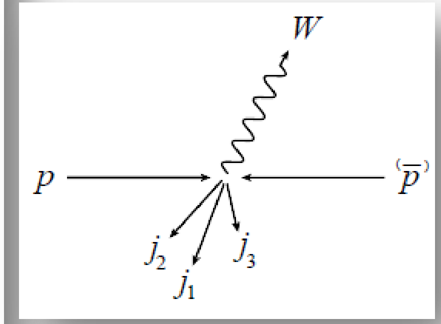
BlackHat: **[arXiv:1103.5445]**

W+n>1 Jet polarization from Factorization?

[Bern, Dixon, FFC, Hoeche, Ita, et al. arXiv:1103.5445]



Indeed properties from the W+1 j amps explain the polarization at large PT, a region where *factorized* configurations are common



Left handed component

Right handed component

Scalar component

CMS W POLARIZATION MEASUREMENT

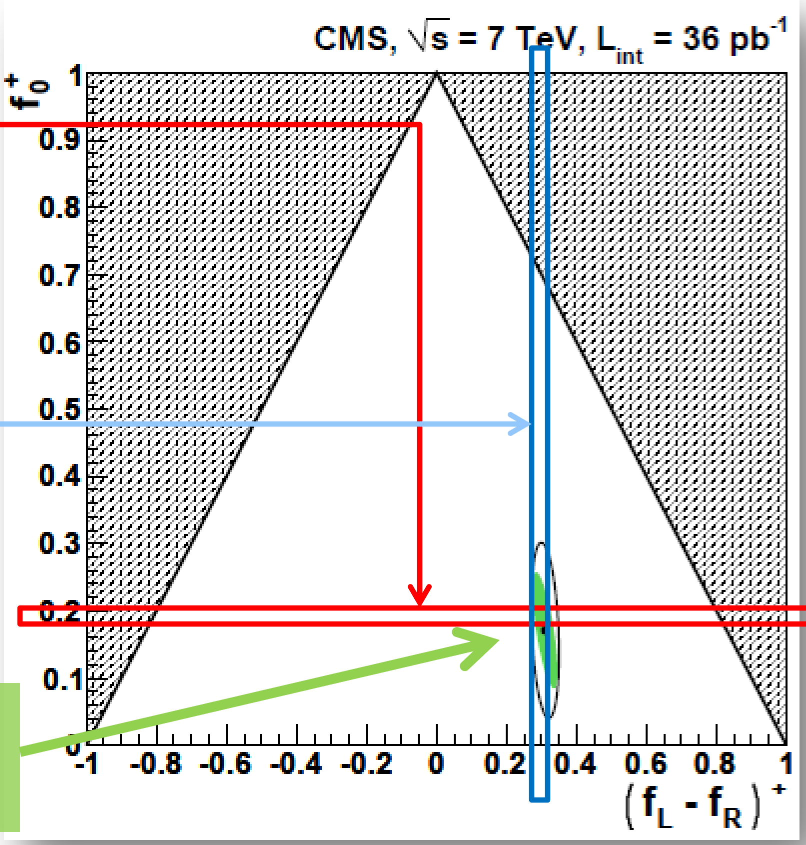
arXiv:1104.3829 [hep-ex]

Polarized W's at CMS

	W^+ NLO	W^+ ME+PS	W^+ LO
f_L	0.554	0.548	0.556
f_R	0.246	0.265	0.246
f_0	0.200	0.187	0.198

Theory predictions by BlackHat+SHERPA collaboration **arXiv:1103.5445 [hep-ph]**

→ In **arXiv:1104.3829 [hep-ex]** CMS reports finding left handed polarized *W*'s
→ Employs 36 pb^{-1} of data collected in 2010
→ Data published in the plane ($f_L - f_R$) *vs.* f_0
→ Results agree with BlackHat's prediction
→ Results shown here for W^+, but similar results for W^-



Excellent theory/experiment agreement within both statistical and total uncertainties

Summary

- QCD is present in all studies performed at Hadron Colliders
- We presented the QCD Lagrangian and its Feynman Rules
- We showed how the strong coupling constant runs and presented recent measurements
- The factorizing properties of QCD amplitudes were discussed in detail with some implications
- We will continue exploring QCD properties and techniques used to describe the messy environment of a hadron collider machin