Three-Loop Soft Functions For Gluon Fusion Higgs Boson And Drell-Yan Lepton Production

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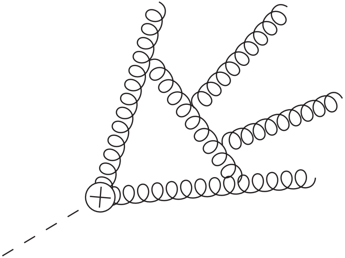
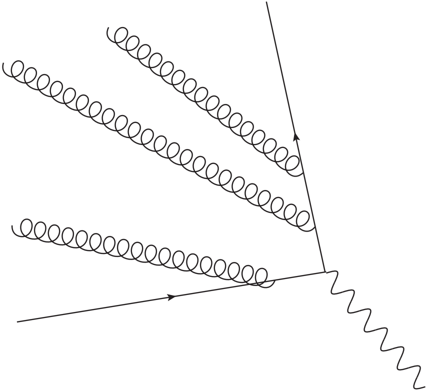
The Current State-Of-The-Art For Soft Functions





Outline

1. Background
   - The Factorization Formula
   - What We Have Calculated
2. Our Calculation Of The Three-Loop Higgs Soft Function
   - Get The Squared Amplitude From Feynman Diagrams
   - Apply Integration By Parts Reduction To The Integrand
   - Derive All-Orders-in-ϵ Expressions For Master Integrals
3. Three-Loop Drell-Yan Soft Function Via Casimir Scaling
4. Outlook

The Threshold Factorization Formula For The Partonic Cross Section For Higgs Boson Production @ LHC

J. Collins, D. Soper, and G. Sterman, Nucl. Phys. **B261**, 104, 1985

The ratio $z = M_H^2/\hat{s}$ is a scale in the partonic cross section which is then convolved with the proton PDFs to obtain a prediction for the total production cross section. The *threshold expansion* of the result is about the limit $z \to 1$ and begins with the so-called *soft-virtual term*:

$$\hat{\sigma}_{gg}^{\mathrm{H}}(z) = \sigma_0^{\mathrm{H}}\, H\Sigma(1-z) + \mathcal{O}\left(1-z\right)$$

In this limit, we say that the partonic cross section *factorizes* into a product of a *hard function*, H, and a *soft function*, $\Sigma(1-z)$.

The Three-Loop Soft Function For Higgs Production

$$\Sigma\left(\ln\left(\frac{2E_{cut}}{\mu}\right)\right) = \int_0^{E_{cut}} d\lambda\, S(\lambda, \mu)$$

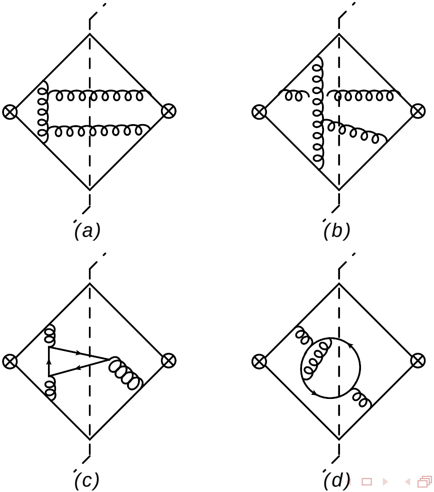
$$S(\lambda, \mu) = \frac{1}{N_c}\sum_{X_s}\delta\Big(\lambda - E_{X_s}\Big)\left|\langle 0|T\left\{Y_n^\dagger Y_{\bar{n}}\right\}|X_s\rangle\right|^2$$

$$n^2 = \bar{n}^2 = 0 \qquad n \cdot \bar{n} = 2$$

This computation was carried out by a different group as well but the one-loop, two-emission part of it was never separately published.

C. Anastasiou *et. al.*, Phys. Lett. **B737**, 325, 2014

Evaluate The Appropriate Squared Sum of Cut Eikonal Feynman Diagrams



Integration By Parts Reduction

F. Tkachov, Phys. Lett. **B100**, 65, 1981; K. Chetyrkin and F. Tkachov, Nucl. Phys. **B192**, 159, 1981

It is well-known that one can generate recurrence relations by considering families of Feynman integrals and then integrating by parts in d spacetime dimensions, *e.g.*

$$\begin{aligned} 0 &= \int \frac{d^d\ell}{(2\pi)^d} \frac{\partial}{\partial \ell_\mu} \left(\frac{\ell_\mu}{(\ell^2 - m^2)^a} \right) \\ &= \int \frac{d^d\ell}{(2\pi)^d} \left(\frac{d}{(\ell^2 - m^2)^a} - \frac{2a\ell^2}{(\ell^2 - m^2)^{a+1}} \right) \\ &= (d - 2a)I(a) - 2am^2 I(a+1) \end{aligned}$$

In this case, the recurrence relation can be solved explicitly but it is one of the few known examples where one can proceed directly.

Apply the Reduze 2 Integration By Parts Identity Solver To Reduce The Integrand

- In all but the simplest examples, the strategy used (S. Laporta, Int. J. Mod. Phys. **A15**, 5087, 2000) to solve integration by parts identities is to build a linear system of equations for the Feynman integrals in the calculation by explicitly substituting particular values of the indices into the recurrence relations.
- The Reduze 2 (A. von Manteuffel and C. Studerus, arXiv:1201.4330) implementation of Laporta's algorithm is robust and well-tested.
- However, the public version of the code was written with virtual corrections in mind and does not support phase space integrals such as those which arise in the calculation under discussion.

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- However, the public version of the code was written with virtual corrections in mind and does not support phase space integrals such as those which arise in the calculation under discussion.

The functionality of the code is straightforward to appropriately extend and we find that there are just 9 master integrals which need to be calculated.

The Art Of Phase Space Integral Evaluation

Y. Li, S. Mantry, and F. Petriello, Phys. Rev. **D84**, 094014, 2011

$$\mathrm{Re}\left\{-i\pi^{3\epsilon-4}e^{3\gamma_E\epsilon}\int \mathrm{d}^dk_1\int \mathrm{d}^dk_2\int \mathrm{d}^dq\frac{\delta\left(\lambda-(k_1+k_2)\cdot(n+\bar{n})\right)\delta\left(k_1^2\right)\delta\left(k_2^2\right)}{q^2\left(k_1+k_2-q\right)^2\ 2q\cdot n\ 2\left(k_1+k_2-q\right)\cdot\bar{n}}\right\}$$

$$=\frac{\pi^{2\epsilon-2}e^{3\gamma_E\epsilon}\Gamma^2(1-\epsilon)\Gamma^2(\epsilon)\cos(\pi\epsilon)}{4\Gamma(-2\epsilon)\Gamma(2+\epsilon)}\int \mathrm{d}^dk_1\int \mathrm{d}^dk_2\times$$

$$\times\frac{\delta\left(\lambda-(k_1+k_2)\cdot(n+\bar{n})\right)\delta\left(k_1^2\right)\delta\left(k_2^2\right){}_2F_1\left(1,1;2+\epsilon;1-\frac{(k_1+k_2)^2}{(k_1+k_2)\cdot n(k_1+k_2)\cdot\bar{n}}\right)}{(k_1+k_2)\cdot n\,(k_1+k_2)\cdot\bar{n}\left((k_1+k_2)^2\right)^{\epsilon}}$$

At first sight, the remaining integrations look challenging because of the non-trivial dependence on the dot product of k_1 and k_2...

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$$\mathrm{Re}\left\{-i\pi^{3\epsilon-4}e^{3\gamma_E\epsilon}\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \int \mathrm{d}^d q \frac{\delta\left(\lambda-(k_1+k_2)\cdot(n+\bar{n})\right)\delta\left(k_1^2\right)\delta\left(k_2^2\right)}{q^2\left(k_1+k_2-q\right)^2\ 2q\cdot n\ 2\left(k_1+k_2-q\right)\cdot\bar{n}}\right\}$$

$$= \frac{\pi^{2\epsilon-2}e^{3\gamma_E\epsilon}\Gamma^2(1-\epsilon)\Gamma^2(\epsilon)\cos(\pi\epsilon)}{4\Gamma(-2\epsilon)\Gamma(2+\epsilon)}\int \mathrm{d}^d k_1 \int \mathrm{d}^d k_2 \times$$

$$\times \frac{\delta\left(\lambda-(k_1+k_2)\cdot(n+\bar{n})\right)\delta\left(k_1^2\right)\delta\left(k_2^2\right)\,{}_2F_1\left(1,1;2+\epsilon;1-\frac{(k_1+k_2)^2}{(k_1+k_2)\cdot n(k_1+k_2)\cdot\bar{n}}\right)}{(k_1+k_2)\cdot n\,(k_1+k_2)\cdot\bar{n}\left(\left(k_1+k_2\right)^2\right)^{\epsilon}}$$

At first sight, the remaining integrations look challenging because of the non-trivial dependence on the dot product of k_1 and k_2...
By inserting $1=\int \mathrm{d}^d p\,\delta\left(p-k_1-k_2\right)$ and then integrating over one of the k_i, we see that this dependence can be eliminated entirely!

The Art Of Phase Space Integral Evaluation Continued

This change of variable completely decouples the phase space integrations in this case. One can simply proceed by carrying out the trivial k_1 phase space integral, followed by the integration over $|\mathbf{p}_T|^2$ from 0 to p^+p^-, p^+ from 0 to $\lambda - p^-$, and p^- from 0 to λ.

$$\frac{\pi^{2\epsilon-2}e^{3\gamma_E\epsilon}\Gamma^2(1-\epsilon)\Gamma^2(\epsilon)\cos(\pi\epsilon)}{4\Gamma(-2\epsilon)\Gamma(2+\epsilon)}\int \mathrm{d}^d p\int \mathrm{d}^d k_1 \times$$

$$\times\frac{\delta\left(\lambda - p\cdot(n+\bar{n})\right)\delta\left(k_1^2\right)\delta\left((p-k_1)^2\right){}_2F_1\left(1,1;2+\epsilon;1-\frac{p^2}{p\cdot n\, p\cdot\bar{n}}\right)}{p\cdot n\, p\cdot\bar{n}\,\left(p^2\right)^{\epsilon}}$$

$$= -\frac{\lambda^{1-6\epsilon}e^{3\gamma_E\epsilon}\Gamma^2(1-3\epsilon)\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)\Gamma(\epsilon)\cos(\pi\epsilon)}{8\Gamma(2-6\epsilon)\Gamma(2-2\epsilon)\Gamma(2-3\epsilon)\Gamma(2+\epsilon)}\times$$

$$\times{}_3F_2\left(1,1,1-\epsilon;2-3\epsilon,2+\epsilon;1\right)$$

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$$\frac{\pi^{2\epsilon-2}e^{3\gamma_E\epsilon}\Gamma^2(1-\epsilon)\Gamma^2(\epsilon)\cos(\pi\epsilon)}{4\Gamma(-2\epsilon)\Gamma(2+\epsilon)}\int \mathrm{d}^d p\int \mathrm{d}^d k_1 \times$$
$$\times\frac{\delta\left(\lambda - p\cdot(n+\bar{n})\right)\delta\left(k_1^2\right)\delta\left((p-k_1)^2\right){}_2F_1\left(1,1;2+\epsilon;1-\frac{p^2}{p\cdot n\, p\cdot\bar{n}}\right)}{p\cdot n\, p\cdot\bar{n}\,\left(p^2\right)^\epsilon}$$
$$= -\frac{\lambda^{1-6\epsilon}e^{3\gamma_E\epsilon}\Gamma^2(1-3\epsilon)\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)\Gamma(\epsilon)\cos(\pi\epsilon)}{8\Gamma(2-6\epsilon)\Gamma(2-2\epsilon)\Gamma(2-3\epsilon)\Gamma(2+\epsilon)}\times$$
$$\times{}_3F_2\left(1,1,1-\epsilon;2-3\epsilon,2+\epsilon;1\right)$$

Remarkably, this transformation is helpful even in situations where some k_1 dependence remains in the integrand after changing variables!

Normal Forms For Single-Scale Integrals

J. Henn, Phys. Rev. Lett. **110** 25, 251601, 2013

It turns out to be straightforward to rotate to a normal form basis.

$$-(d-4)^3(d-3)(3d-11)\frac{e^{3\gamma_E\epsilon}\Gamma^2(1-3\epsilon)\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)\Gamma(\epsilon)\cos(\pi\epsilon)}{8\Gamma(2-6\epsilon)\Gamma(2-2\epsilon)\Gamma(2-3\epsilon)\Gamma(2+\epsilon)}\times$$
$$\times {}_3F_2\left(1,1,1-\epsilon;2-3\epsilon,2+\epsilon;1\right)$$
$$=\frac{e^{3\gamma_E\epsilon}\epsilon^2\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)\Gamma(1-3\epsilon)\,{}_3F_2(1,1+\epsilon,1+2\epsilon;2-\epsilon,2+\epsilon;1)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)\Gamma(1-6\epsilon)\Gamma(2-\epsilon)\Gamma(2+\epsilon)}$$
$$=\zeta_2\epsilon^2+3\zeta_3\epsilon^3-29\zeta_4\epsilon^4+\left(-\frac{229\zeta_2\zeta_3}{2}+\frac{75\zeta_5}{2}\right)\epsilon^5+\left(-\frac{12155\zeta_6}{64}-195\zeta_3^2\right)\epsilon^6+\cdots$$

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Our analysis serves to demonstrate that the explicit cancellation of spurious poles at locations like $d=11/3$ can provide a valuable guide if one seeks a normal form basis for single-scale problems.

From N^{3}LO Higgs To N^{3}LO Drell-Yan

T. Ahmed, M. Mahakhud, N. Rana, and V. Ravindran, Phys. Rev. Lett. **113** 11, 112002, 2014

Through N^{3}LO, Casimir scaling arguments suffice to predict the renormalized Drell-Yan soft function from the result for Higgs.

- $M_{\rm H} \to M_{\gamma^*}$
- $\Gamma^{\rm H}_{\rm cusp}(\alpha_s) \to \frac{C_F}{C_A}\Gamma^{\rm H}_{\rm cusp}(\alpha_s)$
- $\gamma^{\rm H}_s(\alpha_s) \to \frac{C_F}{C_A}\gamma^{\rm H}_s(\alpha_s)$
- Define a generating function for the Higgs matching coefficients:
  $c^{\rm H}_s(\alpha_s) = \exp\Big((\alpha_s/4\pi)c^{\rm H}_1 + (\alpha_s/4\pi)^2\Delta c^{\rm H}_2 + (\alpha_s/4\pi)^3\Delta c^{\rm H}_3 + \cdots\Big)$
- Then, $c^{\rm H}_s(\alpha_s) \to \left(c^{\rm H}_s(\alpha_s)\right)^{\frac{C_F}{C_A}}$

Future Directions

What's next?

- A phenomenological result for $\mathrm{N^3LO}$ gluon fusion Higgs production in the full theory just appeared!

  Anastasiou *et. al.*, arXiv:1503.06056

- Obtain a full $\mathrm{N^3LO} + \mathrm{N^3LL}$ prediction for the production cross section ($i\pi$ resummation?).

  Ahrens *et. al.*, Phys. Rev. **D79**, 033013, 2009 VS. Anastasiou *et. al.*, arXiv:1411.3584

- Generalize and extend the analytical integration techniques used to do the calculations.