CRPA and SRC in neutrino-nucleus scattering

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Overview

Introduction

Continuum random phase approximation

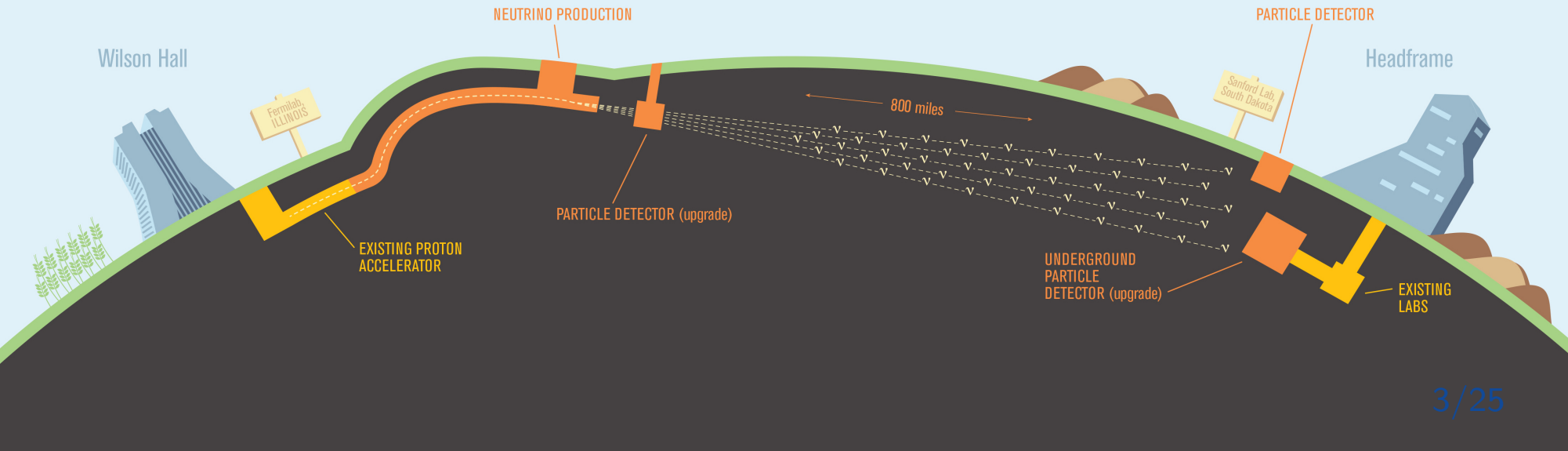
Short-range correlations

Summary and outlook

Motivation

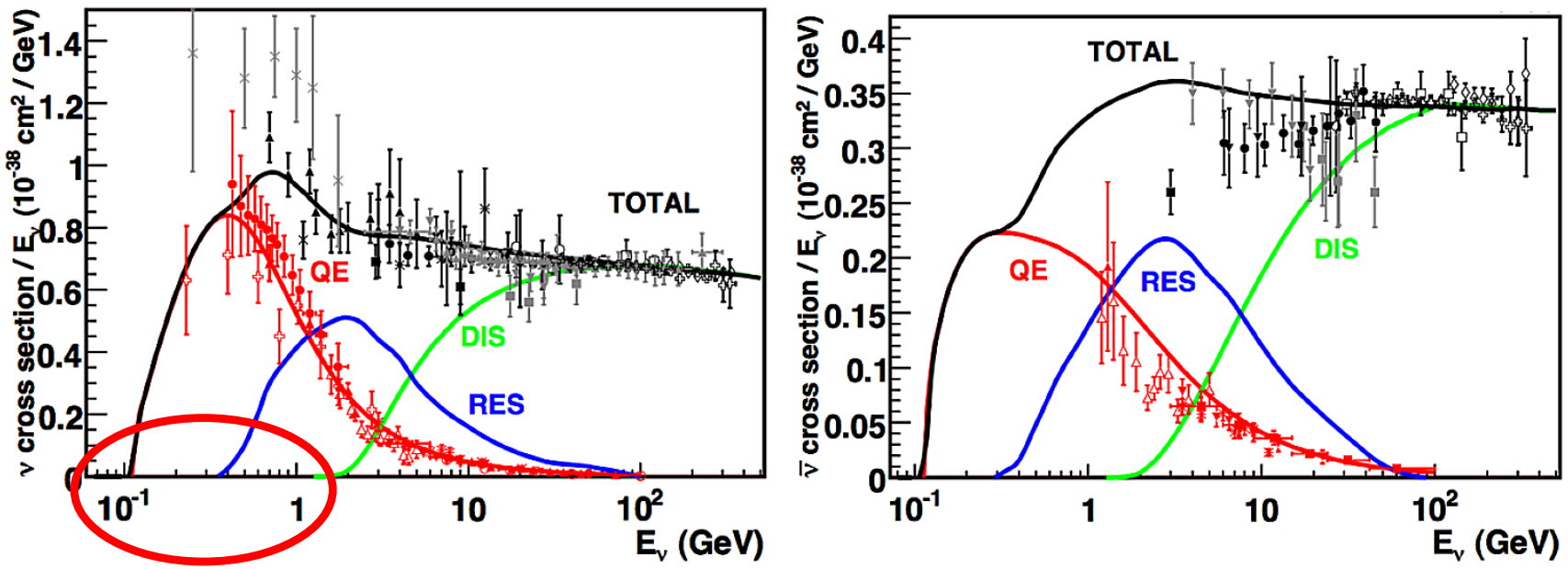
1. ν_μ production at near detector
2. neutrino oscillations occurs between near and far detector
3. count ν_e, ν_μ and ν_τ at near and far detector
4. extract Δm, θ, δ_{CP} from differences between near and far detector

$\rightarrow$ **precise neutrino-nucleus cross-section needed,**
major source of uncertainty is associated with nuclear structure



Ref: J.A. Formaggio, *et al.*
Rev.Mod.Phys. 84 (2012) 1307

Reaction channels



- QE - Quasi-elastic scattering: nucleon stays intact

  $\nu_\mu + n \rightarrow \mu + p$

- RES - Resonance production: nucleon is excited

  $\nu_\mu + n \rightarrow \mu + \Delta$

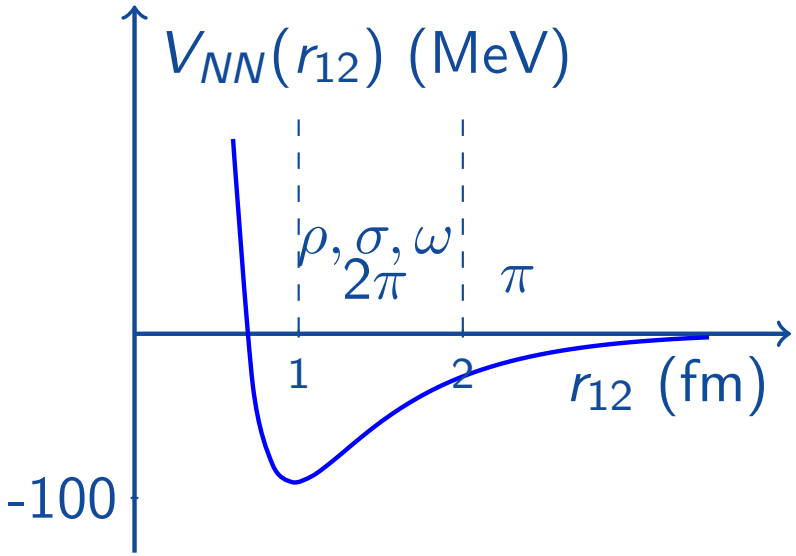
  $\hookrightarrow p + \pi$

- DIS - Deep inelastic scattering: nucleon breaks up

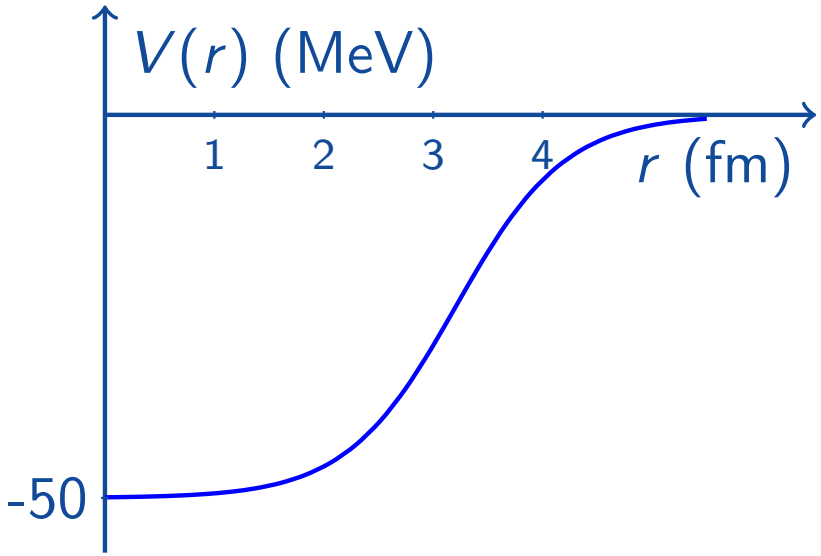
  $\nu_\mu + n \rightarrow \mu + X$

Many-body corrections

Nucleus is more than a sum of nucleons: they are bound



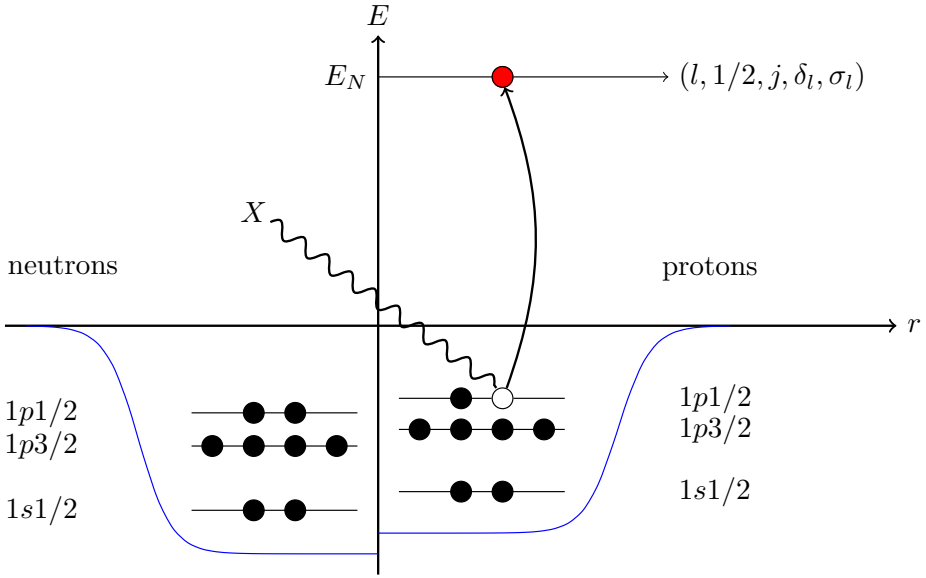
(a) Nucleon-nucleon potential

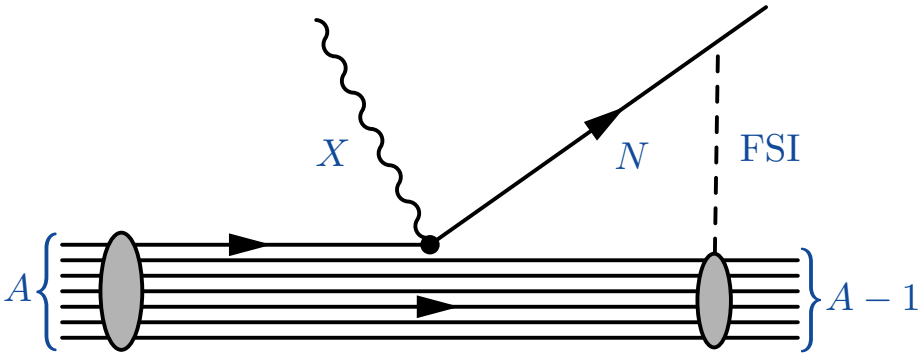


(b) Mean-field potential

- IPM - Independent Particle Model: HF calculations
- CRPA - Continuum random phase approximation: long-range correlations between the nucleons
- SRC - Short-range correlations: repulsive part of the NN potential, short-range behaviour

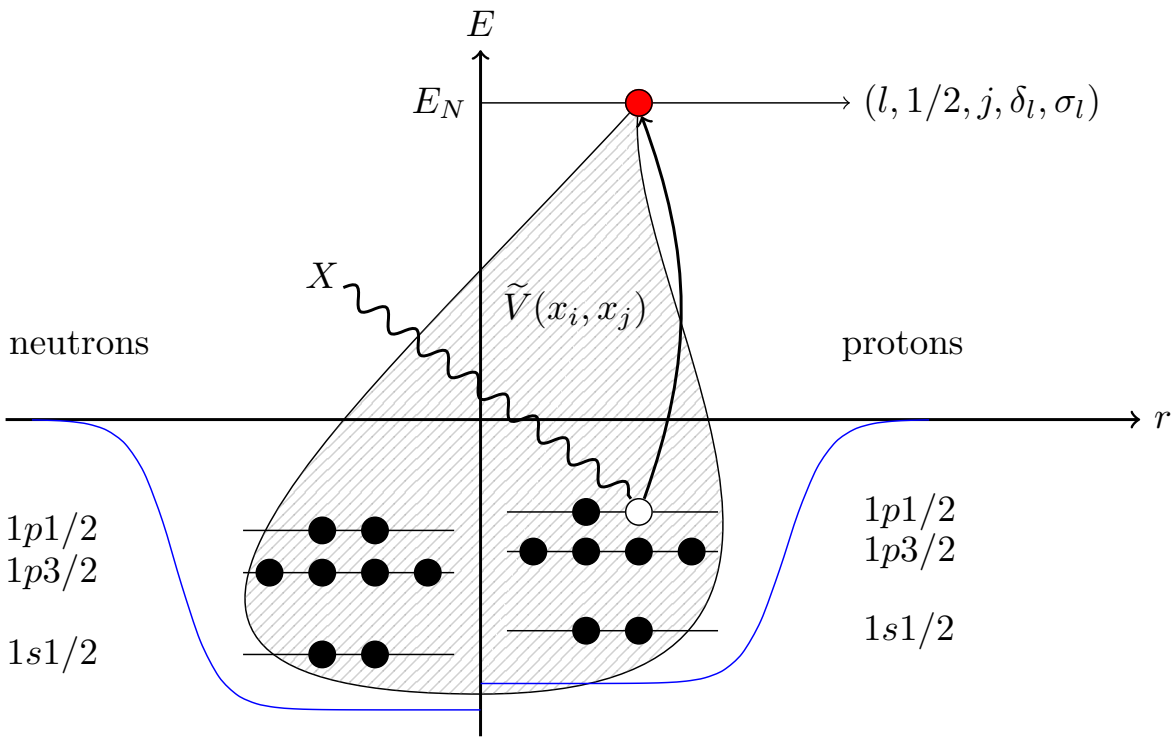
Nuclear model





- Ground state nucleus is a shell model
  - Calculated with a Hartree-Fock (HF) approximation using a Skyrme NN force (SkE2)
  - Accounts for binding energies and nuclear structure
  - Pauli-blocking effects included inherently
- Continuum wave functions are calculated using the same NN potential as the HF calculations
  - Orthogonality is preserved between initial and final states
  - Distortion effects of the residual nucleus on the ejected nucleons are incorporated (FSI)

Nuclear model

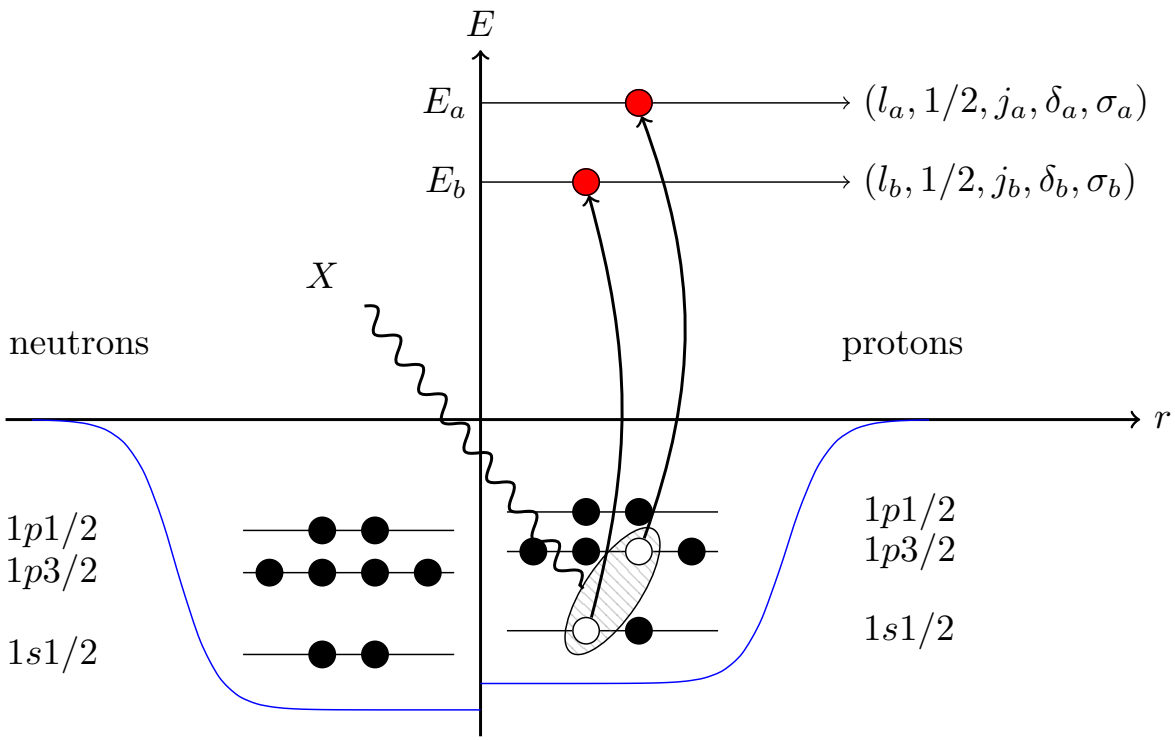


CRPA

Long-range correlations are introduced through a continuum random phase approximation (CRPA)

- nuclear current is a one-body current
- only one-particle knockout ($1p1h$)

Nuclear model



SRC

Short-range correlations are introduced by working with a correlation operator on the (uncorrelated) HF wave-functions

- nuclear current is a two-body current
- one-particle ($1p1h$) and two-particle ($2p2h$) knockout

Overview

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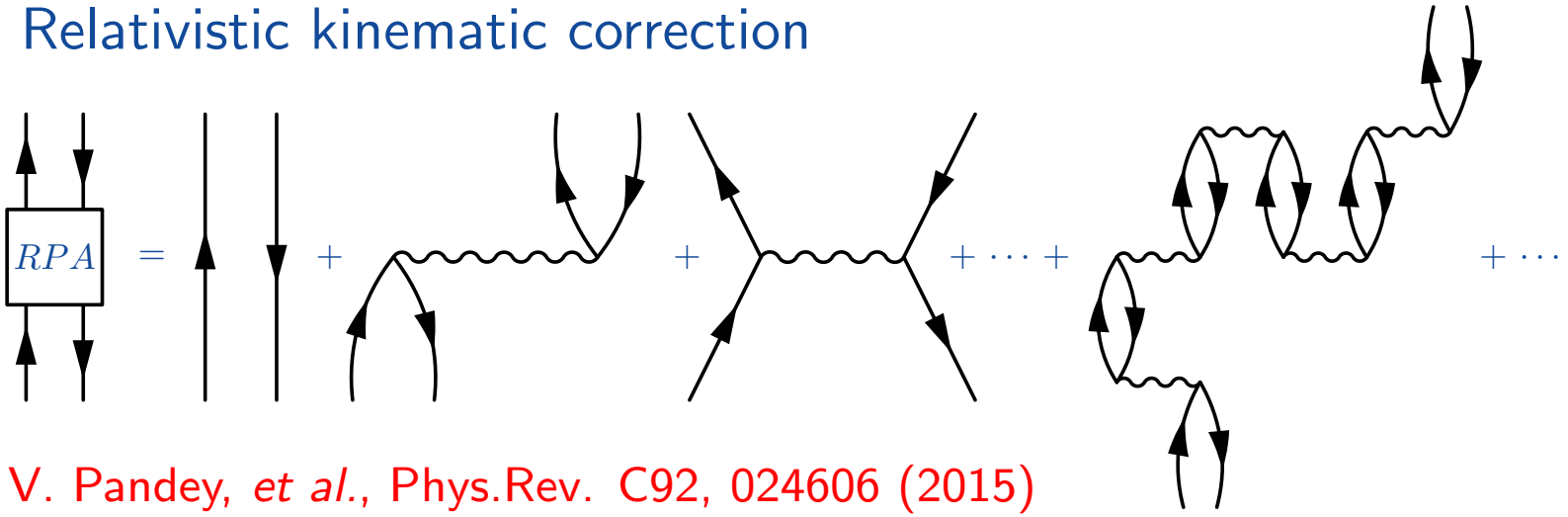
Continuum Random Phase Approximation

Collective behaviour of nucleons cannot be explained with an independent particle model

- Nucleons are able to exchange energy and momentum via a two-body force
- A nuclear excitation is a coherent superposition of individual ph^{-1} and hp^{-1} excitations
- CRPA equations are solved using a Green's function approach in coordinate space
- Relativistic kinematic correction



Ref: V. Pandey, *et al.*, Phys.Rev. C92, 024606 (2015)
V. Pandey, *et al.*, Phys.Rev. C89, 024601 (2014)
N. Jachowicz, *et al.*, Phys.Rev. C65, 025501 (2002)

HF ——

CRPA —

Ref: V. Pandey, *et al.*, Phys.Rev. C92 024606, (2015)

CRPA results

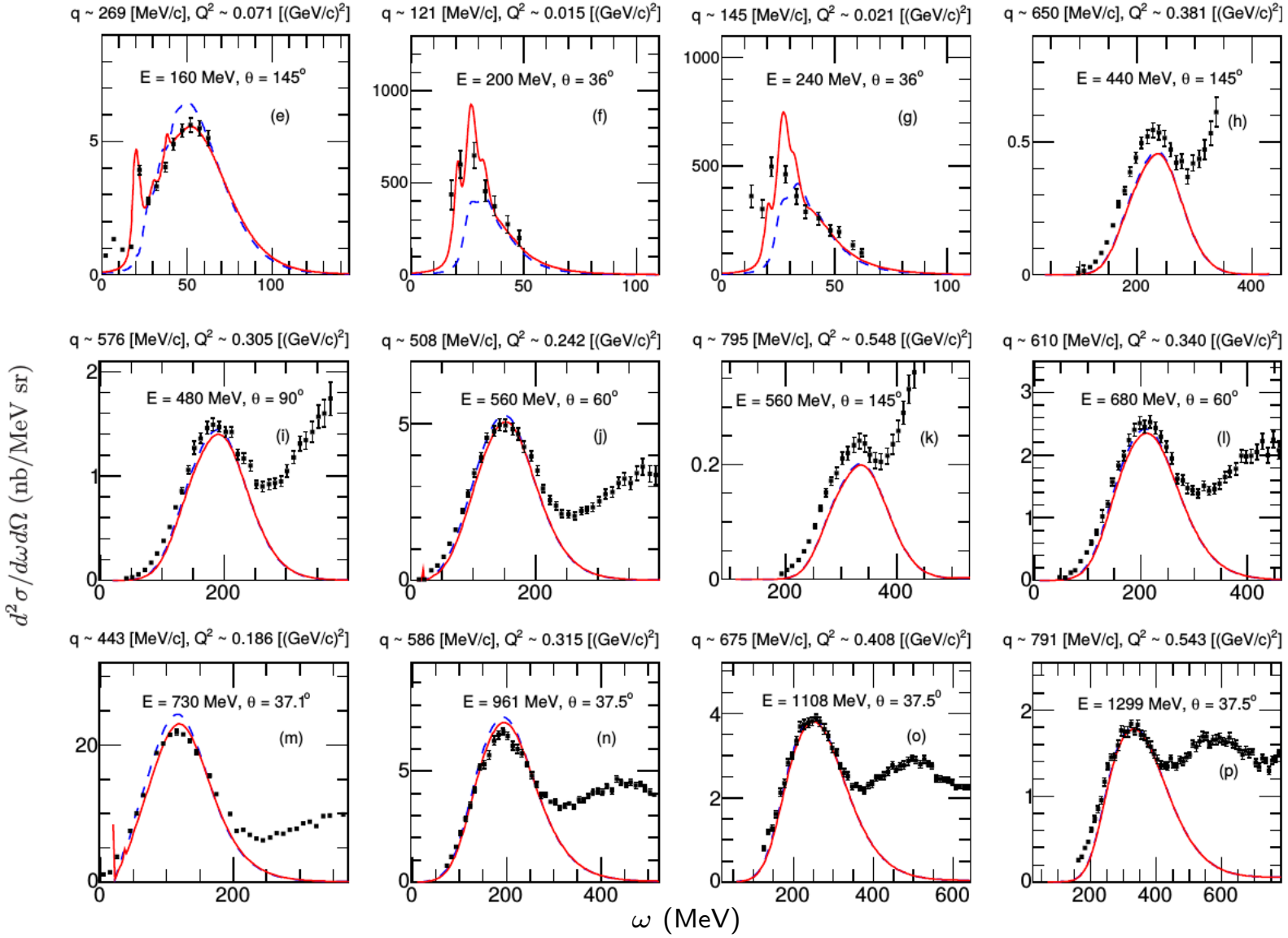


Figure : (e, e') scattering on ^{12}C

RFG ——
CRPA ——

CRPA results

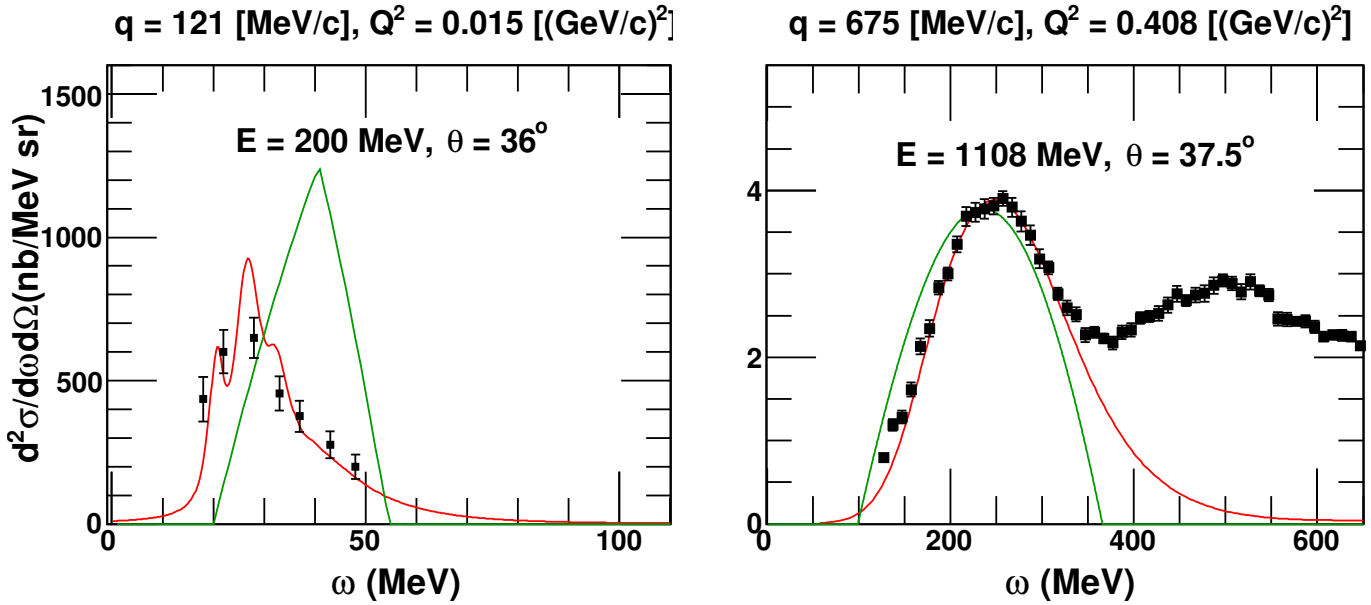


Figure : (e, e') scattering on ^{12}C, comparison between CRPA and RFG

- For higher incoming neutrino energy: a relativistic Fermi-gas (RFG) model reproduces the data reasonably well
- For low incoming neutrino energy: model which accounts for nuclear structure necessary to reproduce data

HF ——
CRPA —

CRPA results

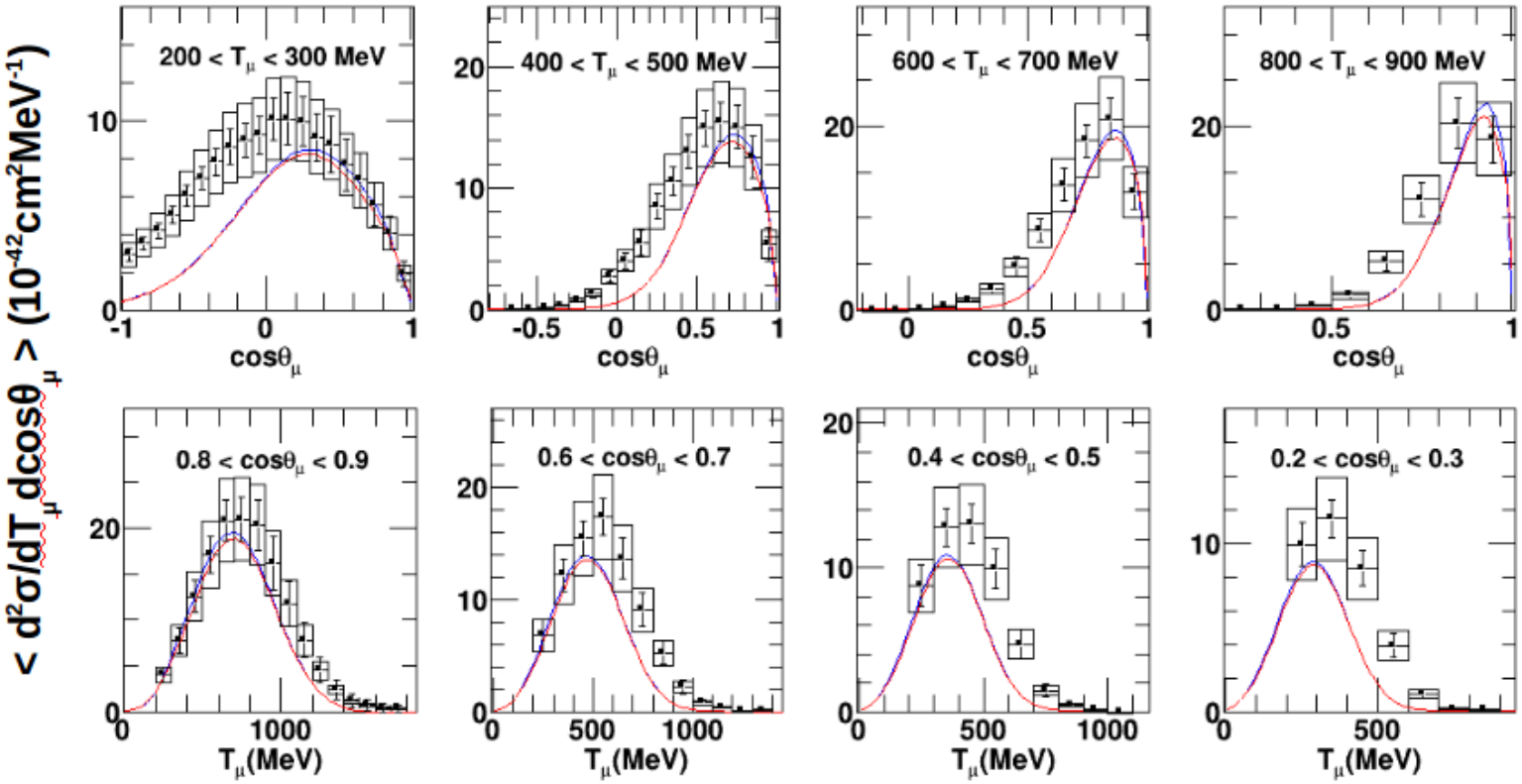


Figure : MiniBooNE CCQE ν_μ measurements

- Successful description of the gross features of the cross section
- The missing strength can be associated with the contribution from multinucleon processes, not included here

HF ——
CRPA —

CRPA results

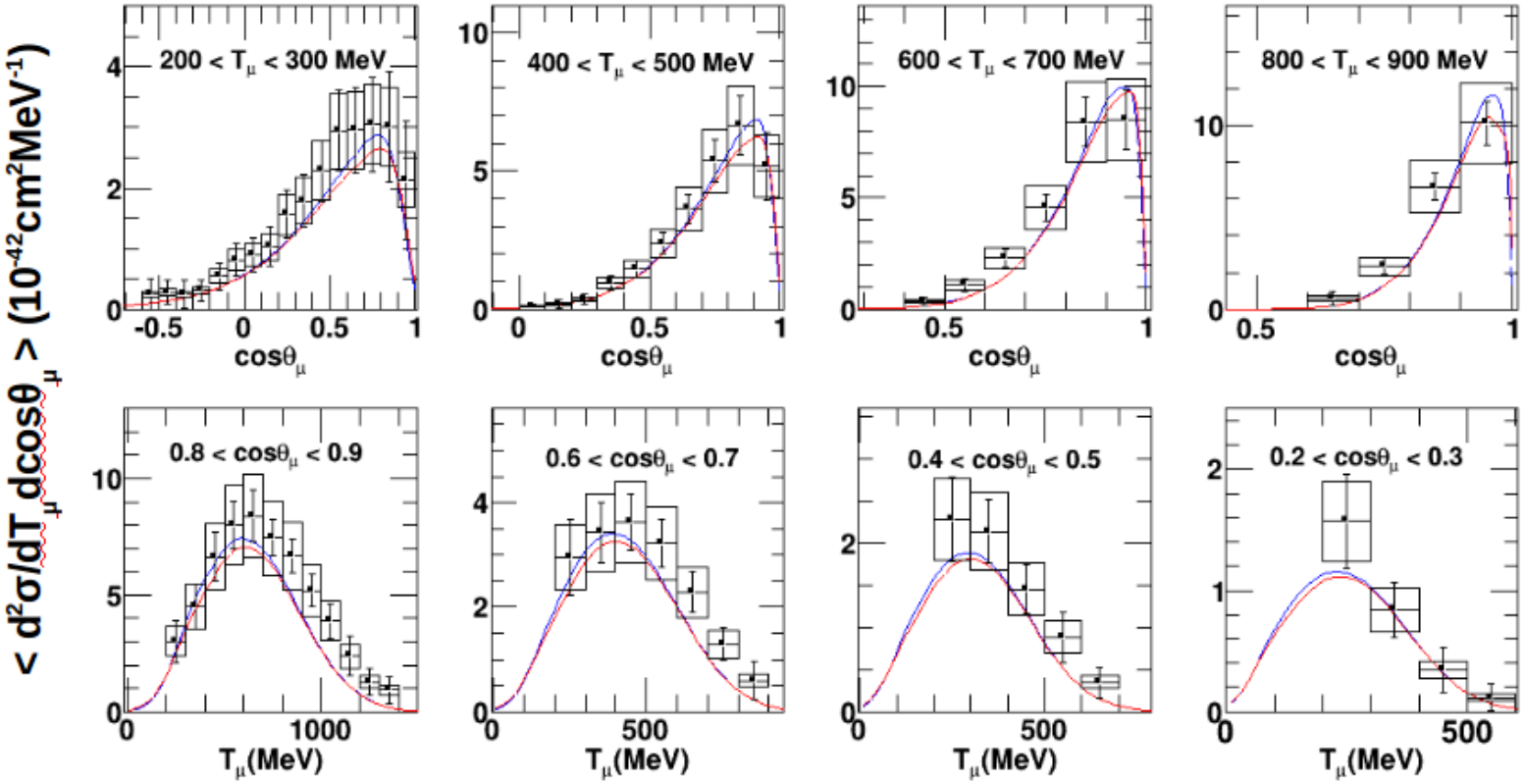


Figure : MiniBooNE CCQE $\overline{\nu}_\mu$ measurements

- Antineutrino predictions are better describing the data than the neutrino ones. But there still seems to be some strength missing.

CRPA results - nuclear excitations

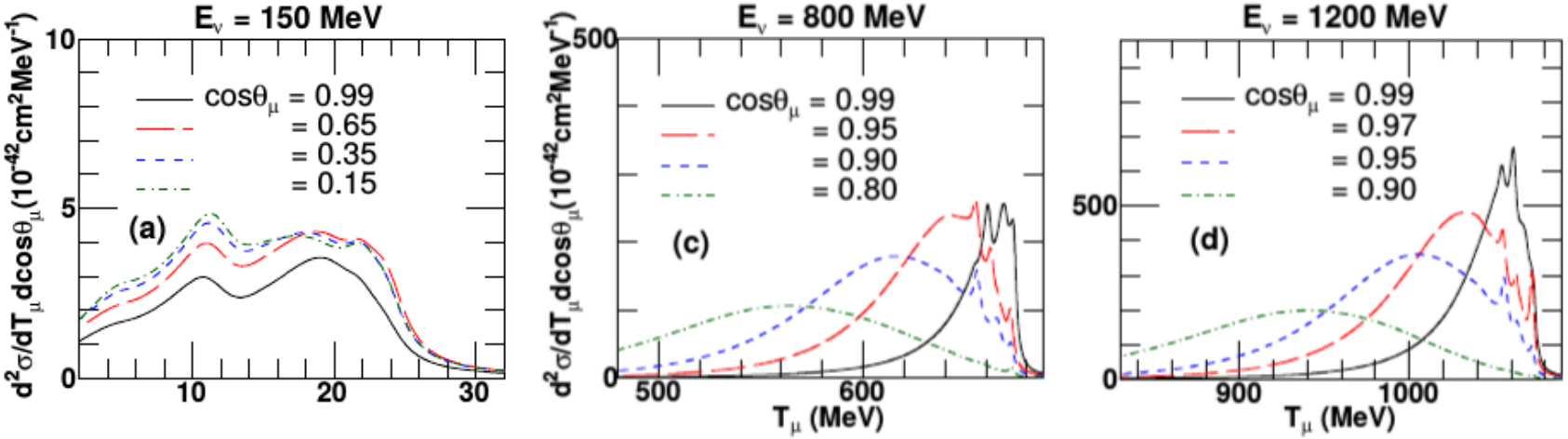


Figure : ν_μ scattering on ^{12}C for fixed neutrino energy
Ref: V. Pandey, *et al.*, Phys.Rev. C92, 024606 (2015)

- For low incoming neutrino energy: nuclear excitations are dominating the cross section
- For higher incoming neutrino energy: important contributions from nuclear excitations at forward scattering angles
- For low Q^2 nuclear excitations important

CRPA results - nuclear excitations

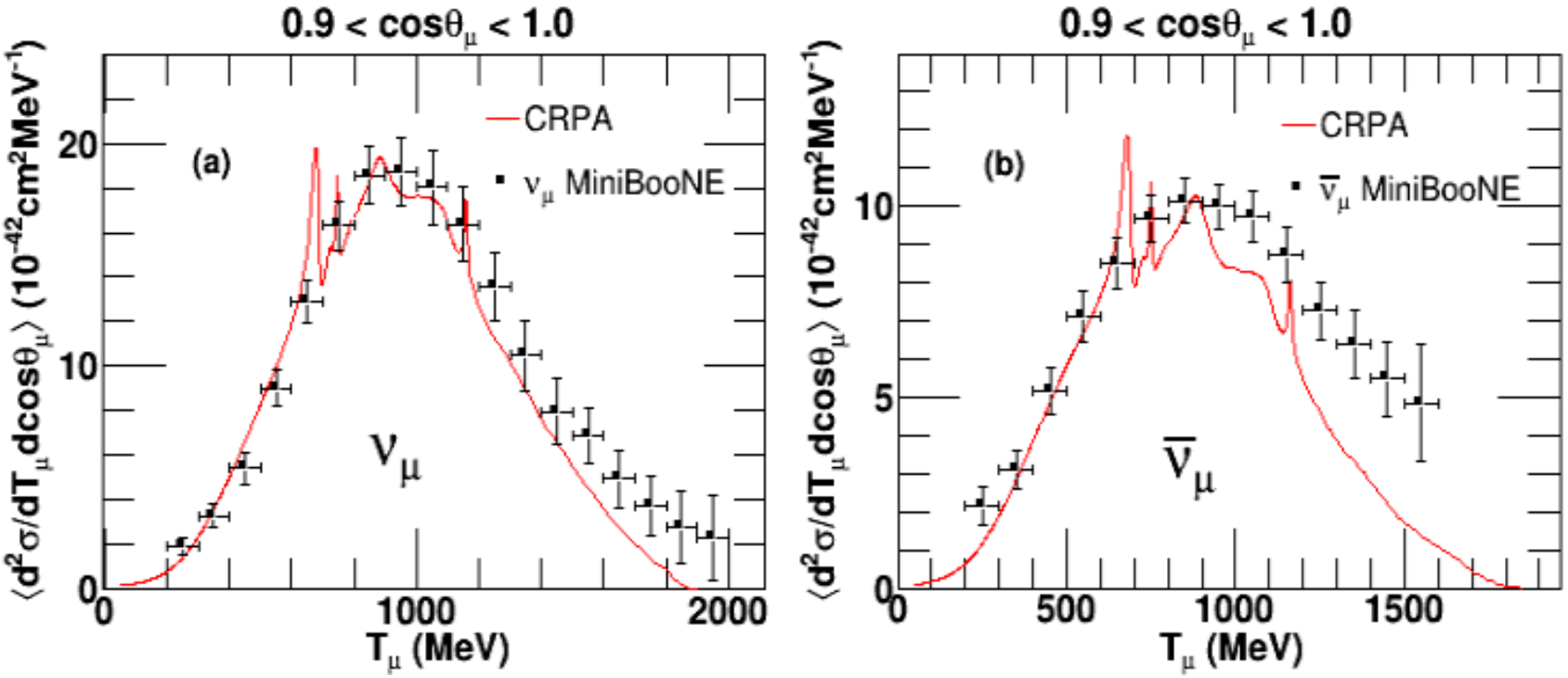


Figure : ν_μ and $\overline{\nu}_\mu$ scattering results for MiniBooNE in the most forward bin (Preliminary results)

- Nuclear excitations are non-neglegible for flux folded results, especially in the most forward bin.

CRPA results - nuclear excitations

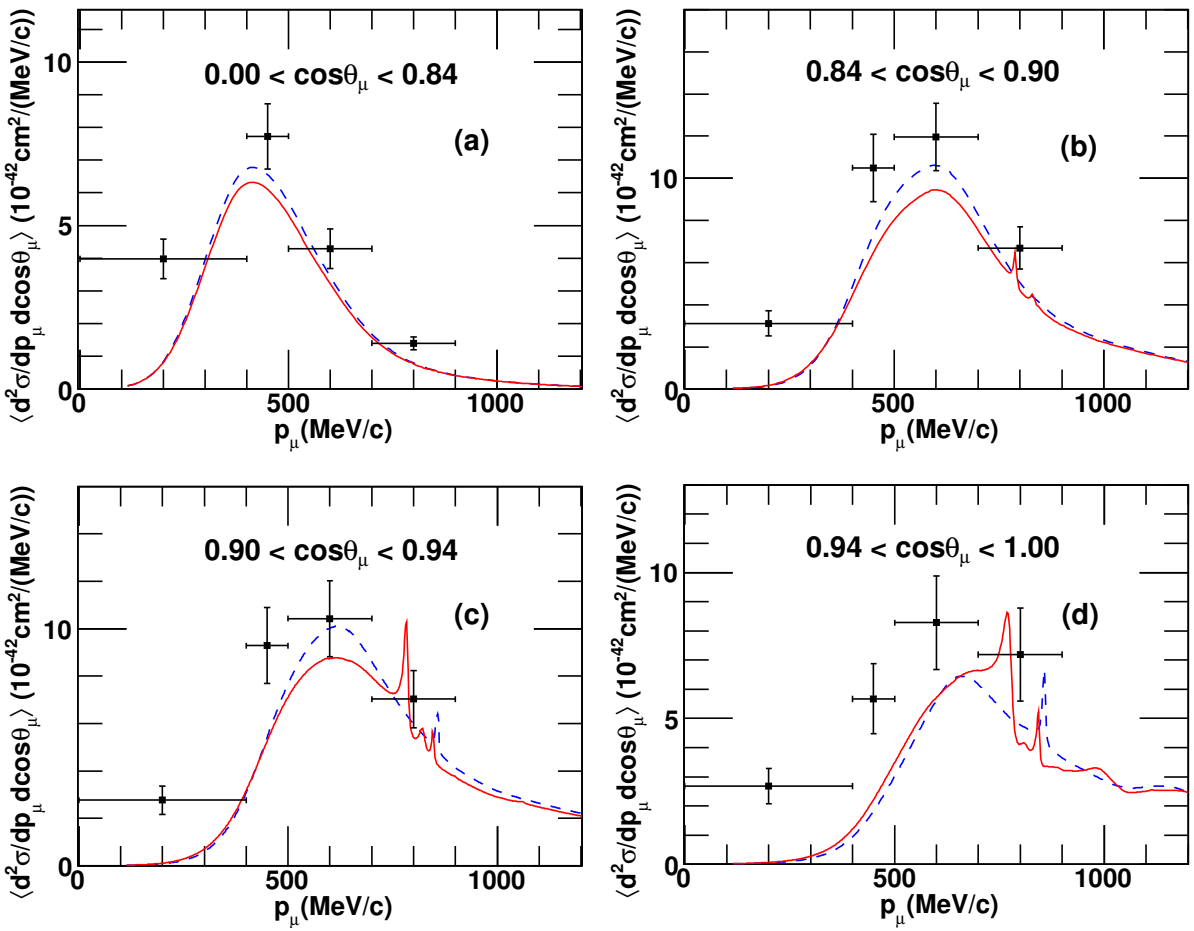


Figure : *NEW* ν_μ scattering results for T2K on ^{12}C

Data: T2K Collaboration, Phys.Rev. D87 092003 (2013)
(Inclusive QE measurements)

CRPA results - nuclear excitations

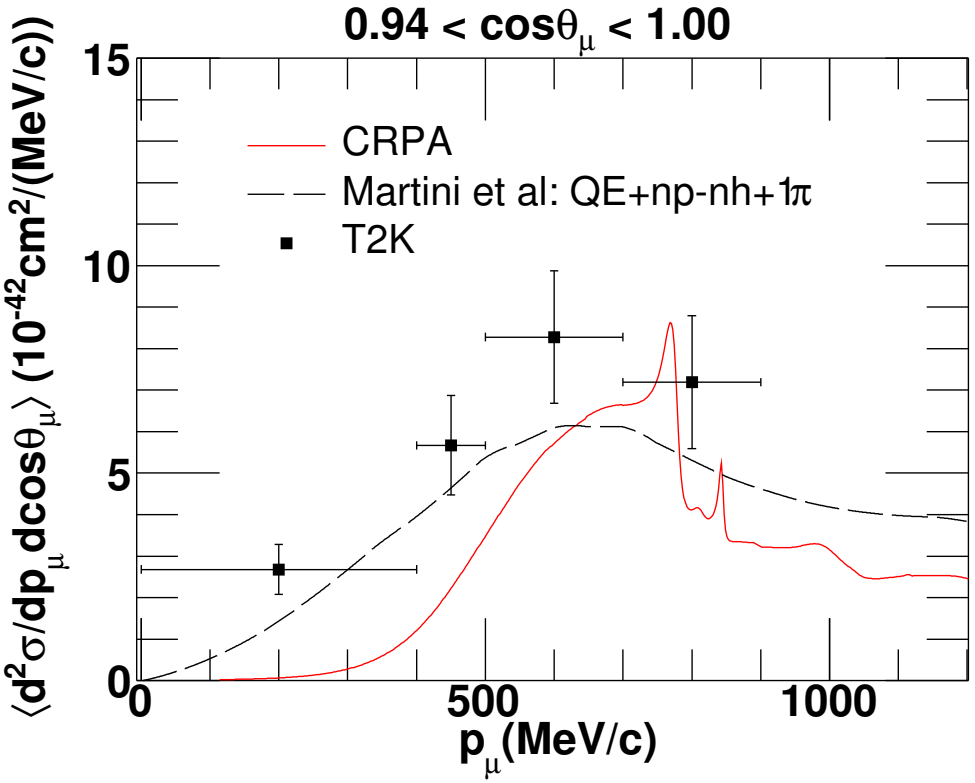


Figure : *NEW* ν_μ scattering results for T2K in the most forward bin

- Martini *et al.* describe all bins very well except most forward
- CRPA yields significantly higher cross-section in region $600 - 800$ MeV/c

Ref: M. Martini, *et al.*, Phys.Rev. C90, 025501 (2014)

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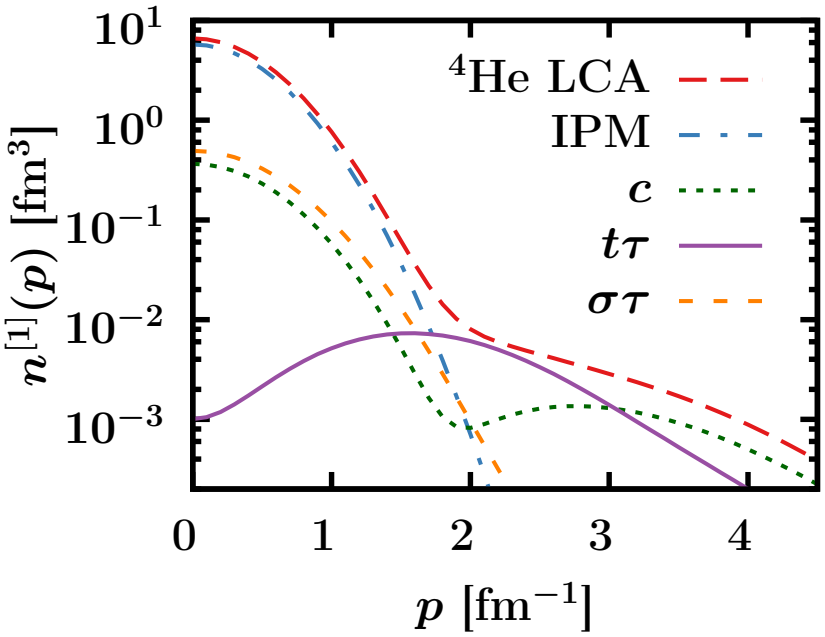
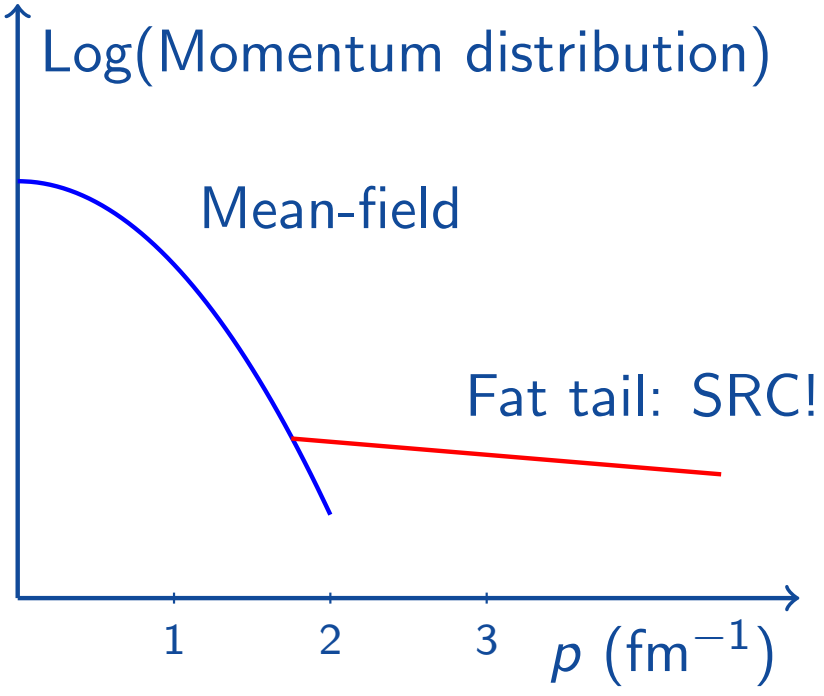
Short-range correlations

Fat tails in the single-nucleon momentum distribution cannot be explained within the impulse approximation



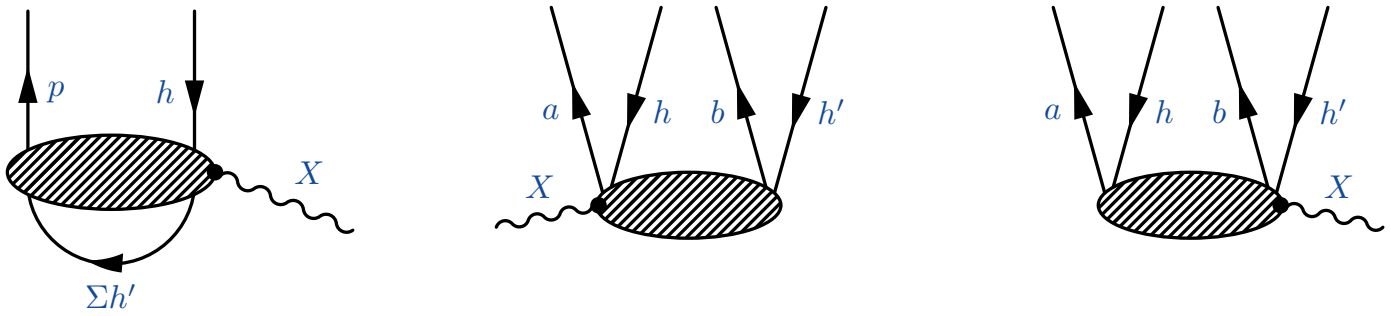


Ref: J. Ryckebusch, *et al.*, J. Phys. G: Nucl. Part. Phys. 42 055104 (2015)

Short-range correlations

Fat tails in the single-nucleon momentum distribution cannot be explained within the impulse approximation

- Nucleons occur in pairs with high relative momenta and low center-of-mass momenta (SRC pairs)
  - tensor correlations dominate at intermediate momenta
  - central correlations dominate at high momenta
- Signature of SRC is back-to-back two-nucleon knockout
- SRC also have an effect on one-nucleon knockout



Ref: J. Ryckebusch, *et al.*, Nucl.Phys. A624, 581 (1997)
S. Janssen, *et al.*, Nucl.Phys. A672, 285 (2000)
(electron-scattering model with SRC + MEC)

Short-range correlations

Correlated wave functions $|\Psi\rangle$ are constructed by acting with a many-body correlation operator $\widehat{\mathcal{G}}$ on the uncorrelated Hartree-Fock wave functions $|\Phi\rangle$

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\widehat{\mathcal{G}}|\Phi\rangle, \qquad \text{with} \qquad \mathcal{N} = \langle\Phi|\widehat{\mathcal{G}}^\dagger\widehat{\mathcal{G}}|\Phi\rangle$$

Central and tensor part are responsible for majority of the strength

$$\widehat{\mathcal{G}} = \widehat{\mathcal{S}}\left(\prod_{i<j}^{A}\left[1+\widehat{l}(i,j)\right]\right),$$

$$\widehat{l}(i,j) = -g_c(r_{ij}) + f_{t\tau}(r_{ij})\widehat{S}_{ij}\left(\vec{\tau}_i\cdot\vec{\tau}_j\right).$$

$g_c(r_{ij})$ and $f_{t\tau}(r_{ij})$ are the central and tensor correlation functions

Ref: C. C. Gearhaert, PhD thesis, Washington University, (1994).
S. C. Pieper, *et al.* Phys.Rev. C46, 1741 (1992).

Short-range correlations

Expectation values between correlated states $|\Psi\rangle$ can be turned into expectation values between uncorrelated states $|\Phi\rangle$, with an effective A-body operator (conservation of misery)

$$\langle\Psi_f|\widehat{\Omega}|\Psi_i\rangle = \frac{1}{\mathcal{N}}\langle\Phi_f|\widehat{\mathcal{G}}^\dagger\widehat{\Omega}\widehat{\mathcal{G}}|\Phi_i\rangle = \frac{1}{\mathcal{N}}\langle\Phi_f|\widehat{\Omega}^{\text{eff}}|\Phi_i\rangle,$$

with

$$\widehat{\Omega}^{\text{eff}} = \widehat{\mathcal{G}}^\dagger\widehat{\Omega}\widehat{\mathcal{G}} = \left(\prod_{j<k}^{A}\left[1+\widehat{I}(j,k)\right]\right)^\dagger \widehat{\Omega}\left(\prod_{l<m}^{A}\left[1+\widehat{I}(l,m)\right]\right).$$

In the IA, the nuclear current operator is the sum of one-body current operators

$$J_\lambda^{\text{nucl,eff}} = \left(\prod_{j<k}^{A}\left[1+\widehat{I}(j,k)\right]\right)^\dagger \sum_{i=1}^{A} J_\lambda^{[1]}(i)\left(\prod_{l<m}^{A}\left[1+\widehat{I}(l,m)\right]\right).$$

Short-range correlations

Use the fact that SRC is a short-range phenomenon

- Probability of finding 3 nucleons close to each other very small
- Terms linear in the correlation operator are retained
- SRC are pair correlations
- A-body operator $\rightarrow$ 2-body operator

$$J_\lambda^{\text{nucl,eff}} \approx J_\lambda^{\text{LCA}}$$
$$= \sum_{i=1}^{A} J_\lambda^{[1]}(i) + \sum_{i<j}^{A} J_\lambda^{[1],\text{in}}(i,j),$$

where

$$J_\lambda^{[1],\text{in}}(i,j) = \left[J_\lambda^{[1]}(i) + J_\lambda^{[1]}(j)\right] \widehat{I}(i,j)$$

- Effective nuclear current is the sum of a one-body (IA) and two-body (SRC) current

SRC electron results - $1p1h$

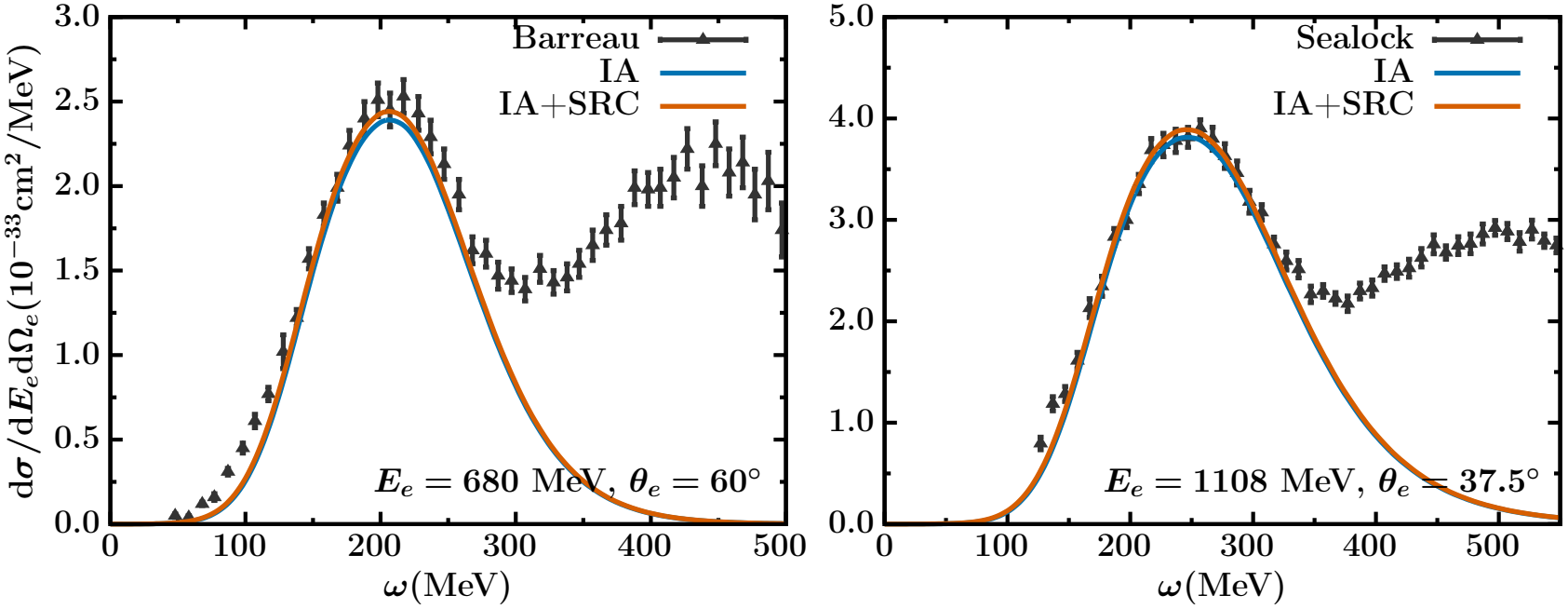


Figure : (e, e') scattering on ^{12}C

- Influence of SRC on $1p1h$ is a small increase.

SRC results - Exclusive $2p2h$

Compare two-nucleon knockout cross section with one-nucleon knockout cross section $\rightarrow$ integrate over outgoing nucleons

- 5D numerical integral is computationally intensive
- exclusive differential cross section shows clear back-to-back knockout signal: use this to calculate some of the integrals analytically

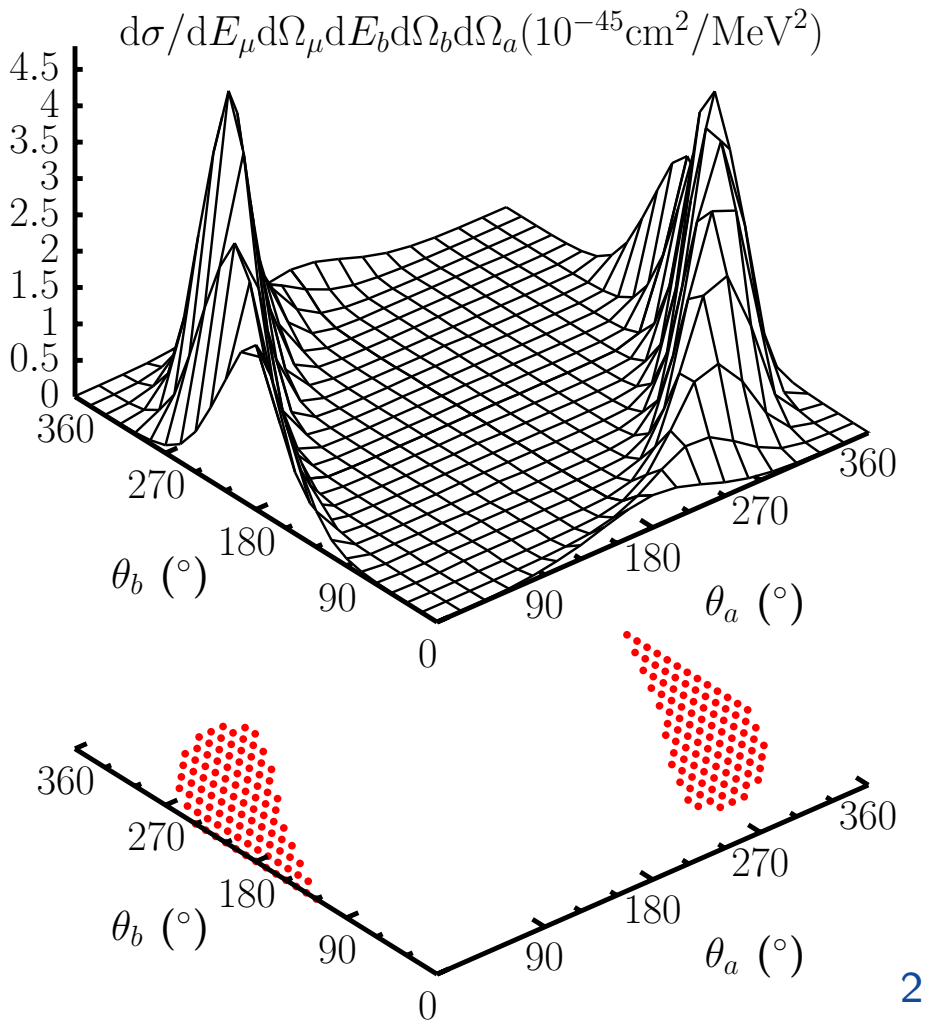


Figure : $E_\nu = 750$ MeV, $E_\mu = 550$ MeV, $\theta_\mu = 15°$ and $T_p = 50$ MeV

SRC electron results - Inclusive $2p2h$

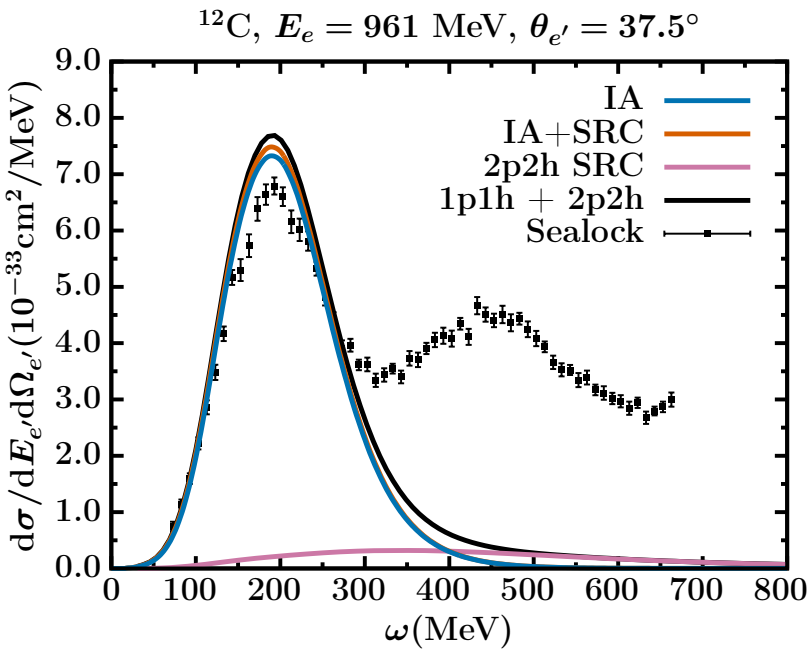


Figure : (e, e') scattering on ^{12}C

Strength of the $2p2h$ contribution

- tensor SRC dominates at small to intermediate ω
- central SRC dominates at large ω
- tensor dominated by pn pairs
- vector, axial and interference terms are equally important

- Inclusion of SRC in the $2p2h$ channel yields a broad background over the whole ω range
  (momentum conservation poses no limits on individual nucleon momenta but on the pair center-of-mass momentum)

SRC neutrino results

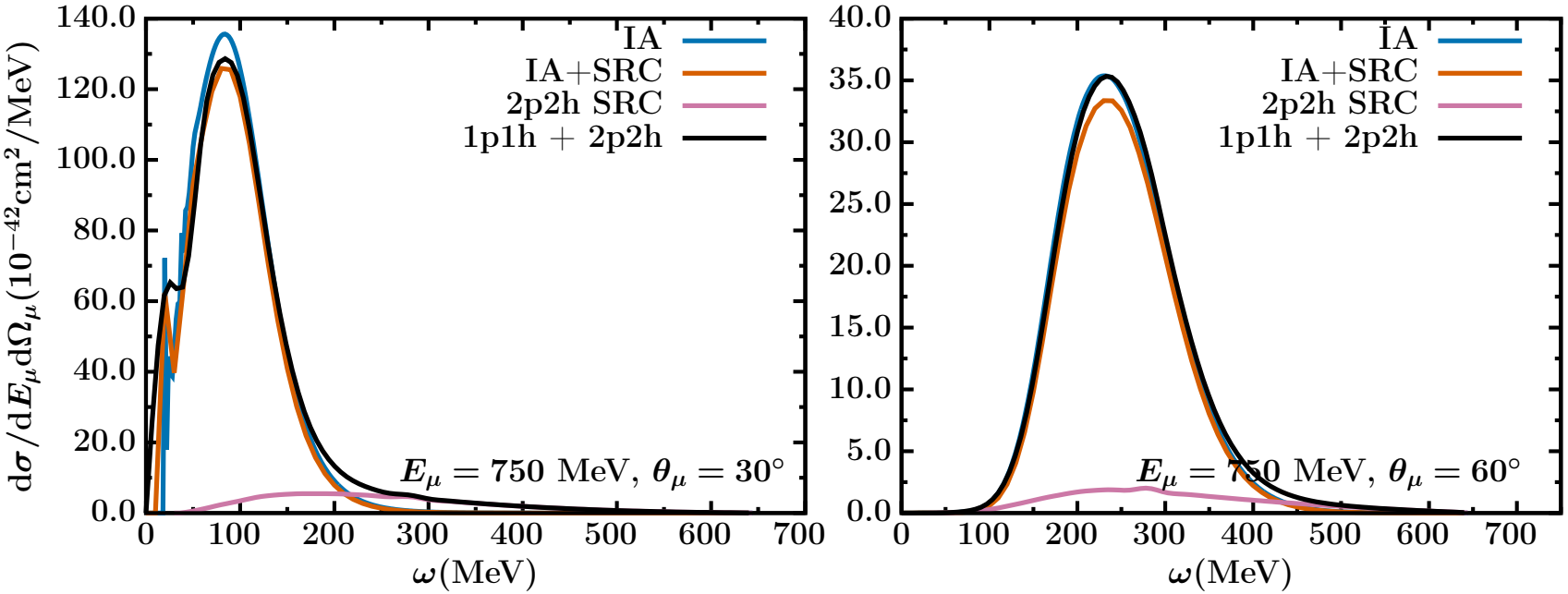


Figure : (ν, μ) scattering on ^{12}C

- Influence of SRC on $1p1h$ now a reduction. This results from the isospin part of the SRC matrix elements and different behaviour of electric and magnetic form factors.
- Contribution of $2p2h$ is a broad background

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Summary

- Our model successfully describes the e-scattering data in the QE channel, so it can be expected to reproduce ν data as well

CRPA

- At low Q^2 nuclear excitations should be accounted for in a CRPA formalism
- For flux-folded results, the nuclear excitations are still non-neglegible, especially in the most forward bin

SRC

- Started from a model for exclusive calculations which was tested against electron scattering data
- Calculated contribution of SRC to double differential QE cross section

Outlook

Outlook

- Extending the model with meson-exchange currents in a consistent approach
  $\rightarrow$ Model exists for electron scattering
  $\rightarrow$ Axial MEC are *challenging*

Parallel research topics at UGent

- CRPA and SRC neutrino scattering results on ^{40}Ar
  $\rightarrow$ Nils Van Dessel
- Inclusion of the Δ-resonance and π production for ν-scattering
  $\rightarrow$ Raúl González-Jiménez

Thank you!

References

CRPA calculations for ν-interactions

- N. Jachowicz, *et al.*, Phys.Rev. C65, 025501 (2002)
- V. Pandey, *et al.*, Phys.Rev. C89, 024601 (2014)
- V. Pandey, *et al.*, Phys.Rev. C92, 024606 (2015)

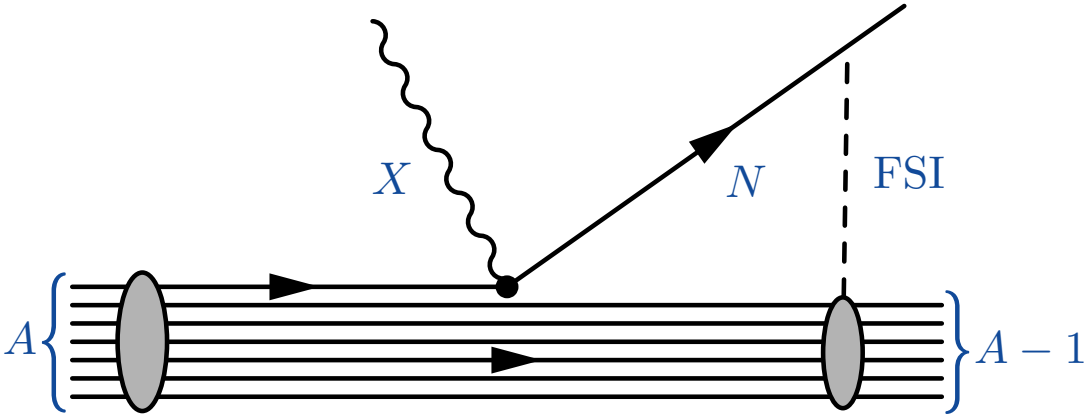
$2p2h$ e-scattering calculations including SRC and MEC

- J. Ryckebusch, *et al.*, Nucl.Phys. A568, 828 (1994)
- J. Ryckebusch, *et al.*, Nucl.Phys. A624, 581 (1997)
- S. Janssen, *et al.*, Nucl.Phys. A672, 285 (2000)

Momentum distributions with SRC

- J. Ryckebusch, *et al.*, J.Phys.G: Nucl.Part.Phys. 42 055104 (2015)

Nuclear model



- Ground state nucleus is described using a Hartree-Fock (HF) approximation. The mean-field (MF) potential is obtained by solving the HF equations using a Skyrme (SkE2) two-body interaction. (Shell model)

  Ref: M. Waroquier *et al.*, Phys.Rept. 148, 249 (1987)
- Continuum wave-functions are calculated using the same MF potential.
  - Orthogonality is preserved between initial and final states
  - Effect of final state interactions (FSI) of the ejected nucleons with the residual nucleus are incorpotated (distortion effects)

Continuum Random Phase Approximation

- CRPA equations are solved using a Green's function approach in coordinate space

$$\Pi^{(\mathrm{CRPA})}(x_1, x_2; E_x) = \Pi^{(0)}(x_1, x_2; E_x) + \frac{1}{\hbar}\int \mathrm{d}x\mathrm{d}x' \Pi^{(0)}(x_1, x; E_x)\widetilde{V}(x, x')\Pi^{(\mathrm{CRPA})}(x', x_2; E_x)$$

- The same nucleon-nucleon interaction which was used for the HF calculations is used for the CPRA calculations (SkE2). This makes it a self-consistent approach.
- SkE2 is designed for ground-state properties of nuclei and for interactions at low E. At high E, its Q^2 behavior is not realistic and the effect of long-range correlations is overestimated. We remedy this shortcoming by introducing a dipole Q^2 running for the interaction

$$V(Q^2) \to V(Q^2 = 0)\frac{1}{\left(1 + \frac{Q^2}{\Lambda^2}\right)^2}, \qquad \Lambda = 455 \text{ MeV}$$

Continuum Random Phase Approximation

- We implemented a relativistic kinematic correction as suggested in the following references. Important for $q > 500$ MeV/c

  Ref: W. Alberico, *et al.* Nucl. Phys. A 512, 541-590 (1990)
  S. Jeschonnek, T.W. Donnelly, Phys. Rev. C57, 2438 (1998)
  J. E. Amaro, *et al.*, Phys. Rev. C71, 065501 (2005).

$$\lambda \to \lambda\,(1+\lambda)\,, \qquad \lambda = \frac{\omega}{2m_N}$$

- Width of the single-particle states is accounted for by folding the responses with a Lorentzian (phenomenological approach)

$$R'(q,\omega') = \int_{-\infty}^{+\infty} \mathrm{d}\omega R(q,\omega) L(\omega,\omega')$$

$$L(\omega,\omega') = \frac{1}{2\pi}\left[\frac{\Gamma}{(\omega-\omega')^2 - (\Gamma/2)^2}\right], \qquad \Gamma = 3 \text{ MeV}$$

Continuum Random Phase Approximation

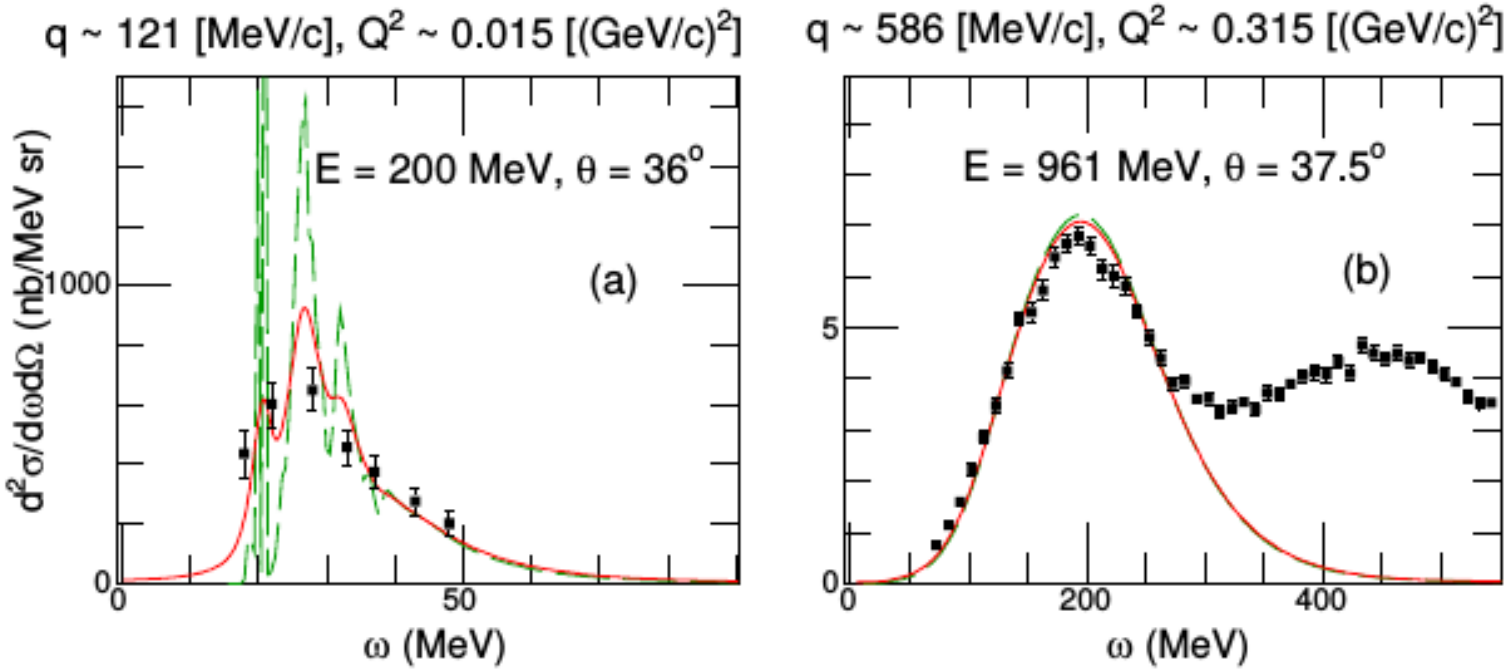


Figure : Cross section with and without the folding procedure

- The overall effect of folding is a redistribution of strength
- Great improvement in the region of the giant-resonance
- Effect very small at higher energy transfers

CRPA - folding procedure

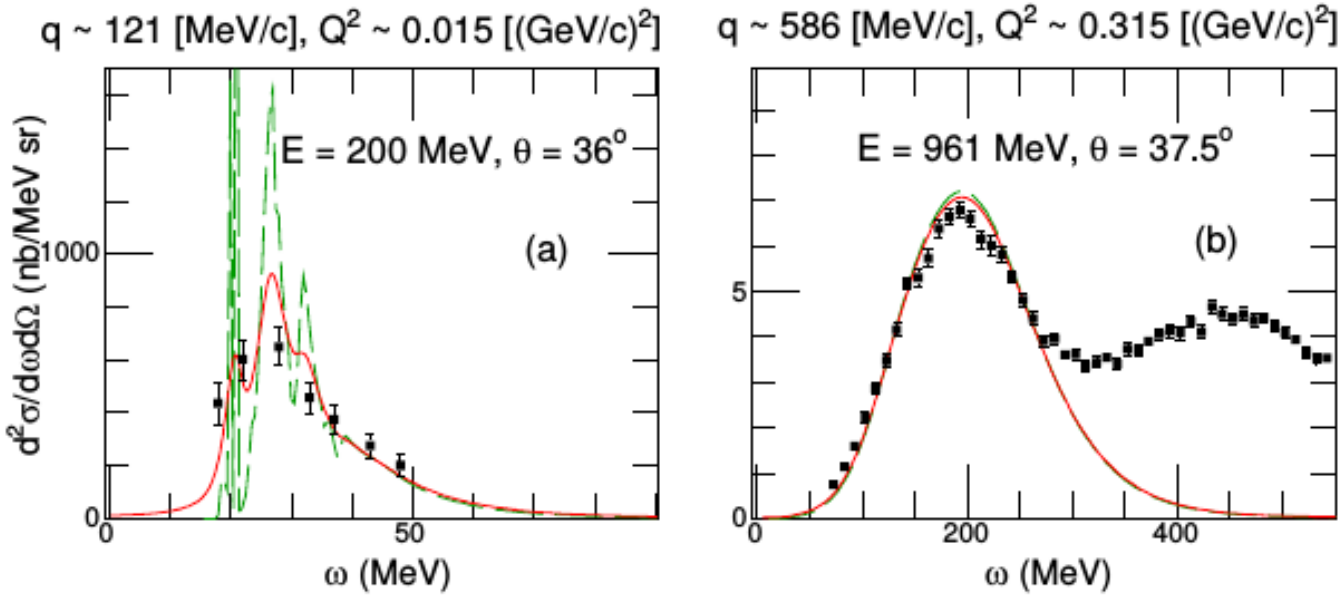


Figure : Cross section with and without the folding procedure

- Width of the single-particle states is accounted for by folding the responses with a Lorentzian (phenomenological approach)

$$R'(q,\omega') = \int_{-\infty}^{+\infty} \mathrm{d}\omega R(q,\omega) L(\omega,\omega')$$

$$L(\omega,\omega') = \frac{1}{2\pi}\left[\frac{\Gamma}{(\omega-\omega')^2-(\Gamma/2)^2}\right], \qquad \Gamma = 3 \text{ MeV}$$

Short-range correlations

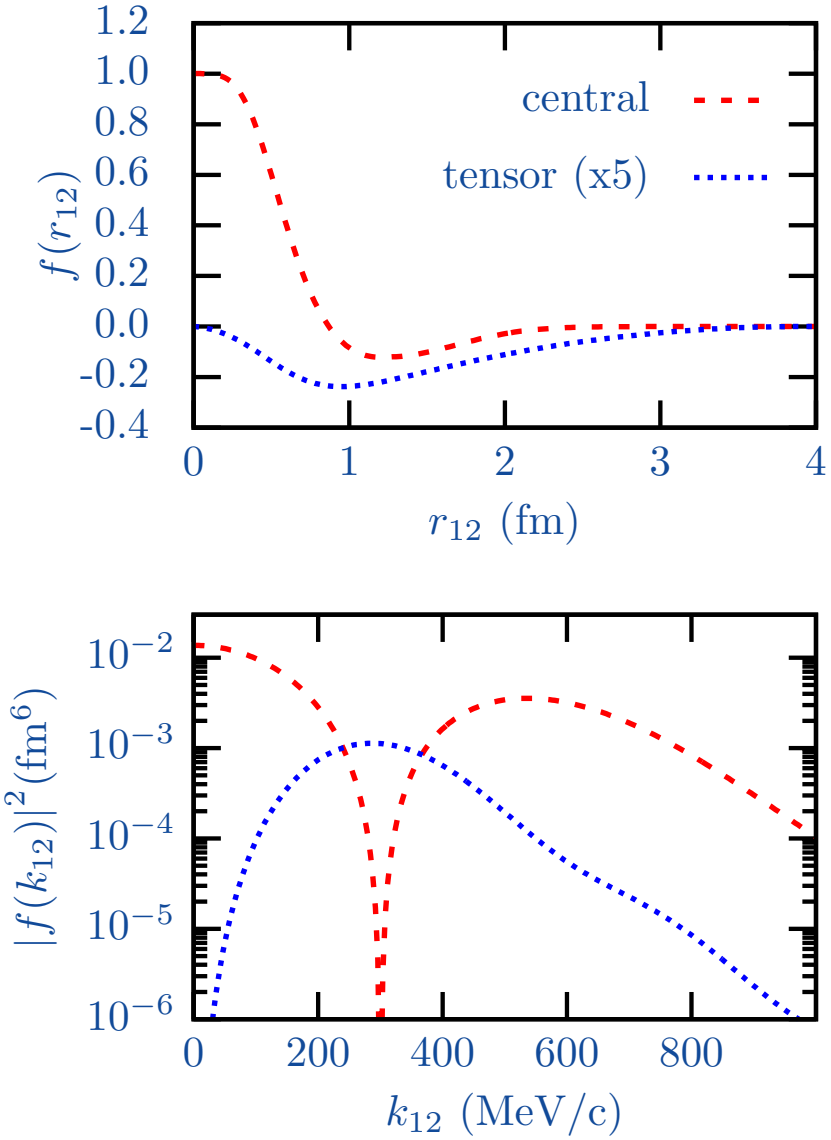


Figure : Correlation functions

- Short-range: $f(r_{ij}) \to 0$ at $r_{ij} > 3$ fm
- Tensor correlation function dominates for intermediate relative momenta $200 - 400$ MeV/c
- Central correlation function dominates at high relative momenta

Ref: C. C. Gearhaert, PhD thesis, Washington University, (1994).
S. C. Pieper, *et al.* Phys.Rev. C46, 1741 (1992).

Short-range correlations

Correlated wave functions $|\Psi\rangle$ are constructed by acting with a many-body correlation operator $\widehat{\mathcal{G}}$ on the uncorrelated Hartree-Fock wave functions $|\Phi\rangle$

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\widehat{\mathcal{G}}|\Phi\rangle, \qquad \text{with} \qquad \mathcal{N} = \langle\Phi|\widehat{\mathcal{G}}^\dagger\widehat{\mathcal{G}}|\Phi\rangle$$

For the correlation operator, we consider a similar structure as the one-boson exchange parameterization of the NN potential

$$\mathcal{O}^c(i,j) = 1 \qquad \mathcal{O}^{\sigma\tau}(i,j) = (\vec{\sigma}_i \cdot \vec{\sigma}_j)(\vec{\tau}_i \cdot \vec{\tau}_j)$$

$$\mathcal{O}^\sigma(i,j) = \vec{\sigma}_i \cdot \vec{\sigma}_j \qquad \mathcal{O}^t(i,j) = \widehat{S}_{ij}$$

$$\mathcal{O}^\tau(i,j) = \vec{\tau}_i \cdot \vec{\tau}_j \qquad \mathcal{O}^{t\tau}(i,j) = \widehat{S}_{ij}(\vec{\tau}_i \cdot \vec{\tau}_j)$$

$$\widehat{S}_{ij} = \frac{3}{r_{ij}^2}(\vec{\sigma}_i \cdot \vec{r}_{ij})(\vec{\sigma}_j \cdot \vec{r}_{ij}) - (\vec{\sigma}_i \cdot \vec{\sigma}_j).$$

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Central and tensor part are responsible for majority of the strength

$$\widehat{\mathcal{G}} = \widehat{\mathcal{S}}\left(\prod_{i<j}^{A}\left[1+\widehat{l}(i,j)\right]\right),$$

$$\widehat{l}(i,j) = -g_c(r_{ij}) + f_{t\tau}(r_{ij})\widehat{S}_{ij}\left(\vec{\tau}_i\cdot\vec{\tau}_j\right).$$

$g_c(r_{ij})$ and $f_{t\tau}(r_{ij})$ are the central and tensor correlation functions

Ref: C. C. Gearhaert, PhD thesis, Washington University, (1994).
S. C. Pieper, *et al.* Phys.Rev. C46, 1741 (1992).

Short-range correlations

When calculating matrix elements, we shift the complexity of the correlations from the wave functions to the operators

$$J_\lambda^{\text{nucl,eff}} = \left(\prod_{j<k}^{A} \left[1 + \widehat{l}(j,k) \right] \right)^\dagger \sum_{i=1}^{A} J_\lambda^{[1]}(i) \left(\prod_{l<m}^{A} \left[1 + \widehat{l}(l,m) \right] \right).$$

(a) **(b)** **(c)**

(d) **(e)** **(f)**

Short-range correlations

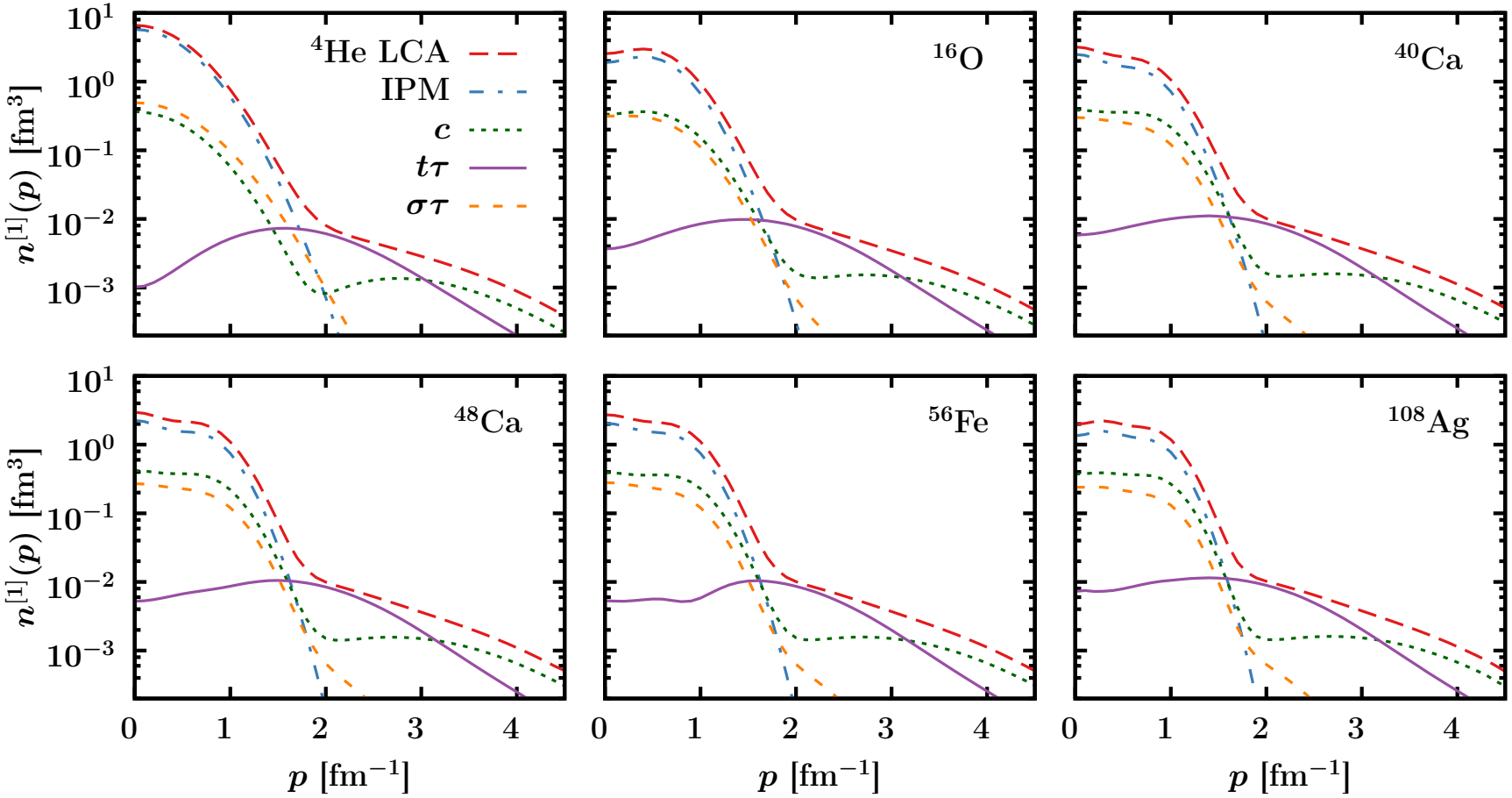
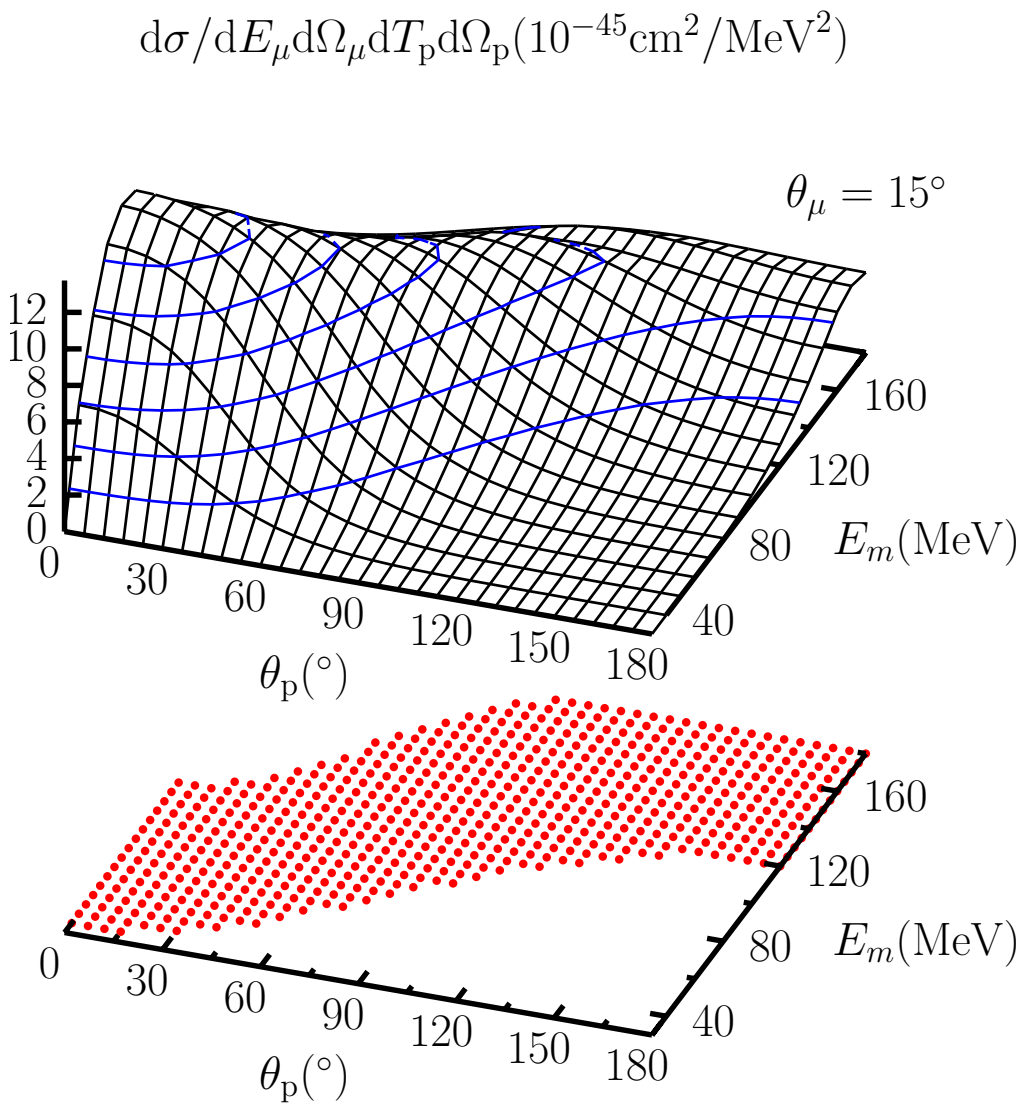


Figure : Nucleon single-nucleon momentum distribution

Ref: J. Ryckebusch *et al.* J. Phys. G: Nucl. Part. Phys. 42 (2015) 055104

SRC results - Semi-exclusive $2p2h$



Cross section is an incoherent sum of pp and pn knockout

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_{l'}\mathrm{d}\Omega_{l'}\mathrm{d}T_p\mathrm{d}\Omega_p}(l, l'p)$$
$$= \int \mathrm{d}\Omega_n \frac{\mathrm{d}\sigma}{\mathrm{d}E_{l'}\mathrm{d}\Omega_{l'}\mathrm{d}T_p\mathrm{d}\Omega_p\mathrm{d}\Omega_n}(l, l'pn)$$
$$+ \int \mathrm{d}\Omega_{p'} \frac{\mathrm{d}\sigma}{\mathrm{d}E_{l'}\mathrm{d}\Omega_{l'}\mathrm{d}T_p\mathrm{d}\Omega_p\mathrm{d}\Omega_{p'}}(l, l'pp')$$

Figure : $E_\nu = 750$ MeV, $E_\mu = 550$ MeV for in-plane kinematics ($\phi_p = 0°$).

SRC results - Inclusive $2p2h$

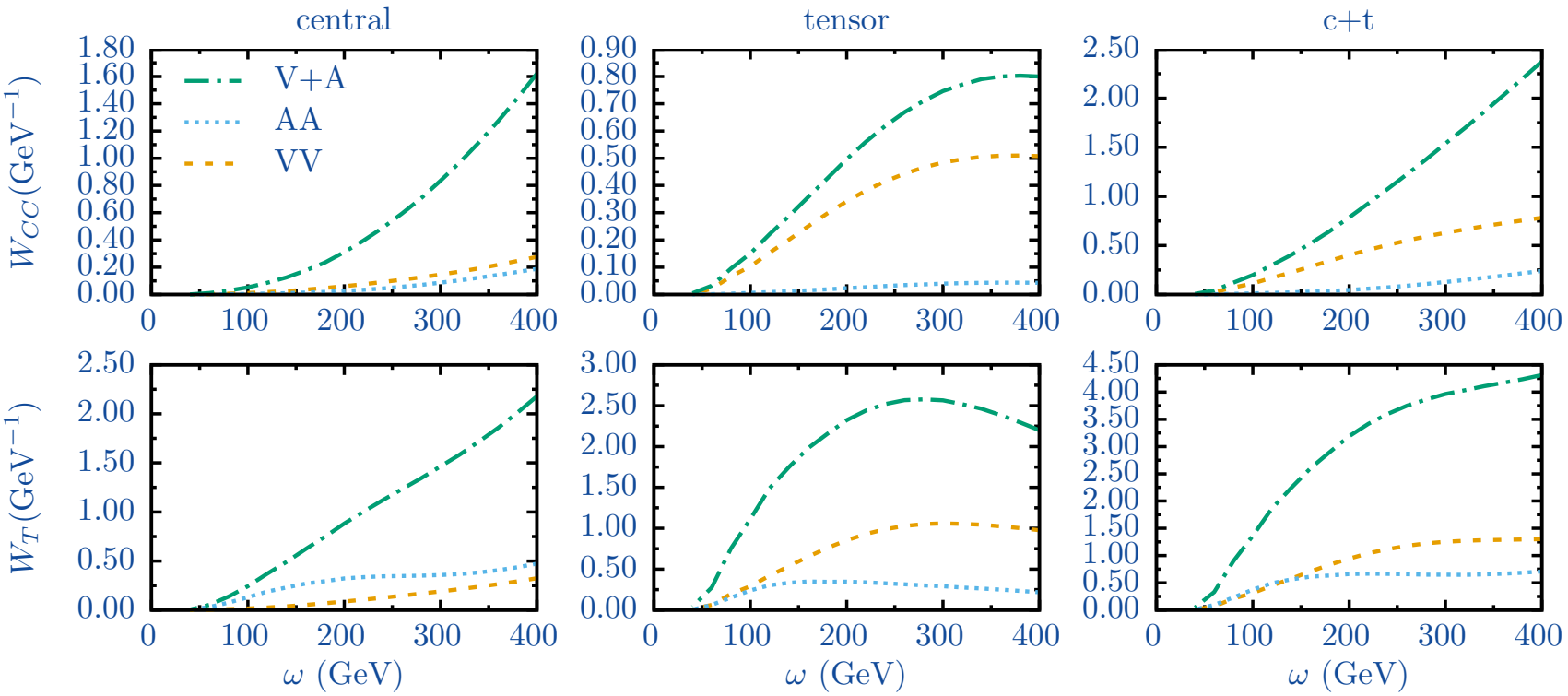


Figure : The dominating $2p2h$ SRC responses for ν-scattering at ^{12}C with fixed momentum transfer $q = 400$ MeV/c

SRC results - Inclusive $2p2h$

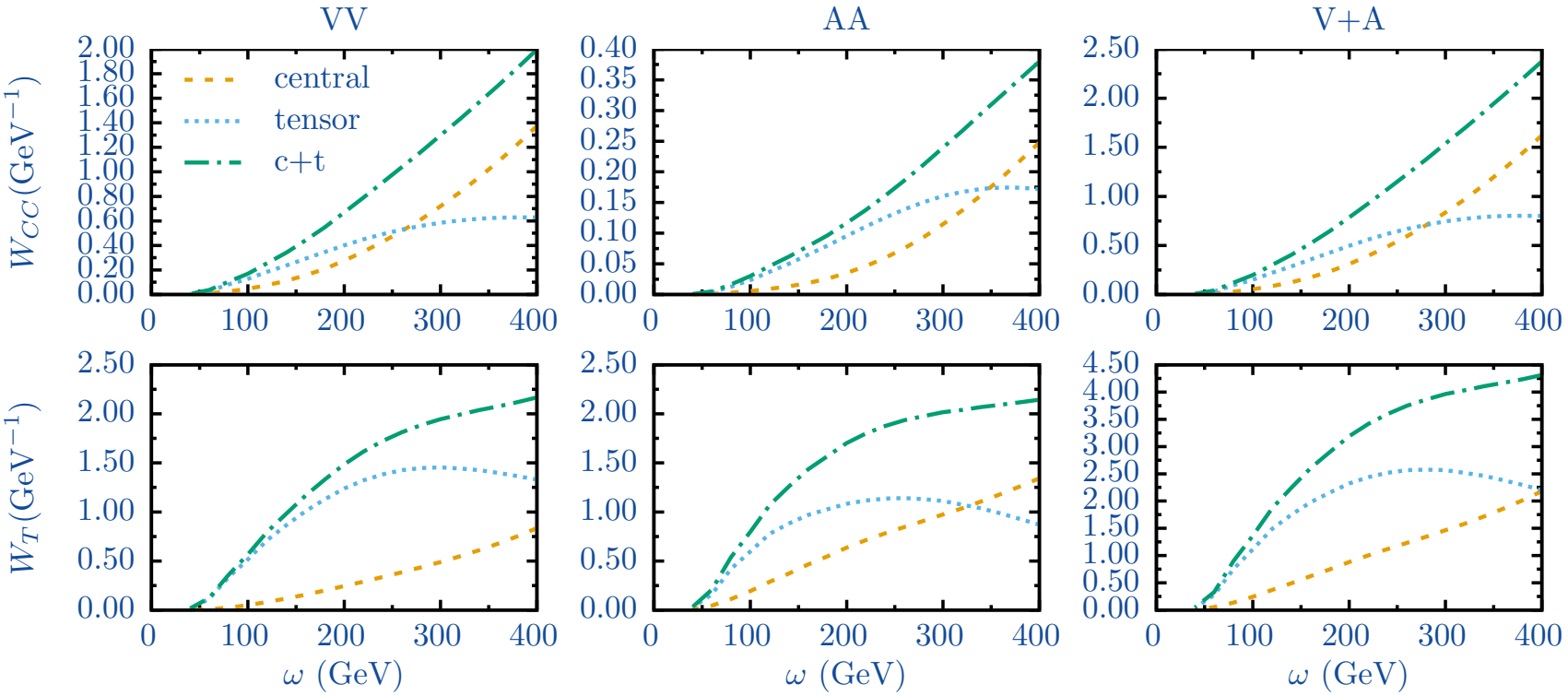


Figure : The dominating $2p2h$ SRC responses for ν-scattering at ^{12}C with fixed momentum transfer $q = 400$ MeV/c

SRC results - Inclusive $2p2h$

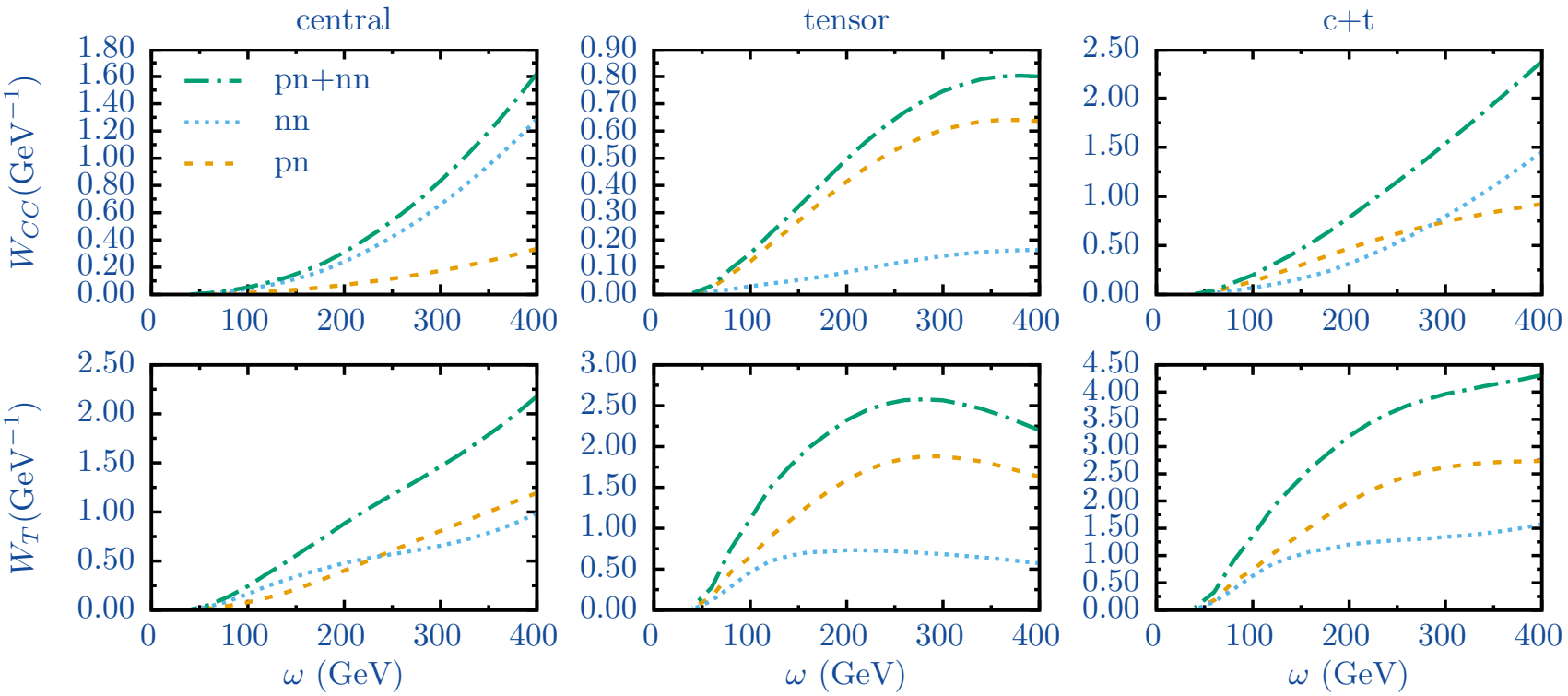


Figure : The dominating $2p2h$ SRC responses for ν-scattering at ^{12}C with fixed momentum transfer $q = 400$ MeV/c

One-nucleon knockout

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_{l'}\mathrm{d}\Omega_{l'}} = 4\pi\sigma^X\zeta\left[v_{CC}W_{CC} + v_{CL}W_{CL} + v_{LL}W_{LL} + v_T W_T - h v_{T'} W_{T'}\right],$$

with v and σ^X containing leptonic information, e.g.

$$\sigma^{\mathrm{Mott}} = \left(\frac{\alpha\cos\left(\theta_{e'}/2\right)}{2E_e\sin^2\left(\theta_{e'}/2\right)}\right)^2, \qquad \sigma^W = \left(\frac{G_F\cos\theta_c E_\mu}{2\pi}\right)^2,$$

and the response functions containing the nuclear information

$$W_{CC} = |\mathcal{J}_0|^2$$
$$W_{CL} = 2\Re\left(\mathcal{J}_0\mathcal{J}_3^\dagger\right)$$
$$W_{LL} = |\mathcal{J}_3|^2$$
$$W_T = |\mathcal{J}_+|^2 + |\mathcal{J}_-|^2$$
$$W_{T'} = |\mathcal{J}_+|^2 - |\mathcal{J}_-|^2$$

$$\mathcal{J}_0 = \langle\Psi_\mathrm{f}|\hat{\mathcal{J}}_0(\vec{q})|\Psi_\mathrm{i}\rangle$$
$$\mathcal{J}_+ = \langle\Psi_\mathrm{f}|\hat{\mathcal{J}}_+(\vec{q})|\Psi_\mathrm{i}\rangle$$
$$\mathcal{J}_- = \langle\Psi_\mathrm{f}|\hat{\mathcal{J}}_-(\vec{q})|\Psi_\mathrm{i}\rangle$$
$$\mathcal{J}_3 = \langle\Psi_\mathrm{f}|\hat{\mathcal{J}}_3(\vec{q})|\Psi_\mathrm{i}\rangle$$

Two-nucleon knockout

$$\frac{\mathrm{d}\sigma}{\mathrm{d}E_{l'}\mathrm{d}\Omega_{l'}\mathrm{d}\,T_b\mathrm{d}\Omega_b\mathrm{d}\Omega_a} = \frac{m_N^2 p_a p_b}{(2\pi)^6}\sigma^X\zeta$$
$$\times\left[v_{CC}W_{CC} + v_{CL}W_{CL} + v_{LL}W_{LL} + v_T W_T + v_{TT}W_{TT} + v_{TC}W_{TC}\right.$$
$$\left. + v_{TL}W_{TL} - h(v_{T'}W_{T'} + v_{TC'}W_{TC'} + v_{TL'}W_{TL'})\right],$$

The leptonic factors v and σ^X are independent of the number of knockout particles and five more response functions appear

$$W_{TT} = 2\Re\left(\mathcal{J}_+\mathcal{J}_-^\dagger\right)$$
$$W_{TC} = 2\Re\left(\mathcal{J}_0\left(\mathcal{J}_+ - \mathcal{J}_-\right)^\dagger\right)$$
$$W_{TL} = 2\Re\left(\mathcal{J}_3\left(\mathcal{J}_+ - \mathcal{J}_-\right)^\dagger\right)$$
$$W_{TC'} = 2\Re\left(\mathcal{J}_0\left(\mathcal{J}_+ + \mathcal{J}_-\right)^\dagger\right)$$
$$W_{TL'} = 2\Re\left(\mathcal{J}_3\left(\mathcal{J}_+ + \mathcal{J}_-\right)^\dagger\right)$$

$$\mathcal{J}_0 = \langle\Psi_\mathrm{f}|\widehat{\mathcal{J}}_0(\vec{q})|\Psi_\mathrm{i}\rangle$$
$$\mathcal{J}_+ = \langle\Psi_\mathrm{f}|\widehat{\mathcal{J}}_+(\vec{q})|\Psi_\mathrm{i}\rangle$$
$$\mathcal{J}_- = \langle\Psi_\mathrm{f}|\widehat{\mathcal{J}}_-(\vec{q})|\Psi_\mathrm{i}\rangle$$
$$\mathcal{J}_3 = \langle\Psi_\mathrm{f}|\widehat{\mathcal{J}}_3(\vec{q})|\Psi_\mathrm{i}\rangle$$