Soft Theorems from Effective Field Theory

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$$\mathcal{A}(1,\ldots,N,s) \to S^{(0)}(s)\mathcal{A}(1,\ldots,N)$$

$$S^{(0)}_{\text{gauge}}(s) = \sum_{i=1}^{N} \left(gT^i\right) \frac{\epsilon_s \cdot p_i}{p_s \cdot p_i}$$

(Prehistory)
Weinberg 1964, 1965

Gauge invariance: Charge conservation

$$S^{(0)}_{\text{grav}}(s) = \sum_{i=1}^{N} \left(\kappa Q^i\right) \frac{\epsilon_s^{\mu\nu} p_{i\mu} p_{i\nu}}{p_s \cdot p_i}$$

Gauge invariance: Momentum conservation & universal coupling of gravity

$$\mathcal{A}(1,\ldots,N,s) \to \left[S^{(0)}(s) + S^{(2)}(s)\right] \mathcal{A}(1,\ldots,N)$$

$$S^{(2)}_{\text{gauge}}(s) = \sum_{i=1}^{N} \left(gT^i\right) \frac{\epsilon_s^\mu p_s^\nu J^i_{\mu\nu}}{p_s \cdot p_i}$$

Low 1958
Burnett, Kroll 1967

Gauge invariance: Anti-symmetry of angular momentum tensor

$$J^i_{\mu\nu} = p_{i[\mu} \frac{\partial}{\partial p_i^{\nu]}} + \Sigma^i_{\mu\nu}$$

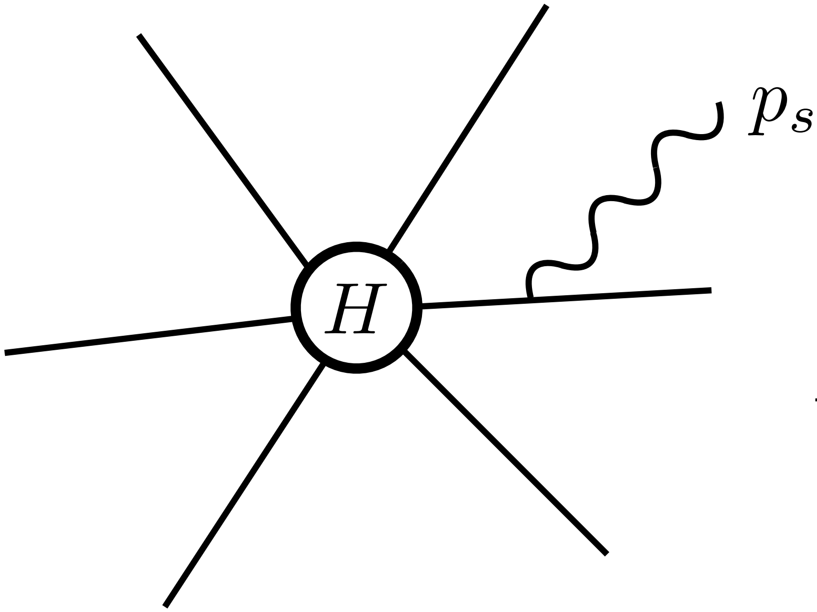
Proofs of Low-Burnett-Kroll at tree-level:

BCFW recursion relations — Casali, 2014

Conformal symmetry of tree-level 4D gauge theory amplitudes — AJL, 2014

Gauge and Lorentz invariance — Bern, Davies, Di Vecchia, Nohles, 2014

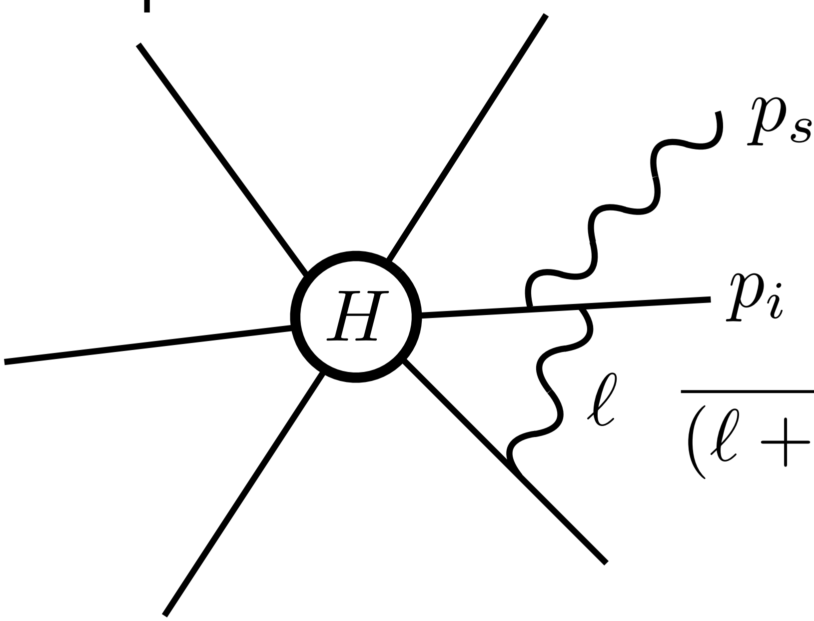
Tree-level:



$$\frac{p_s \cdot p_i}{(p_j + p_k)^2} \ll 1$$

$$\frac{1}{(p_H + p_s)^2} = \frac{1}{p_H^2} - \frac{2p_H \cdot p_s}{p_H^4} + \cdots$$

Loop-level:



$$\frac{p_s \cdot p_i}{(p_i + \ell)^2} \sim 1$$

$$\frac{1}{(\ell + p_i + p_s)^2} = \frac{1}{(\ell + p_i)^2 + 2(\ell + p_i) \cdot p_s}$$

loop-order λ order

$$\mathcal{A}^{[0]}(1,\ldots,N,s) \to \mathcal{A}^{[0](0)}(1,\ldots,N,s) \qquad (\sim \lambda^{-2})$$

$$+\mathcal{A}^{[0](1)}(1,\ldots,N,s) \qquad (\sim \lambda^{-1})$$

$$+\mathcal{A}^{[0](2)}(1,\ldots,N,s) \qquad (\sim \lambda^{0})$$

$$+\mathcal{O}(\lambda^1)$$

$p_s \sim Q\lambda^2$ "soft"

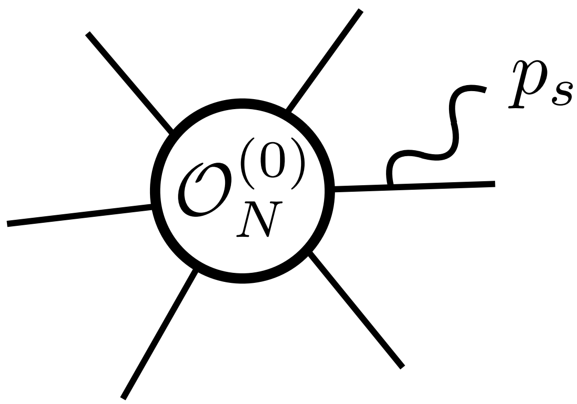
$p_c \sim Q(1,\lambda^2,\lambda)$ external collinear

$\mathcal{A}^{[0](0)}(1,\ldots,N,s)$:

Tree-level

$$\mathcal{A}(1,\ldots,N,s) \to S^{(0)}(s)\mathcal{A}(1,\ldots,N)$$

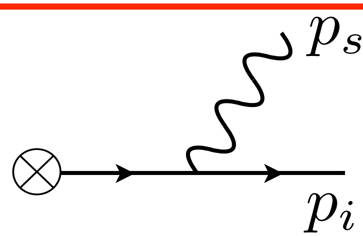
In SCET:



$$\langle 0|\mathcal{O}_N^{(0)}|p_1,\ldots,p_N\rangle = \mathcal{A}_N^{[0]} + \cdots$$

$$\langle 0|\mathcal{O}_N^{(0)}|p_1,\ldots,p_N,p_s\rangle = \langle 0|T\{\mathcal{O}_N^{(0)},\sum_i \mathcal{L}^{(0)}_{n_i,\mathrm{soft}}\}|p_1,\ldots,p_N,p_s\rangle_{\mathrm{int}} + \ldots$$

$$= S^{(0)}(s)\,\mathcal{A}_N^{[0]} + \ldots\ ,$$

$S^{(0)}(s) =$  $= \bar{u}(p_i)\cdot\left[-gT_i\,\frac{(p_i^- n_i)\cdot\epsilon_s}{(p_i^- n_i)\cdot p_s}\right]$

for fermions

$$\mathcal{A}^{[0](1)}(1,\ldots,N,s)\,,\ \mathcal{A}^{[0](2)}(1,\ldots,N,s)\,:$$

Tree-level

Power-count Low-Burnett-Kroll Operator

$$S^{(2)}_{\rm gauge}(s) = \sum_{i=1}^{N} \left(g T^i\right)\ \frac{\epsilon_s^\mu p_s^\nu J^i_{\mu\nu}}{p_s \cdot p_i}$$

Propagator factor:

$$p_i \cdot p_s = \underbrace{\frac{(\bar{n}\cdot p_i)(n\cdot p_s)}{2}}_{\sim\lambda^2} + \underbrace{p_{i\perp}\cdot p_{s\perp}}_{\sim\lambda^3} + \underbrace{\frac{(n\cdot p_i)(\bar{n}\cdot p_s)}{2}}_{\sim\lambda^4}$$

$$\frac{1}{p_i\cdot p_s} = \frac{2}{(\bar{n}\cdot p_i)(n\cdot p_s)} - \frac{4 p_{i\perp}\cdot p_{s\perp}}{(\bar{n}\cdot p_i)^2 (n\cdot p_s)^2} + \mathcal{O}(\lambda^0)$$

$$\mathcal{A}^{[0](1)}(1,\ldots,N,s)\,,\;\mathcal{A}^{[0](2)}(1,\ldots,N,s)\,:$$

Tree-level

Power-count Low-Burnett-Kroll Operator

$$S^{(2)}_{\rm gauge}(s) = \sum_{i=1}^{N} \left(gT^i\right)\,\frac{\epsilon_s^\mu p_s^\nu J^i_{\mu\nu}}{p_s\cdot p_i}$$

Angular momentum factor:

$$J_{i\mu\nu} = p_{i[\mu}\frac{\partial}{\partial p_i^{\nu]}} + \Sigma_{i\mu\nu}$$

$$\begin{aligned} J_{i\mu\nu} = &\left\{p_{i\perp[\mu}n_{\nu]}\frac{\partial}{\partial(n\cdot p_i)} + n_{[\mu}\frac{\bar{n}\cdot p_i}{2}\frac{\partial}{\partial p_{i\perp}^{\nu]}}\right\} \\ &+ \left\{p_{i\perp[\mu}\frac{\partial}{\partial p_{i\perp}^{\nu]}} + \bar{n}_{[\mu}n_{\nu]}\frac{n\cdot p_i}{2}\frac{\partial}{\partial(n\cdot p_i)} + n_{[\mu}\bar{n}_{\nu]}\frac{\bar{n}\cdot p_i}{2}\frac{\partial}{\partial(\bar{n}\cdot p_i)} + \Sigma_{i\mu\nu}\right\} \\ &+ \mathcal{O}(\lambda^1) \end{aligned}$$

$$\mathcal{A}^{[0](1)}(1,\ldots,N,s)\,,\ \mathcal{A}^{[0](2)}(1,\ldots,N,s)\,:$$

Tree-level

Power-count Low-Burnett-Kroll Operator

$$S^{(2)}_{\rm gauge}(s) = \sum_{i=1}^{N} (gT^i)\ \frac{\epsilon_s^\mu p_s^\nu J^i_{\mu\nu}}{p_s\cdot p_i}$$

Putting it together:

RPI choice: $p_{i\perp} = 0$

On-shell: $n\cdot p_i = 0$

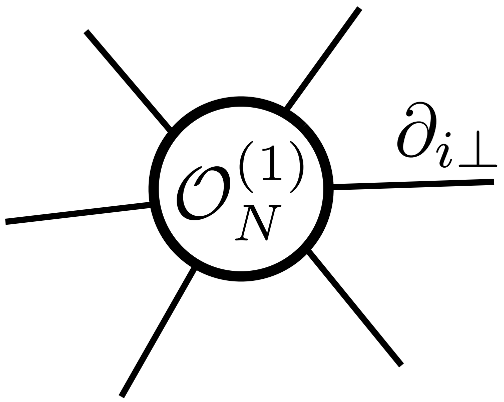
$$S^{(2)}_{\rm gauge}(s) \overset{\rm RPI}{\simeq} \sum_{i=1}^{N} (gT^i)\ \frac{2\epsilon_s^\mu p_s^\nu}{(n\cdot p_s)(\bar{n}\cdot p_i)}\left[n_{[\mu}\bar{n}_{\nu]}\,\frac{\bar{n}\cdot p_i}{2}\,\frac{\partial}{\partial(\bar{n}\cdot p_i)} + \Sigma^i_{\mu\nu}\right]$$

$(\sim\lambda^0)$

$$\mathcal{A}^{[0](1)}(1,\ldots,N,s) = \langle 0| T\{\mathcal{O}_N^{(1)}, \sum_i \mathcal{L}^{(0)}_{n_i,\mathrm{soft}}\} + T\{\mathcal{O}_N^{(0)}, \sum_i \mathcal{L}^{(1)}_{n_i,\mathrm{soft}}\} |p_1,\ldots,p_N,p_s\rangle$$

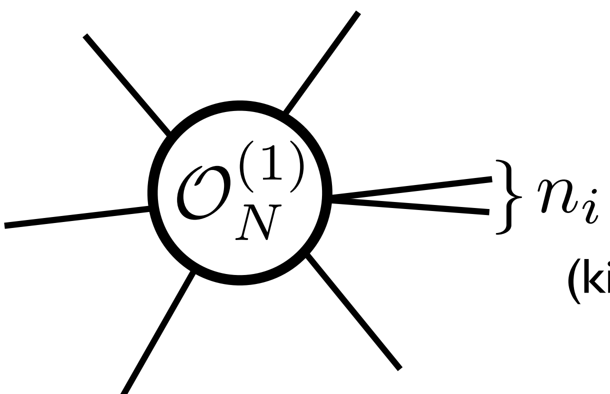
Tree-level

Operators:



$\simeq 0$

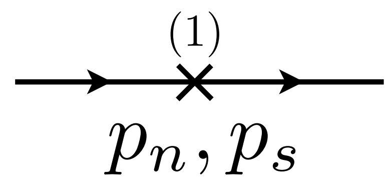
(RPI)



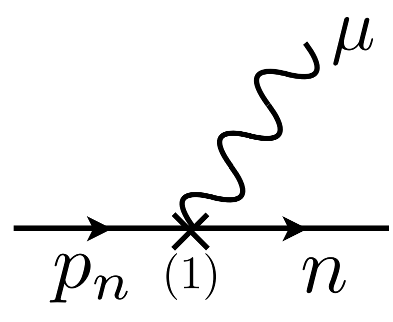
$\simeq 0$

(kinematics)

Lagrangians:



$$= i\frac{\bar{n}\!\!\!/}{2}\frac{2p_{n\perp}\cdot p_{s\perp}}{\bar{n}\cdot p_n}$$



$$= ig\frac{\bar{n}\!\!\!/}{2}\frac{2p^{\mu}_{n\perp}}{\bar{n}\cdot p_n}$$

No non-trivial contribution at λ^{-1}

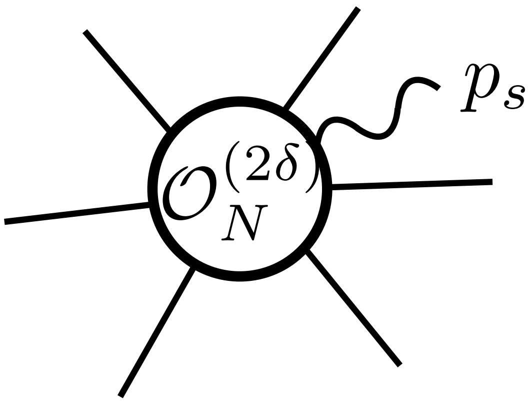
for fermions

$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) \simeq \langle 0| T\{\mathcal{O}_N^{(2)}, \sum_i \mathcal{L}^{(0)}_{n_i,\text{soft}}\}$$

Tree-level

$$+ T\{\mathcal{O}_N^{(0)}, \sum_i \mathcal{L}^{(2)}_{n_i,\text{soft}}\}|p_1,...,p_N,p_s\rangle$$

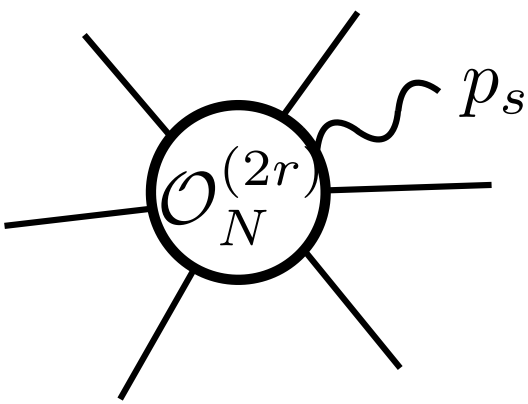
Operators:



Generated by RPI expansion of label δ-functions in $\mathcal{O}_N^{(0)}$

$$= -\sum_{k=1}^{N} \frac{\partial}{\partial \bar{n}_k \cdot Q_k} C_N^{(0)}(\{Q_i\}) \otimes \prod_{i=1}^{N} \Big[\delta(\bar{n}_i \cdot Q_i - \bar{n}\cdot i\partial_n) X^{\kappa_i}_{n_i}(0)\Big]$$

$$\otimes \,\mathrm{T}\Big\{\bar{n}_k \cdot g B_s^{(n_k)A} T^{\kappa_k A} \prod_{i=1}^{N} Y^{\kappa_i}_{n_i}(0)\Big\}$$



Generated by RPI expansion of collinear fields in $\mathcal{O}_N^{(0)}$

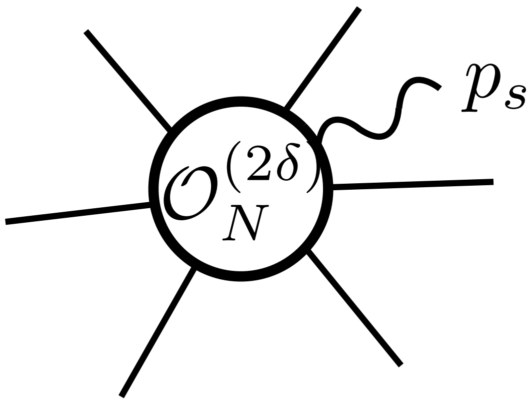
$$= C_N^{(0)}(\{Q_i\}) \otimes \sum_{k=1}^{N} \prod_{i=1, i\neq k}^{N} \Big[\delta(\bar{n}_i \cdot Q_i - \bar{n}\cdot i\partial_n) X^{\kappa_i}_{n_i}(0)\Big]$$

$$\times \Big[\delta(\bar{n}_k \cdot Q_k - \bar{n}\cdot i\partial_n) \frac{t_k^\mu}{\bar{n}_k \cdot Q_k} X^{\kappa_k}_{n_k}(0)\Big] \otimes \mathrm{T}\Big\{g B_{s\mu}^{(n_k)A} T^{\kappa_k A} \prod_{i=1}^{N} Y^{\kappa_i}_{n_i}(0)\Big\}$$

$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) \simeq \langle 0| T\{\mathcal{O}_N^{(2)}, \sum_i \mathcal{L}^{(0)}_{n_i,\mathrm{soft}}\} + T\{\mathcal{O}_N^{(0)}, \sum_i \mathcal{L}^{(2)}_{n_i,\mathrm{soft}}\} |p_1,\ldots,p_N,p_s\rangle$$

Tree-level

Operators:

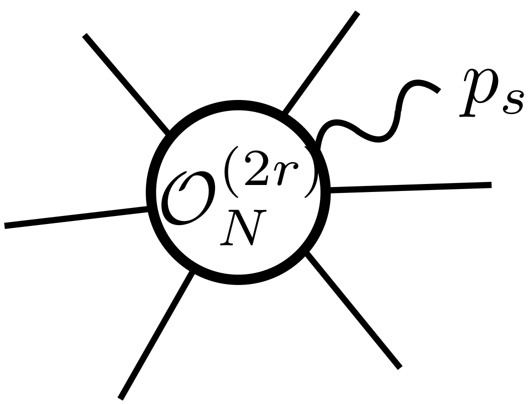


Generated by RPI expansion of label δ-functions in $\mathcal{O}_N^{(0)}$

$$= -\sum_{k=1}^{N} \frac{\partial}{\partial \bar{n}_k \cdot Q_k} C_N^{(0)}(\{Q_i\}) \otimes \prod_{i=1}^{N} \Big[\delta(\bar{n}_i\cdot Q_i - \bar{n}\cdot i\partial_n) X^{\kappa_i}_{n_i}(0)\Big]$$

orbital angular momentum

$$\otimes\, \mathrm{T}\Big\{\bar{n}_k \cdot g B_s^{(n_k)A} T^{\kappa_k A} \prod_{i=1}^{N} Y^{\kappa_i}_{n_i}(0)\Big\}$$



Generated by RPI expansion of collinear fields in $\mathcal{O}_N^{(0)}$

$$= C_N^{(0)}(\{Q_i\}) \otimes \sum_{k=1}^{N} \prod_{i=1, i\neq k}^{N} \Big[\delta(\bar{n}_i\cdot Q_i - \bar{n}\cdot i\partial_n) X^{\kappa_i}_{n_i}(0)\Big]$$

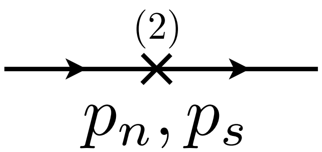
$$\times \Big[\delta(\bar{n}_k\cdot Q_k - \bar{n}\cdot i\partial_n) \frac{t_k^{\mu}}{\bar{n}_k\cdot Q_k} X^{\kappa_k}_{n_k}(0)\Big] \otimes \mathrm{T}\Big\{g B_{s\mu}^{(n_k)A} T^{\kappa_k A} \prod_{i=1}^{N} Y^{\kappa_i}_{n_i}(0)\Big\}$$

only non-zero for fermions

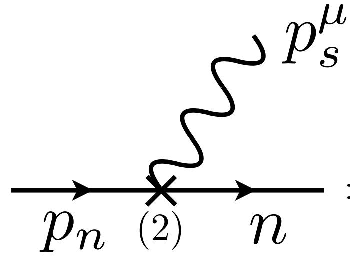
$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) \simeq \langle 0|T\{\mathcal{O}_N^{(2)}, \sum_i \mathcal{L}^{(0)}_{n_i,\mathrm{soft}}\} + T\{\mathcal{O}_N^{(0)}, \sum_i \mathcal{L}^{(2)}_{n_i,\mathrm{soft}}\}|p_1,\ldots,p_N,p_s\rangle$$

Tree-level

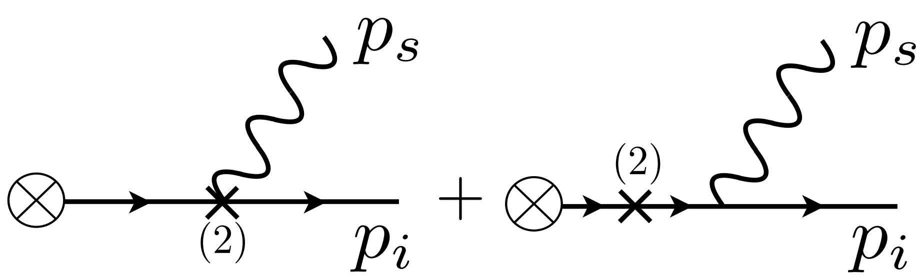
Lagrangians:



$$= i\frac{\bar{n}\!\!\!/}{2}\frac{p_{s\perp}^2}{\bar{n}\cdot p_n}$$



$$= ig\frac{\bar{n}\!\!\!/}{2}\frac{p_{s\perp}^\mu}{\bar{n}\cdot p_n} + ig\frac{\bar{n}\!\!\!/}{2}\frac{p_{s\perp\nu}}{\bar{n}\cdot p_n}\frac{1}{2}[\gamma_\perp^\nu,\gamma_\perp^\mu]$$



$$= \bar{u}(p_i)\cdot(-g)\frac{\epsilon_{s\mu}p_{s\nu}}{p_i^-(n_i\cdot p_s)}\left(p_{s\perp}^\mu\frac{n_i^\nu}{n_i\cdot p_s} - p_{s\perp}^\nu\frac{n_i^\mu}{n_i\cdot p_s} + \frac{1}{2}[\gamma_\perp^\nu,\gamma_\perp^\mu]\right)$$

for fermions

$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) \simeq \langle 0|T\{\mathcal{O}_N^{(2)}, \sum_i \mathcal{L}^{(0)}_{n_i,\text{soft}}\}$$ Tree-level

$$+\,T\{\mathcal{O}_N^{(0)}, \sum_i \mathcal{L}^{(2)}_{n_i,\text{soft}}\}|p_1,\ldots,p_N,p_s\rangle$$

Putting it all together:

$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) = S^{(2)}(s)\mathcal{A}(1,\cdots,N)$$

$$\mathcal{A}_N = \bar{u}(p_i)\tilde{\mathcal{A}}_N$$

for fermions

$$S^{(2)}_{i\psi}\mathcal{A}_N = g\frac{2\epsilon_{s\mu}p_{s\nu}}{(\bar{n}_i\cdot p_i)(n_i\cdot p_s)}\bar{u}(p_i)T_i\Big\{n_i^{[\mu}\bar{n}_i^{\nu]}\frac{\bar{n}_i\cdot p_i}{2}\frac{\partial}{\partial(\bar{n}_i\cdot p_i)} + \gamma_\perp^{[\mu}n_i^{\nu]}\frac{\bar{\not{n}}_i}{4} + p_{s\perp}^{[\mu}\frac{n_i^{\nu]}}{2(n_i\cdot p_s)} + \frac{1}{4}[\gamma_\perp^\mu,\gamma_\perp^\nu]\Big\}\tilde{\mathcal{A}}_N$$

$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) \simeq \langle 0|T\{\mathcal{O}_N^{(2)}, \sum_i \mathcal{L}^{(0)}_{n_i,\text{soft}}\}$$ Tree-level

$$+ T\{\mathcal{O}_N^{(0)}, \sum_i \mathcal{L}^{(2)}_{n_i,\text{soft}}\}|p_1,\ldots,p_N,p_s\rangle$$

Putting it all together:

$$\mathcal{A}^{[0](2)}(1,\ldots,N,s) = S^{(2)}(s)\mathcal{A}(1,\cdots,N)$$

$$\mathcal{A}_N = \bar{u}(p_i)\tilde{\mathcal{A}}_N$$

orbital angular momentum $\mathcal{O}_N^{(2\delta)}$ for fermions

$$S^{(2)}_{i\psi}\mathcal{A}_N = g\frac{2\epsilon_{s\mu}p_{s\nu}}{(\bar{n}_i\cdot p_i)(n_i\cdot p_s)}\bar{u}(p_i)T_i\Big\{n_i^{[\mu}\bar{n}_i^{\nu]}\frac{\bar{n}_i\cdot p_i}{2}\frac{\partial}{\partial(\bar{n}_i\cdot p_i)} + \gamma_\perp^{[\mu}n_i^{\nu]}\frac{\not{\bar{n}}_i}{4} + p_{s\perp}^{[\mu}\frac{n_i^{\nu]}}{2(n_i\cdot p_s)} + \frac{1}{4}[\gamma_\perp^\mu,\gamma_\perp^\nu]\Big\}\tilde{\mathcal{A}}_N$$

spin angular momentum

$\mathcal{O}_N^{(2r)}$ $\mathcal{L}^{(2)}$

RPI was necessary for universal factorized form!

One-loop soft theorem

$\mathcal{A}^{[1](0)}(1,\ldots,N,s)$:

Loop-level

$$\mathcal{A}^{[1](0)}_{N+1_s} = S^{[0](0)}(s)\,\mathcal{A}^{[1](0)}_{N} + S^{[1](0)}(s)\,\mathcal{A}^{[0](0)}_{N}$$

Universality of leading soft factor persists to one-loop

$\mathcal{A}^{[1](1)}(1,\ldots,N,s)$:

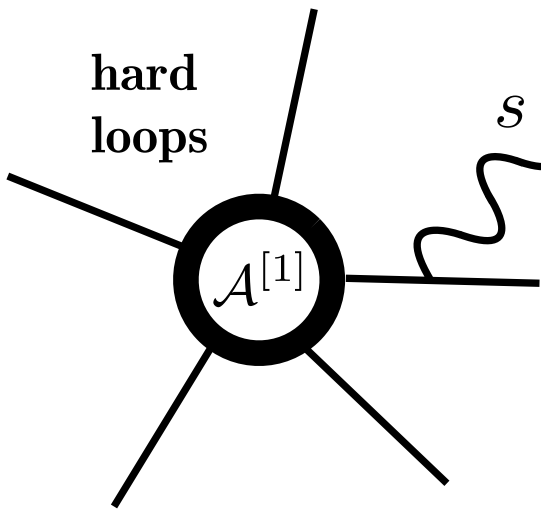
$$\mathcal{A}^{[1](1)}_{N+1_s} \simeq 0$$

All possible operator and Lagrangian contributions
can be set to zero by RPI

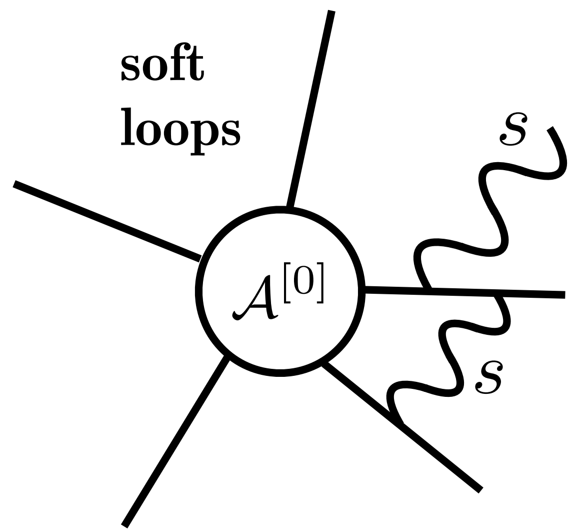
$$
\begin{aligned}
\mathcal{A}^{[1](2)}_{N+1_s} &= S^{[0](2)}(s)\mathcal{A}^{[1,\mathrm{hard}](0)}_N \\
&+ \mathcal{A}^{[0](0)}_N \mathcal{I}^{[0](2L)}_N\, S^{[1](0)}(s) \\
&+ \mathcal{A}^{[0](0)}_N \sum_{k=1}^{N} \int d^d x \left\{ \mathcal{I}^{[0](2L)k}_{N\,\mu}(x)\, \mathcal{E}^{[1]\mu\,\vec{\kappa}}_{s(n_k)}(x) + \mathcal{I}^{[0](2L)k}_{N\,\mu\nu}(x)\, \mathcal{E}^{[1]\mu\nu\,\vec{\kappa}}_{s(n_k)(n_k)}(x,x) \right\} \\
&+ \sum_{k=1}^{N} \left\{ -\frac{\partial \mathcal{A}^{[0](0)}_N}{\partial \bar{n}_k \cdot Q_k}\, \mathrm{Split}^{[0](0)}\, \mathcal{E}^{[1]\mu\,\vec{\kappa}}_{s(n_k)}(0)\, \bar{n}_{k\mu} + \mathcal{A}^{[0](0)}_N \mathcal{I}^{[0](0r)k}_{N\,\mu}(0)\, \mathcal{E}^{[1]\mu\,\vec{\kappa}}_{s(n_k)}(0) \right\} \\
&+ \sum_{k=1}^{N} \overline{\mathrm{Split}}^{[1](2)}(P_k \to k, s)\mathcal{A}^{[0](0)}_N(1,\ldots,P_k,\ldots,N) \\
&+ \sum_{\substack{k=1\\ l\neq k}}^{N} \overline{\mathrm{Split}}^{[0](2)}(P_k \to k, s)\, \mathrm{Split}^{[1](0)}(l \to l)\, \mathcal{A}^{[0](0)}_N(1,\ldots,l,\ldots,P_k,\ldots,N) \\
&+ \sum_{k=1}^{N} \left\{ -\frac{\partial \mathcal{A}^{[0](0)}_N}{\partial \bar{n}_k \cdot Q_k}\, \mathrm{Split}^{[1](0)}\, \mathcal{E}^{[0]\mu\,\vec{\kappa}}_{s(n_k)}(0)\, \bar{n}_{k\mu} + \mathcal{A}^{[0](0)}_N \mathcal{I}^{[1](0r)k}_{N\,\mu}\, \mathcal{E}^{[0]\mu\,\vec{\kappa}}_{s(n_k)}(0) \right\} \\
&+ \sum_{k=1}^{N} \left\{ \left(\mathcal{J}^{[1](2X_k L)}_N + \mathcal{J}^{[1](2X_k \partial)}_N \right) E^{[0]\vec{\kappa}}_{s[n_k]2} + \mathcal{J}^{[1](2X_k^2)}_N E^{[0]\vec{\kappa}}_{s[n_k]3} \right\} \\
&+ \sum_{k,k'=1}^{N} \int d^d x\, \mathcal{J}^{[1](2\,X_k L_{k'})\,\mu}_N(x)\, \mathcal{E}^{[0]\vec{\kappa}}_{s(n_{k'})[n_k]\mu}(x)
\end{aligned}
$$

$$\mathcal{A}^{[1](2)}_{N+1_s} = S^{[0](2)}(s)\mathcal{A}^{[1,\mathrm{hard}](0)}_{N}$$ (Low-Burnett-Kroll)

Loop-level



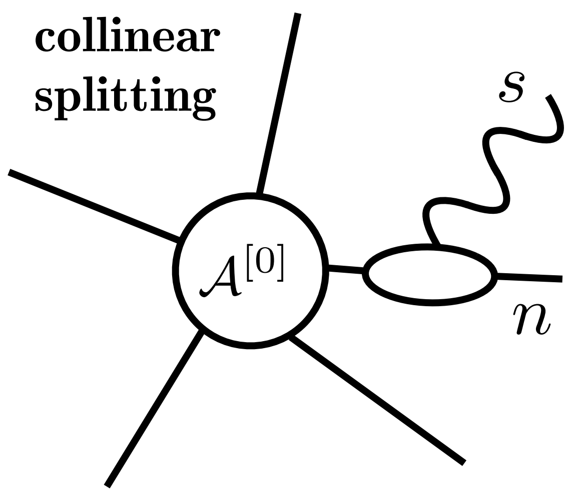
$$
\begin{aligned}
\mathcal{A}^{[1](2)}_{N+1_s} = \\
&+ \mathcal{A}^{[0](0)}_N \mathcal{I}^{[0](2L)}_N \, S^{[1](0)}(s) \\
&+ \mathcal{A}^{[0](0)}_N \sum_{k=1}^{N} \int d^d x \left\{ \mathcal{I}^{[0](2L)k}_{N\,\mu}(x)\, \mathcal{E}^{[1]\mu\,\vec{\kappa}}_{s(n_k)}(x) + \mathcal{I}^{[0](2L)k}_{N\,\mu\nu}(x)\, \mathcal{E}^{[1]\mu\nu\,\vec{\kappa}}_{s(n_k)(n_k)}(x,x) \right\} \\
&+ \sum_{k=1}^{N} \left\{ -\frac{\partial \mathcal{A}^{[0](0)}_N}{\partial \bar{n}_k \cdot Q_k} \,\mathrm{Split}^{[0](0)}\, \mathcal{E}^{[1]\mu\,\vec{\kappa}}_{s(n_k)}(0)\, \bar{n}_{k\mu} + \mathcal{A}^{[0](0)}_N \mathcal{I}^{[0](0r)k}_{N\,\mu}(0)\, \mathcal{E}^{[1]\mu\,\vec{\kappa}}_{s(n_k)}(0) \right\}
\end{aligned}
$$



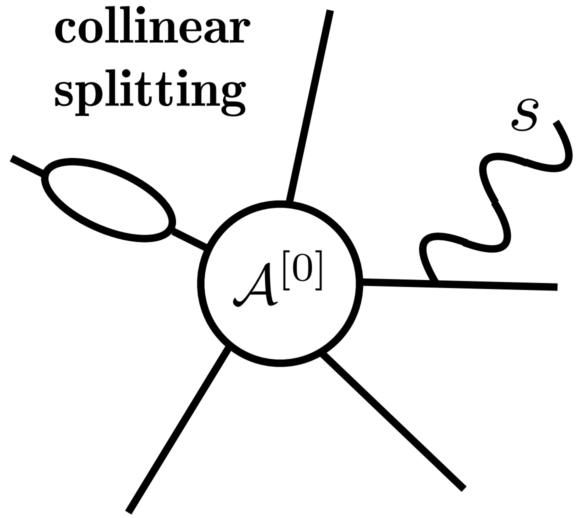
$$\mathcal{A}^{[1](2)}_{N+1_s} =$$

Loop-level

$$+\sum_{k=1}^{N} \overline{\text{Split}}^{[1](2)}(P_k \to k, s)\mathcal{A}^{[0](0)}_N(1, \ldots, P_k, \ldots, N)$$



$$\mathcal{A}^{[1](2)}_{N+1_s} =$$

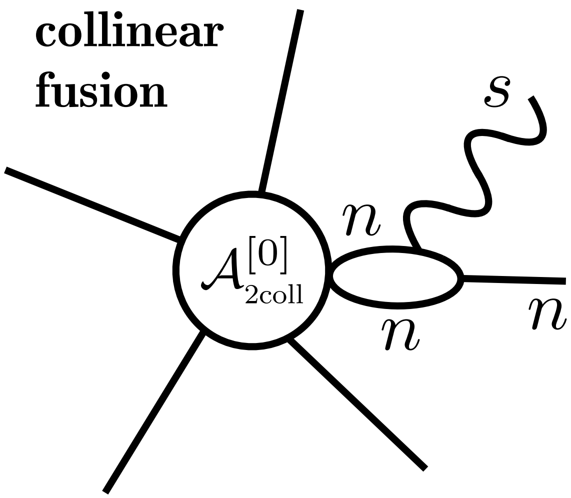


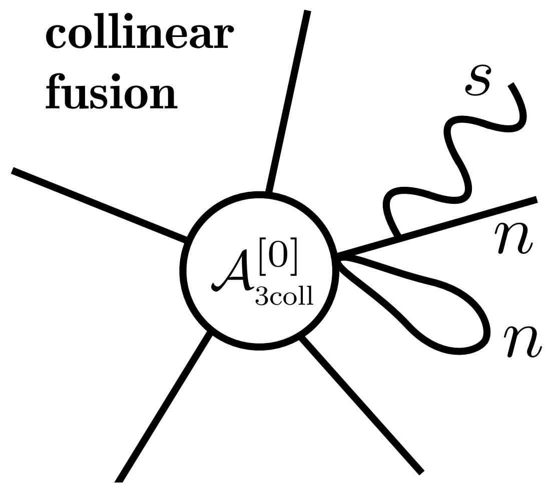
$$+\sum_{\substack{k=1\\ l\neq k}}^{N} \overline{\mathrm{Split}}^{[0](2)}(P_k \to k, s)\,\mathrm{Split}^{[1](0)}(l \to l)\,\mathcal{A}^{[0](0)}_{N}(1,\dots,l,\dots,P_k,\dots,N)$$

$$+\sum_{k=1}^{N}\left\{-\frac{\partial \mathcal{A}^{[0](0)}_{N}}{\partial \bar{n}_k \cdot Q_k}\,\mathrm{Split}^{[1](0)}\,\mathcal{E}^{[0]\mu\,\vec{\kappa}}_{s(n_k)}(0)\,\bar{n}_{k\mu} + \mathcal{A}^{[0](0)}_{N}\mathcal{I}^{[1](0r)k}_{N\,\mu}\,\mathcal{E}^{[0]\mu\,\vec{\kappa}}_{s(n_k)}(0)\right\}$$

$$\mathcal{A}^{[1](2)}_{N+1_s} =$$

Loop-level





$$+ \sum_{k=1}^{N} \left\{ \left(\mathcal{J}_N^{[1](2X_k L)} + \mathcal{J}_N^{[1](2X_k \partial)} \right) E^{[0]\vec{\kappa}}_{s[n_k]2} + \mathcal{J}_N^{[1](2X_k^2)} E^{[0]\vec{\kappa}}_{s[n_k]3} \right\}$$

$$+ \sum_{k,k'=1}^{N} \int d^d x \, \mathcal{J}_N^{[1](2\, X_k L_{k'})\, \mu}(x)\, \mathcal{E}^{[0]\vec{\kappa}}_{s(n_{k'})[n_k]\mu}(x)$$

Explicit example:

$$\mathcal{A}^{[1]}(1^-,2^+,3^+,4^+,5^+) = \frac{i}{48\pi^2}\frac{1}{\langle 34\rangle^2}\left(-\frac{\langle 13\rangle^3[32]\langle 42\rangle}{\langle 15\rangle\langle 54\rangle\langle 32\rangle^2} + \frac{\langle 14\rangle^3[45]\langle 35\rangle}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} - \frac{[25]^3}{[12][51]}\right)$$

Expand in the $p_5 \to 0$ limit:

$$\begin{aligned}\mathcal{A}^{[1]}(1^-,2^+,3^+,4^+,5_s^+) &= \left[\frac{\langle 41\rangle}{\langle 45\rangle\langle 51\rangle}\right]\left(\frac{i}{48\pi^2}\frac{\langle 13\rangle^3\langle 24\rangle[12]}{\langle 23\rangle^2\langle 34\rangle^3}\right)\\ &+\left[\frac{[52]}{\langle 51\rangle[12]}\right]\left(\frac{i}{48\pi^2}\frac{\langle 13\rangle^3\langle 24\rangle[12]}{\langle 23\rangle^2\langle 34\rangle^3}\right)\\ &+\left[\frac{-i}{48\pi^2}\frac{\langle 35\rangle[45]}{\langle 34\rangle\langle 45\rangle^2}\right]\left(\frac{\langle 14\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right) + \mathcal{O}(\lambda^1)\end{aligned}$$

$$\begin{aligned}\mathcal{A}^{[1]}(1^-,2^+,3^+,4^+,5_s^+) &= S^{[0](0)}(5^+)\mathcal{A}^{[1]}(1^-,2^+,3^+,4^+)\\ &+ S^{[0](2)}(5^+)\mathcal{A}^{[1]}(1^-,2^+,3^+,4^+)\\ &+ \mathrm{Split}^{[1](2)}(P^+\to 4^+,5^+)\mathcal{A}^{[0]}(1^-,2^+,3^+,P^-) + \mathcal{O}(\lambda^1)\end{aligned}$$

Conclusions

SCET is powerful for understanding soft expansion of fixed-order amplitudes

Low-Burnett-Kroll theorem violated at tree-level with collinear splittings
(see back-up for explicit calculation)

RPI $\simeq$ Möbius Group $\simeq$ Lorentz Transformation on the celestial sphere

SCET of gravity for understanding soft limits of gravity amplitudes?

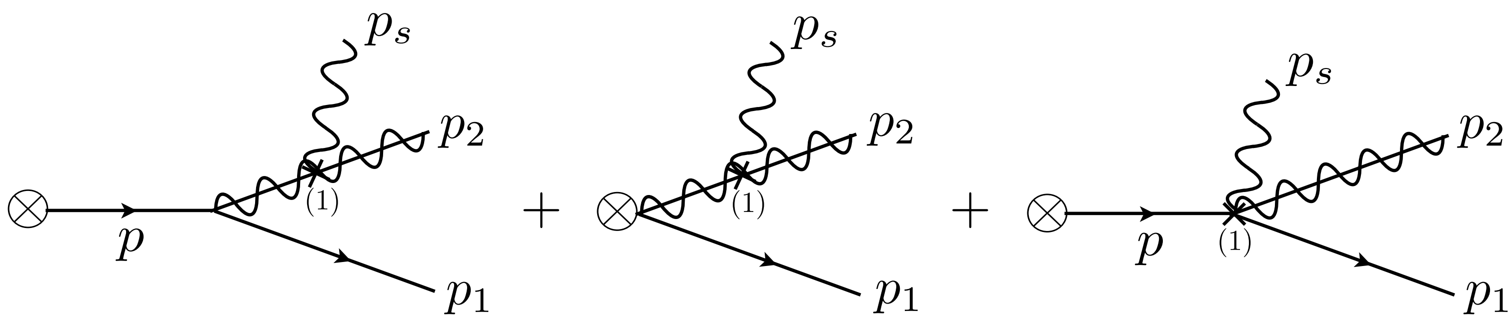
Beneke, Kirilin 2012

Bonus Slides

Tree-level collinear splitting

$$\mathcal{A}^{[0,\text{coll}](1)}_{N+1_s}(1_{\mathbin{/\!/}}, 2_{\mathbin{/\!/}}, 3, \ldots) = C^{[0](1X)}_{N} \left\langle \hat{\mathcal{O}}^{(1,X)}_{N-1} \right\rangle^{[0]}_{\text{coll}} E^{[0](N-1)\vec{\kappa}}_{s[n]2}$$
$$+ \overline{\text{Split}}^{[0](1)}(P \to 1, 2, s)\mathcal{A}^{[0]}(P, 3, \cdots, N)\,,$$

$$\overline{\text{Split}}^{[0](1)}(P \to 1, 2, s) =$$



$$= g^2\, \bar{u}(p_1) T^A T^B \left[\left(\frac{n \cdot \epsilon_2}{n \cdot p} - \frac{\bar{n} \cdot \epsilon_2}{\bar{n} \cdot p_2} + \frac{\not{p}_{1\perp} \not{\epsilon}_{2\perp}}{n \cdot p\, \bar{n} \cdot p_1} \right) 2p^{\rho}_{2\perp} + 2\epsilon^{\rho}_{2\perp} \frac{n \cdot p_s}{n \cdot p} \right.$$
$$\left. - \left(\not{\epsilon}_{2\perp} \frac{\bar{n} \cdot p_2}{\bar{n} \cdot p} + \not{p}_{1\perp} \frac{\bar{n} \cdot \epsilon_2}{\bar{n} \cdot p} \right) \frac{n \cdot p_s}{n \cdot p} \gamma^{\rho}_{\perp} \right] \frac{\epsilon^{\mu}_s\, p^{\nu}_s}{(\bar{n} \cdot p_2)(n \cdot p_s)} \left(g^{\perp}_{\mu\rho} \frac{n_\nu}{n \cdot p_s} - g^{\perp}_{\nu\rho} \frac{n_\mu}{n \cdot p_s} \right)$$