Soft-gluon current at two loops

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and work in progress

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Factorization of gauge-theory amplitudes

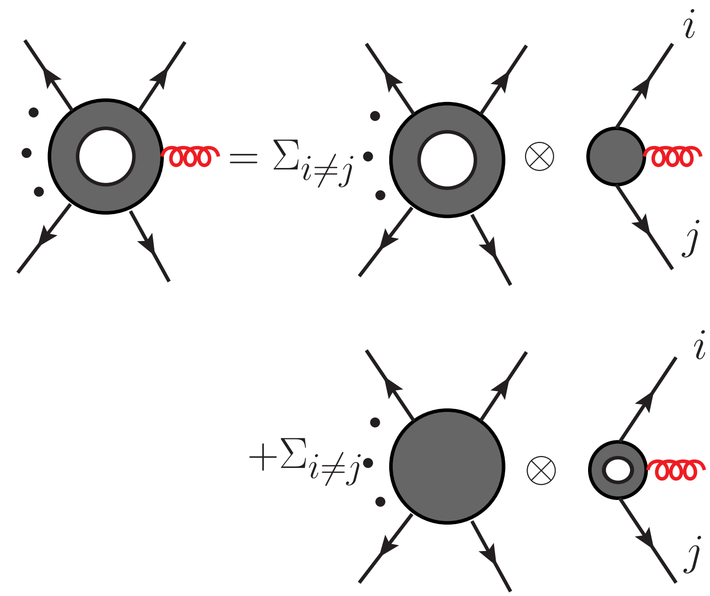
- A remarkable property of scattering amplitudes in gauge theory is factorization
  - ★ Regge factorization
  - ★ Collinear factorization
  - ★ soft factorization

Factorization of gauge-theory amplitudes

- A remarkable property of scattering amplitudes in gauge theory is factorization
  - Regge factorization
  - Collinear factorization
  - soft factorization
- Many of our understanding of QCD rely on these factorization properties
  - Evolution of parton distribution function at small and large x
  - Resummation of various event shape and jet structures
  - Building block for calculation in fixed order perturbation theory

Single soft-gluon factorization

- QCD amplitudes in the single soft gluon limit factorize Berends, Giele, 89; Bern, Del Duca, Kilgore, Schmidt, 99; Catani Grazzini, 00; Feige, Schwartz, 14



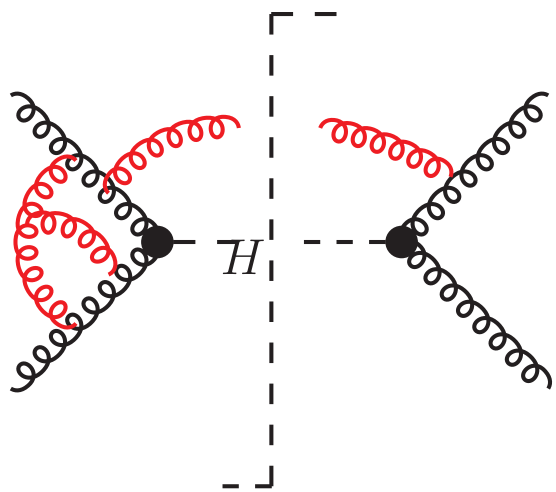
$$\langle a|\mathcal{M}(q,p_1,\ldots,p_m)\rangle \simeq \varepsilon^{\mu}(q) J^a_{\mu}(q,\epsilon)|\mathcal{M}(p_1,\ldots,p_m)\rangle$$

- $J^a_\mu(q,\epsilon)$ is called the one-gluon soft current

$$J^a_\mu(q,\epsilon) = g_S\mu^{\epsilon}[J^{a(0)}_\mu(q) + g_S^2 J^{a(1)}_\mu(q,\epsilon) + g_S^4 J^{a(2)}_\mu(q,\epsilon) + \ldots]$$

Why study one-gluon soft current at two loops?

- Useful approximation of complicated loop amplitudes
- Essential ingredient of soft function calculation in SCET. Example: threshold soft function Higgs production at $\mathrm{N}^3\mathrm{LO}$.



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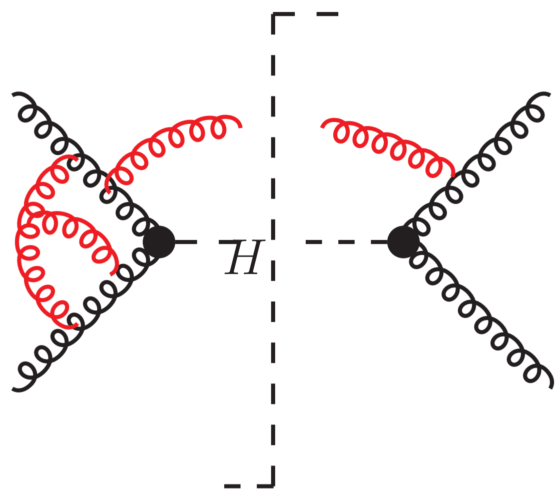

- Contribute to evolution kernel of non-global logarithms (BMS equation) at NNLO Caron-Huot, 15; Larkoski, Moult, Neill, 15

Some history

- Tree-level one-gluon soft current well-knwon,

$$J^{\mu(0)}(q) = \sum_{i=1}^{n} T_i^a \frac{p_i^\mu}{p_i \cdot q}$$

$(T_i^a)_{\alpha\beta} = (t^a)_{\alpha\beta}$ final-state quark

$(T_i^a)_{\alpha\beta} = (-t^a)_{\beta\alpha}$ final-state anti-quark

$(T_i^a)_{bc} = -i f_{abc}$ gluon

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- Direct calculation using Wilson lines

  - One loop massless Catani, Grazzini, 00 , One loop massive Bierenbaum, Czakon, Mitov, 11

  - Two loops massless large N_c Y. Li, HXZ, 13

Type 3 RPI; gauge invariance; maximally non-Abelian

- The soft current are manifestly invariant under type 3 reparametrization (rescaling invariance):

$$\frac{p^\mu}{p\cdot q} \quad \Rightarrow \quad \text{invariant under } p_i^\mu \to (1+\alpha)p_i^\mu$$

- Gauge invariance

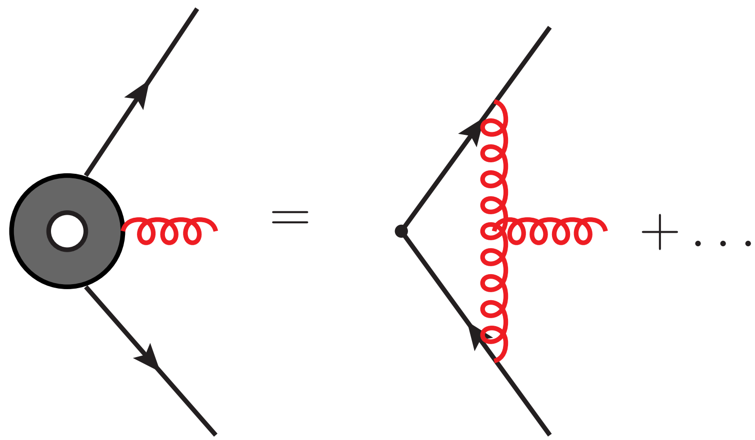
$$q_\mu J_a^\mu = 0$$

- Soft current non-vanishing only when it's maximally non-Abelian. It is a variant of non-Abelian exponentionation theorem Gatheral, 83; Frenkel, Taylor 84; Gardi, Smillie, White, 13

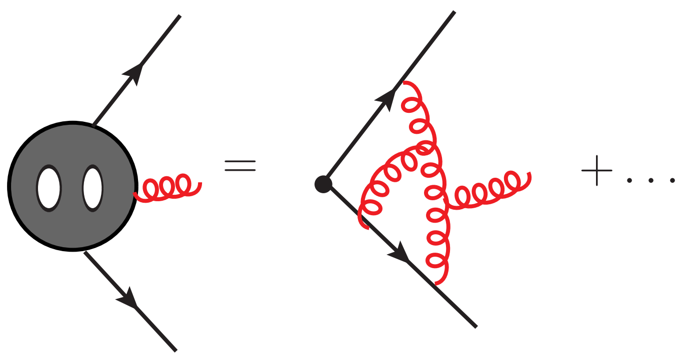
- The soft current calculates automatically the amplitude in the exponent

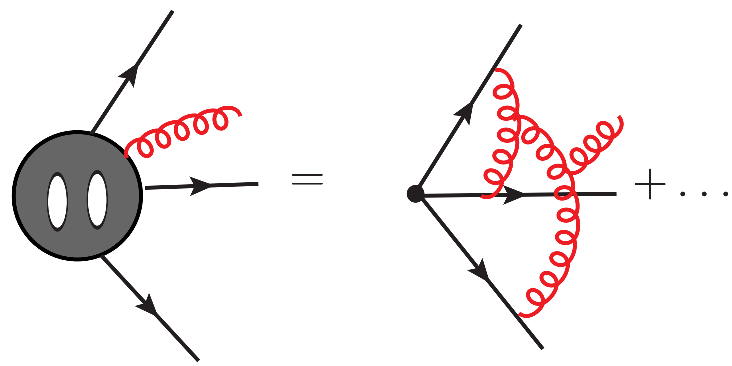
Diagrams contributed to soft current at two loops

- One-loop diagrams: only dipole contribution

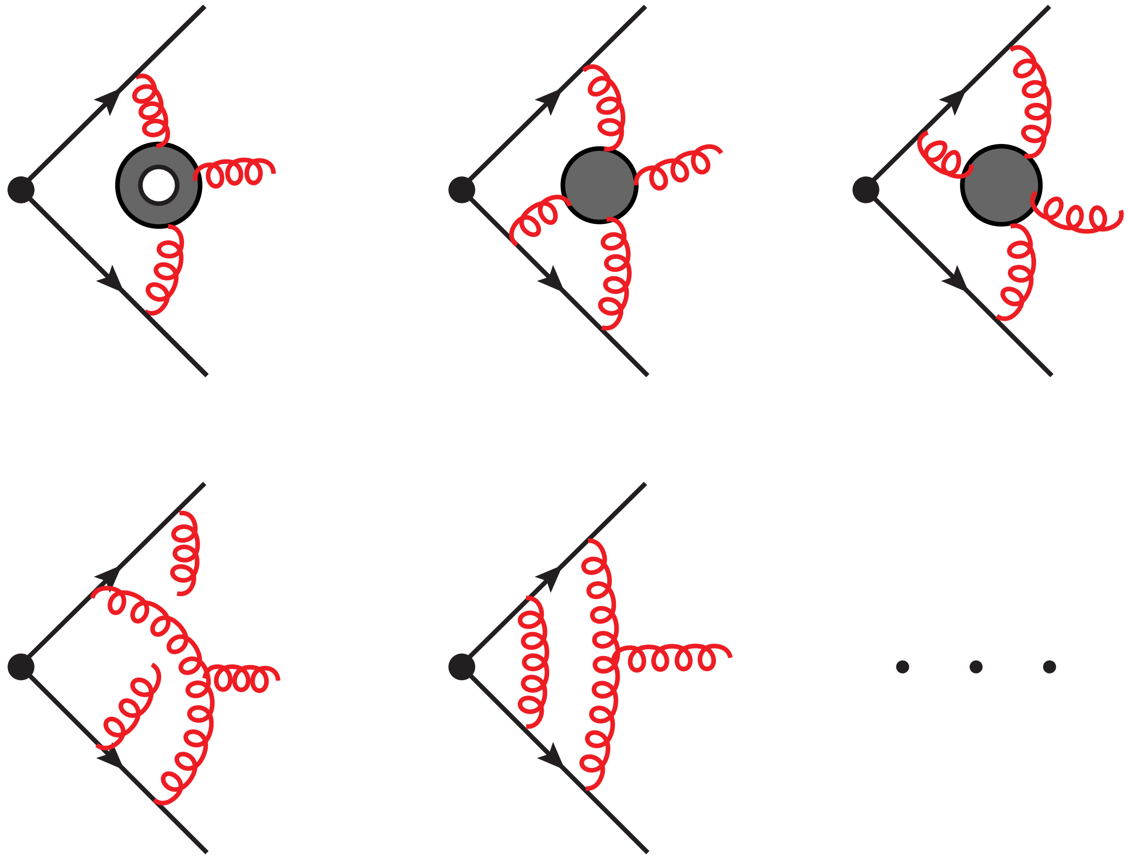


- Two-loop diagrams: both dipole and tripole contributions



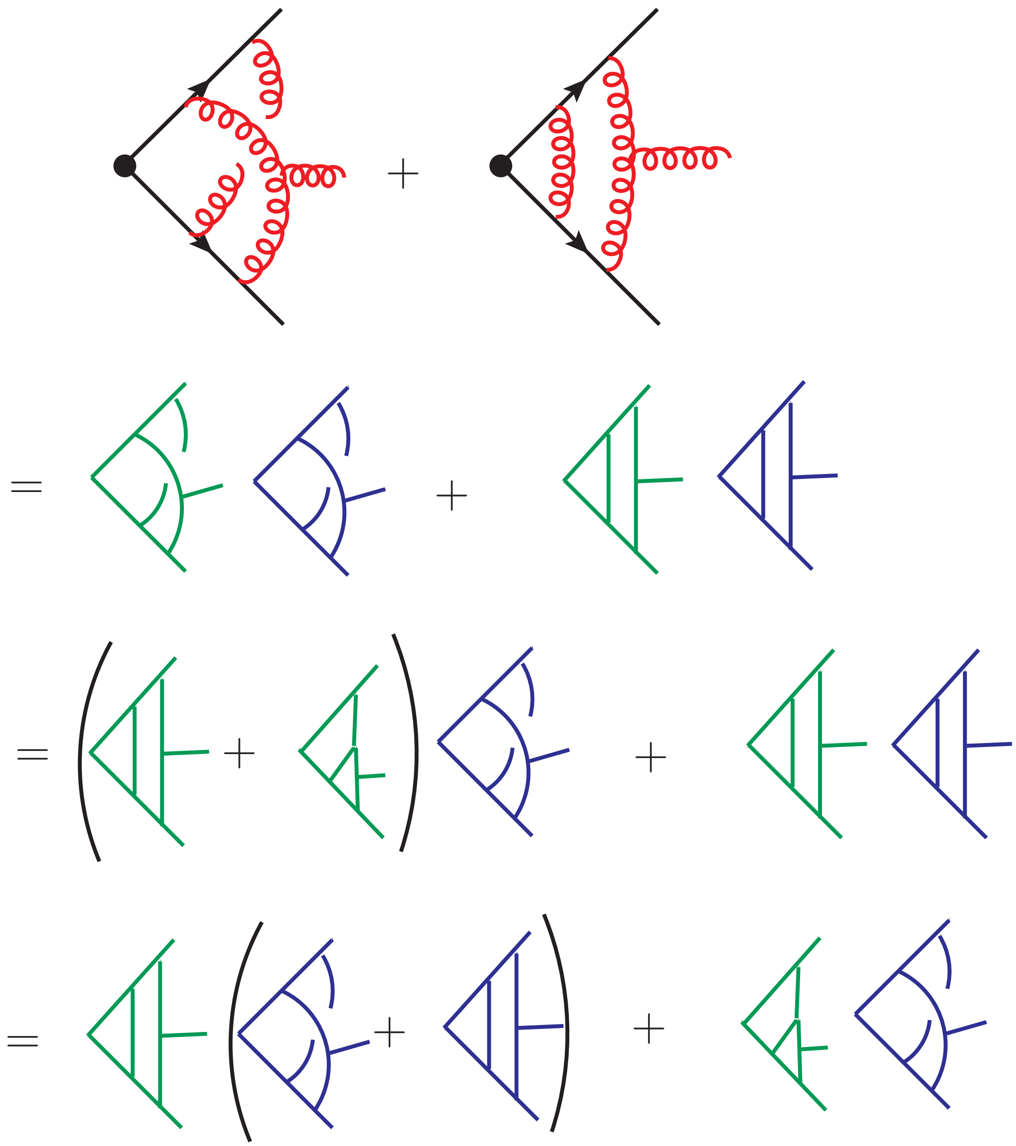


Two-loops diagrams for dipole correlation



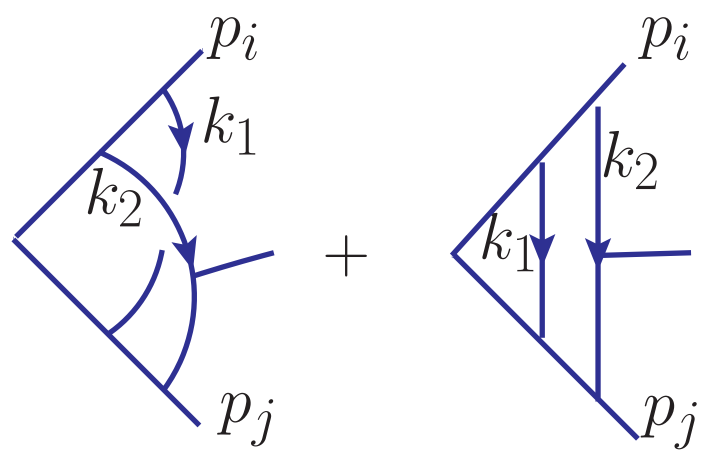
- Diagrams categorized according to the number of insertion on the Wilson lines
- Diagrams with four insertions on the Wilson lines are not color connected.

Simplifying the color structure



$$T^aT^b = T^bT^a + if^{abc}T^c$$

Simplifying the color structure



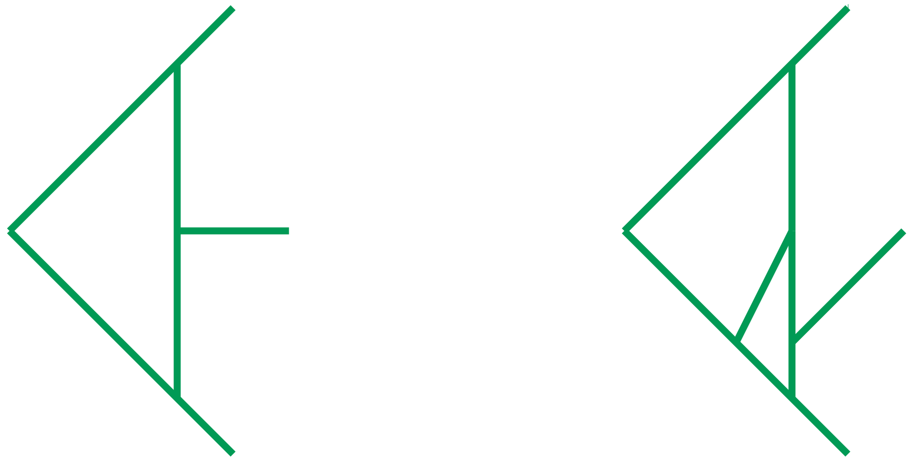
$$\frac{1}{[k_1\cdot p_i]\,[(k_1+k_2)\cdot p_i]}+\frac{1}{[k_2\cdot p_i]\,[(k_1+k_2)\cdot p_i]}=\frac{1}{[k_1\cdot p_i]\,[k_2\cdot p_i]}$$

$\int d^{4-2\epsilon}k_1\,\frac{1}{[k_1^2]\,[k_1 p_i]\,[(-k_1-k_2+q)p_j]}=0$ because it depends only on $(-k_2+q)\cdot p_j$ and $p_i\cdot p_j$. Violate type 3 RPI.

- Only the color connected part survives

Two-loop dipole correlation

- Two color structure from two-loop calculation for the dipole-like diagrams



$$J_{a,ij}^{\mu(2)} = \left(\frac{p_i^\mu}{p_i \cdot q} - \frac{p_j^\mu}{p_j \cdot q} \right) \left(\frac{s_{ij}}{s_{iq} s_{jq}} \right)^{2\epsilon} e^{-2i\pi\epsilon(\lambda_{ij} - \lambda_{iq} - \lambda_{jq})}$$

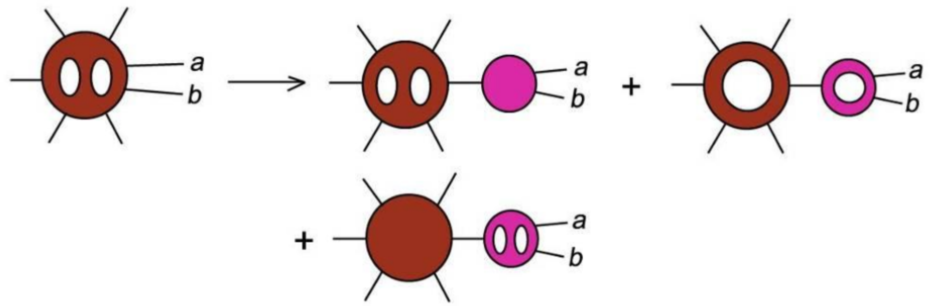
$$\times \left(i f_{abc} T_i^b T_j^c \, B_{1,ij}(\epsilon) + f_{ace} f_{bde} T_{[i,}^b T_{j]}^{cd} \, B_{2,ij}(\epsilon) \right) \qquad T^{cd} = T^c T^d$$

$s_{kl} = 2|p_k \cdot p_l|$ $\qquad \lambda_{kl} = 1$ if both k, l are incoming or outgoing, otherwise 0

- Violation of "strict" collinear factorization made manifest

Strict collinear factorization violation

- “strict” collnear factorization: collinear singular factor only depends on the momenta and quantum number of the collinear partons



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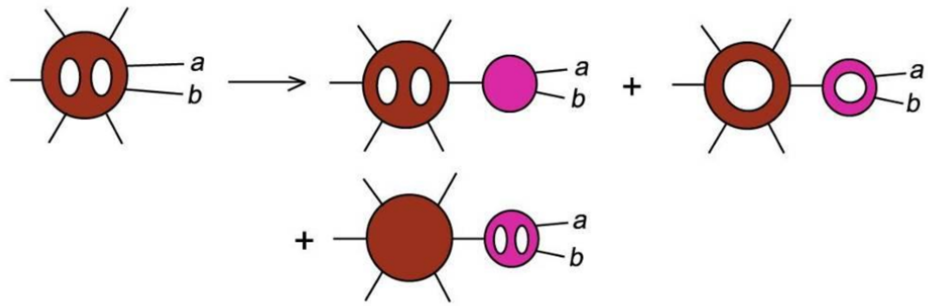

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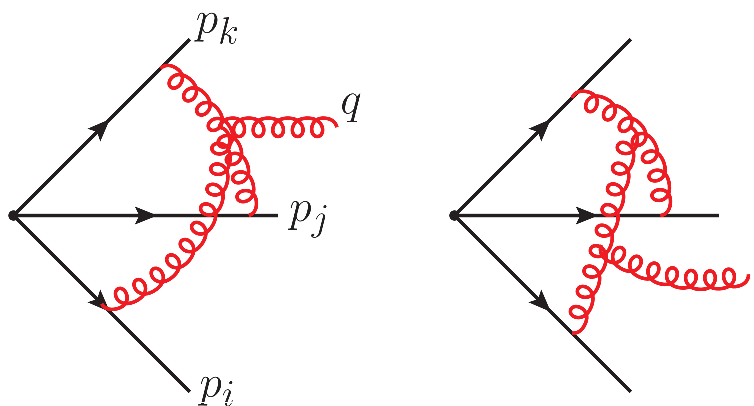
- "Strict" collinear factorization violation is also transparent from one-gluon soft current

$$\lim_{p_i \| q} J^{\mu(2)}_{a,ij} = \left(\frac{p_i^\mu}{p_i \cdot q} - \frac{p_j^\mu}{p_j \cdot q} \right) \left(\frac{s_{ij}}{s_{iq} s_{jq}} \right)^{2\epsilon} e^{-2i\pi\epsilon(\lambda_{ij} - \lambda_{iq} - \lambda_{jq})}$$

$$\frac{s_{ij}}{s_{jq}} = \frac{1}{w}$$

$$\times \left(i f_{abc} T_i^b T_j^c B_{1,ij}(\epsilon) + f_{ace} f_{bde} T^b_{[i,} T^{cd}_{j]} B_{2,ij}(\epsilon) \right)$$

Tripole correlated soft-current



$+ \quad permutation$

$$J^{\mu(2)}_{a,ijk} = \sum_{i\neq j\neq k} \left(\frac{p_i^\mu}{p_i\cdot q} C_1(u,v) + \frac{p_j^\mu}{p_j\cdot q} C_2(u,v) + \frac{p_k^\mu}{p_k\cdot q} C_3(u,v) \right) \times f_{abe} f_{cde} T_i^b T_j^c T_k^d$$

- Depends on dimensionless conformal cross ratio (consider Euclidean region only)

$$u = \frac{s_{ik} s_{jq}}{s_{ij} s_{kq}} \qquad v = \frac{s_{jk} s_{iq}}{s_{ij} s_{kq}}$$

Kinematics of tripole correlation

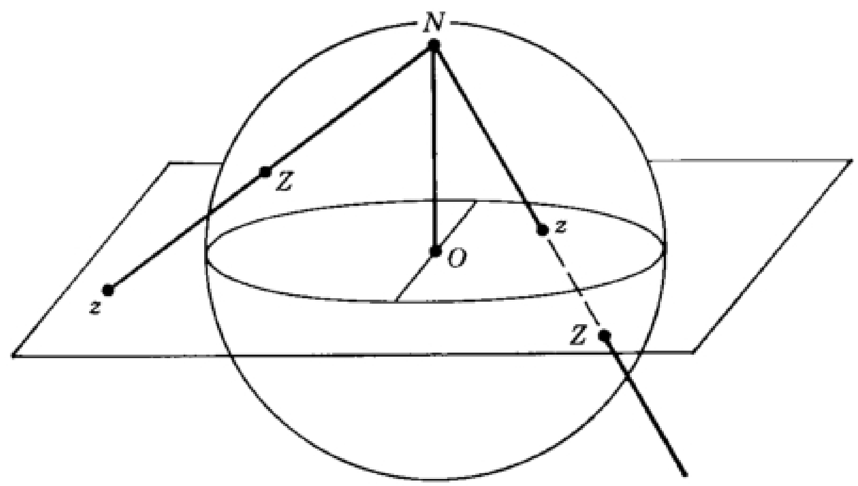
- u and v are not free variables. Constrained by the equation

$$1 - 2u - 2v + (u - v)^2 < 0$$

- p_i, p_j, p_k, q are lightlike momenta. Can be identified as points on two-sphere.

$$p^\mu = (1, \sin\theta \sin\phi, \sin\theta \cos\phi, \cos\theta)$$

- Stereographic projection help simplify the constraints



$$z = \frac{\sin\theta}{1+\cos\theta} e^{i\phi}$$

Kinematics of tripole correlation

- Under the stereographic projection, the conformal cross ratios become

$$u = \frac{|z_i - z_k|^2 |z_j - z_q|^2}{|z_i - z_j|^2 |z_k - z_q|^2} \qquad v = \frac{|z_j - z_k|^2 |z_i - z_q|^2}{|z_i - z_j|^2 |z_k - z_q|^2}$$

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- u, v are invariant under $SL(2, C)$ on the complex plane. Can map z_i, z_j, z_k to $0, 1, \infty$. Let $z = z_q$,

$$u = (1 - z)(1 - z^*) \qquad v = zz^*$$

- Besides being constraint-free variables, z, z^* also simplify the calculation of integrals

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- Square root function appear in the integral calculation with u, v variables, becomes rational function in z, z^* parametrization

$$\sqrt{1 - 2u - 2v + (u - v)^2} = z - z^*$$

Integrals for tripole correlation

- The integrals are functions of external variables, z and z^*
- The method of differential equation is very suitable for calculating Feynman integrals with multiple scales Kotikov, 91; Remiddi, 97; Gehrmann, Remiddi, 99

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- Can cast the resulting system of differential equation into canonical form Henn, 13

$$d\vec{f}(z, z^*, \epsilon) = \epsilon \Big(\sum_m A_m d \ln \alpha_m(z, z^*) \Big) \vec{f}(z, z^*, \epsilon)$$

- A_m are constant matrices, independent of ϵ
- Alphabet of the differential equation $\alpha_m(z, z^*) \in \{z, z^*, 1 - z, 1 - z^*, z - z^*\}$

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- Expand the integrals in ϵ, $\vec{f}(z, z^*, \epsilon) = f_0(z, z^*) + \epsilon f_1(z, z^*) + \epsilon^2 f_2(z, z^*) + \ldots$, the differential can be solved immediately by integrating the right-hand-side, up to some boundary constants

Integrals for tripole correlation

- All the integrals can be expressed through multiple polylogarithm

$$G(w_1, \ldots, w_n; x) = \int_0^x \frac{dt}{t - w_1} G(w_2, \ldots, w_n; t)$$

$$G(;x) = 1$$

$$G(\underbrace{0, \ldots, 0}_{n}; x) = \frac{1}{n!} \ln^n x$$

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- In Euclidean region, all the master integrals are real without branch cut. They form a special class of multiple polylogarithms called single-valued multiple polylogarithms. A famous example is Bloch-Wigner function

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1 - z) \ln(z)$$

- Very similar function appear in other context: three-mass triangle integral Chavez, Duhr, 12 ; Four-point off-shell conformal integral Drummond, Duhr, Eden, Heslop, Pennington, Smirnov, 13

Summary

- Gauge theory amplitudes factorized in the limit of single soft-gluon emission
- The one-gluon soft-current appears as building block in many places, including fixed order, resummation, and evolution kernel calculation
- Two-loop one-gluon soft-current is computed with full color dependence
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