Removing Overlapping Phase-space Singularities

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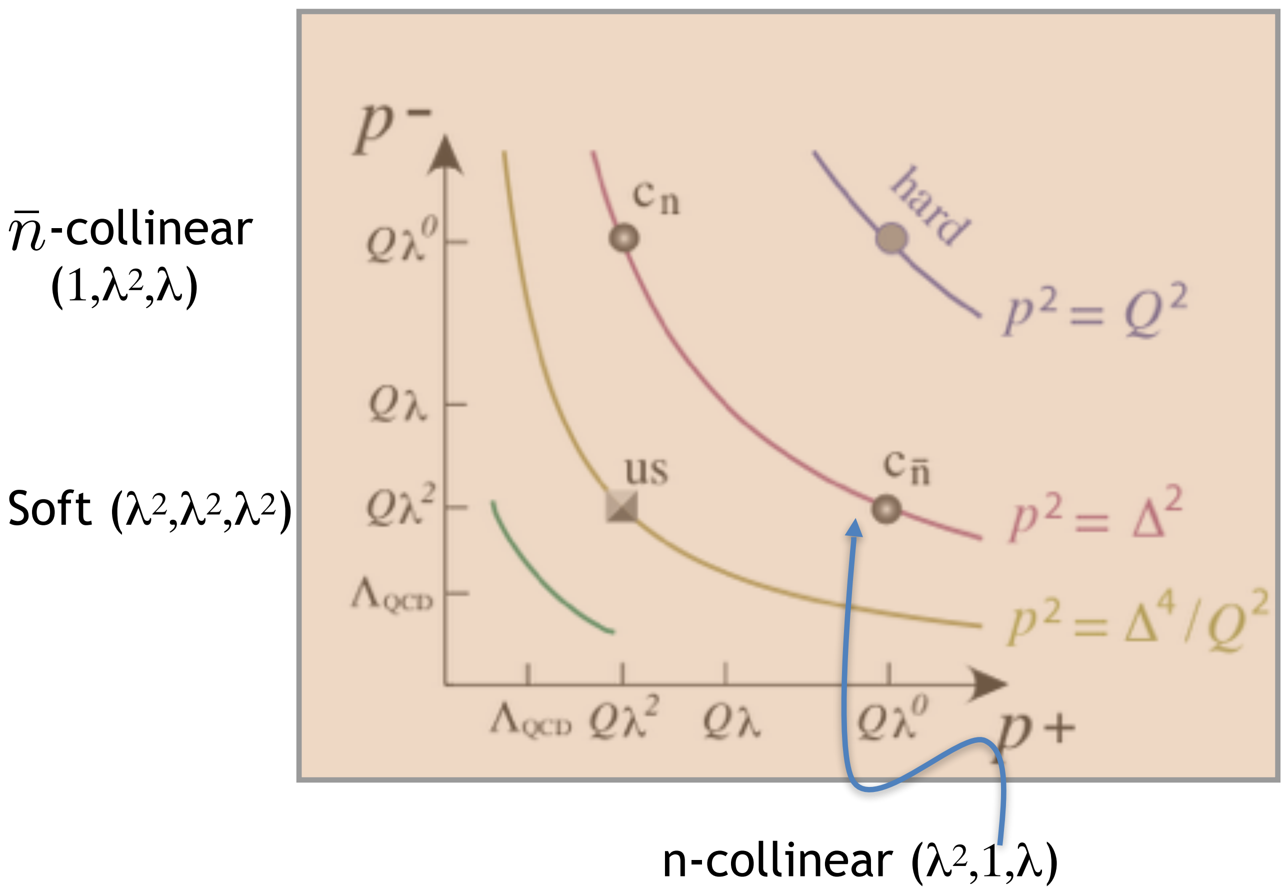
Based on arXiv:1502.05411
with Ilya Feige, Matthew Schwartz

Overview

Soft-collinear overlapping



- Operator-only SCET
  (Luke-Freeman, Feige-Schwartz)

No sum over labels, any regulators;
Overlap not yet addressed.

[Lee,Sterman,Mehen,Idibi]
shown equivalent

- Traditional QCD approach:

  soft collinear subtracted from the soft function with the eikonal jet function

- Label SCET

  excluding zero-bin from collinear integral: diagram by diagram subtraction.

- Method of regions

  when regulating via analytic continuation soft-collinear region does not contribute

Overview

- **Effective Theory Formulation**

Lagrangian is just multiple copies of QCD Lagrangian : $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{soft}} + \sum_{j=1}^{N} \mathcal{L}_j$

Operator-level matching in EFT at leading power in N-jet limit

$$\langle X_1 X_2; X_s | \bar{\psi}\gamma^\mu\psi |0\rangle \cong \mathcal{C}_2 \langle X_1 X_2; X_s | \frac{\bar{\psi}_1 W_1}{\text{tr}\langle 0| Y_1^\dagger W_1 |0\rangle / N_c} Y_1^\dagger \gamma^\mu Y_2 \frac{W_2^\dagger \psi_2}{\text{tr}\langle 0| W_2^\dagger Y_2 |0\rangle / N_c} |0\rangle_{\mathcal{L}_{\text{eff}}}$$

matching coefficient

provided use $\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{soft}} + \mathcal{L}_1 + \mathcal{L}_2$

The validity regime of matching is the N-jet regime.

Can we do inclusive phase-space integrals in each sector?

Amplitude-level factorization

Factorization with designated final state

external momenta prescribed to take collinear or soft scaling:

$|X_s\rangle = |k, \ldots\rangle$ with $k^\mu \sim \mathcal{O}(\lambda_s^2)\sqrt{s}$

$|X_j\rangle = |p, q, \ldots\rangle$ with $p \cdot q \sim \mathcal{O}(\lambda_c^2) s$

$$\langle X_1 \cdots X_N; X_s | \mathcal{O} |0\rangle \cong \mathcal{C}_{\mathcal{O}}(S_{ij}) \frac{\langle X_1 | \psi W_1 |0\rangle}{\langle 0| Y_1^\dagger W_1 |0\rangle} \cdots \frac{\langle X_N | W_N^\dagger \psi |0\rangle}{\langle 0| W_N^\dagger Y_N |0\rangle} \langle X_s | Y_1^\dagger \cdots Y_N |0\rangle$$

leading λ_s and λ_c

IR insensitive, does not depend on final state

contain only collinear singularities

contains all soft singularities

Factorization with designated final state

external momenta prescribed to
take collinear or soft scaling:

$$|X_s\rangle = |k, \ldots\rangle \text{ with } k^\mu \sim \mathcal{O}(\lambda_s^2)\sqrt{s}$$

$$|X_j\rangle = |p, q, \ldots\rangle \text{ with } p \cdot q \sim \mathcal{O}(\lambda_c^2)s$$

$$\langle X_1 \cdots X_N; X_s| \mathcal{O} |0\rangle \cong \mathcal{C}_\mathcal{O}(S_{ij}) \frac{\langle X_1| \bar{\psi} W_1 |0\rangle}{\langle 0| Y_1^\dagger W_1 |0\rangle} \cdots \frac{\langle X_N| W_N^\dagger \psi |0\rangle}{\langle 0| W_N^\dagger Y_N |0\rangle} \langle X_s| Y_1^\dagger \cdots Y_N |0\rangle$$

leading λ_s and λ_c

remove virtual soft/collinear overlapping

- Soft-collinear particles can be included in either soft or collinear sector. Factorization still works at leading power.
- subtraction is needed when phase space of different sectors overlap

Amplitude-level subtractions

- Construct an amplitude where q can go into arbitrary sectors

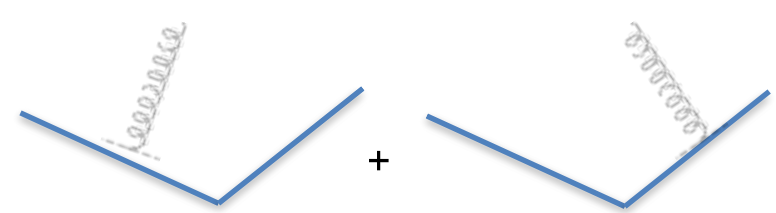
$$
\begin{aligned}
\mathcal{M}_{\rm sub}(p_1,p_2,q) \equiv &\left\{ \frac{\langle p_1; q|\, \bar\psi W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle} - \frac{\langle p_1|\, \bar\psi W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle}\frac{\langle q|\, Y_1^\dagger W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle} \right\} \frac{\langle p_2|\, W_2^\dagger \psi\,|0\rangle}{\langle 0|\, W_2^\dagger Y_2\,|0\rangle} \langle 0|\, Y_1^\dagger Y_2\,|0\rangle \\
&+ \frac{\langle p_1|\, \bar\psi W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle} \left\{ \frac{\langle p_2; q|\, \bar\psi W_2\,|0\rangle}{\langle 0|\, Y_2^\dagger W_2\,|0\rangle} - \frac{\langle p_2|\, \bar\psi W_2\,|0\rangle}{\langle 0|\, Y_2^\dagger W_2\,|0\rangle}\frac{\langle q|\, Y_2^\dagger W_2\,|0\rangle}{\langle 0|\, Y_2^\dagger W_2\,|0\rangle} \right\} \langle 0|\, Y_1^\dagger Y_2\,|0\rangle \\
&+ \frac{\langle p_1|\, \bar\psi W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle}\frac{\langle p_2|\, W_2^\dagger \psi\,|0\rangle}{\langle 0|\, W_2^\dagger Y_2\,|0\rangle} \langle q|\, Y_1^\dagger Y_2\,|0\rangle
\end{aligned}
$$

$$
\frac{\langle p_1; q|\, \bar\psi W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle} \overset{q\ \rm soft}{\cong} \frac{\langle p_1|\, \bar\psi W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle}\frac{\langle q|\, Y_1^\dagger W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle}
$$

$$
\langle q|\, Y_1^\dagger Y_2\,|0\rangle \overset{q\,\parallel\,p_1}{\cong} \frac{\langle q|\, Y_1^\dagger W_1\,|0\rangle}{\langle 0|\, Y_1^\dagger W_1\,|0\rangle} \times \langle 0|\, Y_1^\dagger Y_2\,|0\rangle
$$

cancel with the subtraction term in the top line

agrees with



= M_{QCD}

Amplitude-level subtractions

$$\mathcal{M}_{\rm sub}(X_1,\cdots,X_N,X_s;q_1,q_2) \equiv \frac{\langle X_1|\,\bar\psi W_1\,|0\rangle}{\langle 0|\,Y_1^\dagger W_1\,|0\rangle}\cdots\frac{\langle X_N|\,W_N^\dagger\psi\,|0\rangle}{\langle 0|\,W_N^\dagger Y_N\,|0\rangle}\langle X_s,q_1,q_2|\,Y_1^\dagger\cdots Y_N\,|0\rangle$$
$$+\sum_{i=1}^{N}\cdots\left\{\frac{\langle X_i,q_1|\,W_i^\dagger\psi\,|0\rangle}{\langle 0|\,W_i^\dagger Y_i\,|0\rangle}\right\}^{q_1}_{\substack{\rm soft\\ \rm sub}}\cdots\langle X_s,q_2|\,Y_1^\dagger\cdots Y_N\,|0\rangle$$
$$+\sum_{i=1}^{N}\cdots\left\{\frac{\langle X_i,q_2|\,W_i^\dagger\psi\,|0\rangle}{\langle 0|\,W_i^\dagger Y_i\,|0\rangle}\right\}^{q_2}_{\substack{\rm soft\\ \rm sub}}\cdots\langle X_s,q_1|\,Y_1^\dagger\cdots Y_N\,|0\rangle$$
$$+\sum_{i,j=1}^{N}\cdots\left\{\frac{\langle X_i,q_1|\,W_i^\dagger\psi\,|0\rangle}{\langle 0|\,W_i^\dagger Y_i\,|0\rangle}\right\}^{q_1}_{\substack{\rm soft\\ \rm sub}}\cdots\left\{\frac{\langle X_j,q_2|\,W_j^\dagger\psi\,|0\rangle}{\langle 0|\,W_j^\dagger Y_j\,|0\rangle}\right\}^{q_2}_{\substack{\rm soft\\ \rm sub}}\cdots\langle X_s|\,Y_1^\dagger\cdots Y_N\,|0\rangle$$

$\{\}^{q}_{\substack{\rm soft\\ \rm sub}}$: recursively defined operator-level soft subtraction

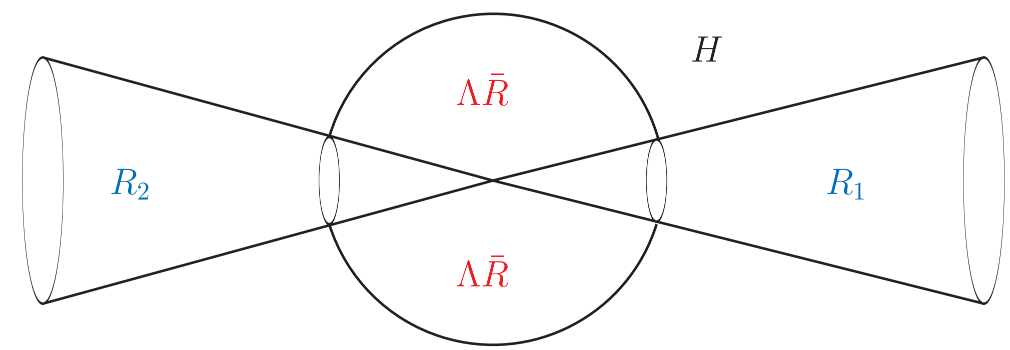
- agrees with full-QCD amplitude in any leading power IR limit
- can be integrated over all momentum region of q1, q2
- However, it is not a factorized formula!

Cross-section level factorization

- **Factorization with a hard-cutoff prescription**

  Λ : the size of the ball at the origin

  $R = \tan^2 \frac{\theta}{2}$, with θ the opening angle of the cone.



in cone

out of cone, in ball

$$\langle X_1, X_2; X_s |\, \bar\psi \gamma^\mu \psi \,|0\rangle \cong \mathcal{C}_2 \langle X_1, X_2; X_s |\, \frac{\bar\psi W_1}{\langle 0|Y_j^\dagger W_j|0\rangle} Y_1^\dagger \gamma^\mu Y_2 \frac{W_2^\dagger \psi}{\langle 0|W_2^\dagger Y_2|0\rangle} |0\rangle \quad (1+\mathcal{O}(\lambda_s,\lambda_c))$$

$$\frac{d\sigma}{d\tau} \cong H \times S^{\Lambda\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

up to $\mathcal{O}(R,\Lambda)$

$$\sum_{X_1} \int d\Pi_{X_1} \left| \frac{\langle X_1|\, \bar\psi W_1 \,|0\rangle}{\langle 0|\, Y_1^\dagger W_1 \,|0\rangle} \right|^2 \delta(\tau - p_{X_1}^+)$$

$$\sum_{X_s} \int d\Pi_{X_s} \left| \langle X_s |\, Y_1^\dagger \cdots Y_N \,|0\rangle \right|^2 \delta\Big(\tau - \frac{1}{2Q}\Omega_\tau(p_{X_s})\Big)$$

- **Problems with hard-cutoff prescription**

$$
\begin{aligned}
S^{\Lambda\overline{R}}(\tau) = \delta(\tau) + C_F\frac{\alpha_s}{\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\Bigg\{\delta(\tau)\left(-\frac{1}{\varepsilon^2}-\frac{7\pi^2}{12}+\frac{2}{\varepsilon}\ln\omega+2\ln\omega\ln R-2\ln^2\omega+\mathcal{O}(R)\right) \\
-\left[\frac{2}{\tau}\ln R\right]_+\Bigg\}\theta\left(\Lambda-\frac{\tau}{R}\right) \\
+C_F\frac{\alpha_s}{\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\Bigg\{\delta(\tau)\left[-\frac{1}{\varepsilon^2}-\frac{\pi^2}{4}-2\left(-\frac{1}{\varepsilon}\ln\omega+\ln\omega\ln\frac{\Lambda}{Q}+\frac{1}{2}\ln^2\omega\right)\right]-\left[\frac{2}{\tau}\ln\frac{\tau Q}{\Lambda}\right]_+\Bigg\}\theta\left(\frac{\tau}{R}-\Lambda\right)
\end{aligned}
$$

UV-IR poles

leading power dependence on the cut-offs

- Regularization scheme: offshellness (ω) + Δ regulator for IR + DR for UV

$$
\frac{n_j^\mu}{n_j\cdot k}\to\frac{p_j^\mu}{p_j\cdot k+\frac{Q^2\omega}{2}}
$$

$$
\frac{1}{t_j\cdot k}\to\frac{1}{t_j\cdot k+\frac{\Delta}{t_j\cdot p_j}}\equiv\frac{1}{t_j\cdot k+\delta_j(t_j\cdot p_j)}
$$

UV-IR poles

leading power dependence on the cut-offs

$$J^{R_j}(\tau) \cong \delta(\tau) + \frac{\alpha_s C_F}{2\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\left\{\delta(\tau)\left(\frac{2}{\varepsilon^2}+\frac{3}{2\varepsilon}+\frac{7}{2}+\frac{\pi^2}{6}\right) + \delta(\tau)\left(-\frac{2}{\varepsilon}\ln\omega - 2\ln\omega\ln R + 2\ln^2\omega + \mathcal{O}(R)\right) - \left(\frac{3}{2} - 2\ln R\right)\left[\frac{1}{\tau}\right]_+ - 2\left[\frac{\ln\tau}{\tau}\right]_+\right\}$$

$$S^{\Lambda\overline{R}} \otimes J^{R_1} \otimes J^{R_2} \cong \delta(\tau) + C_F\frac{\alpha_s}{\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\left\{\delta(\tau)\left(\frac{1}{\varepsilon^2}+\frac{3}{2\varepsilon}+\frac{7}{2}-\frac{5\pi^2}{12}+\mathcal{O}(R)\right) - \frac{3}{2}\left[\frac{1}{\tau}\right]_+ - 2\left[\frac{\ln\tau}{\tau}\right]_+\right\}$$

R dependence do not exactly cancel between sectors

- **The hard-cutoff prescription obscures factorization**

 cannot define factorized sectors that are both IR safe and independent on physical cut-offs.

does not hold exactly at leading power of \tau

- Steps to remove the cut-offs

1. $$\frac{d\sigma}{d\tau} \cong H \times S^{\Lambda\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

2. $$\frac{d\sigma}{d\tau} \cong H \times S^{\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

3. $$\frac{d\sigma}{d\tau} \otimes J^{R_1}_{\rm eik} \otimes J^{R_2}_{\rm eik} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$

4. $$\frac{d\sigma}{d\tau} \otimes J^{1}_{\rm eik} \otimes J^{2}_{\rm eik} \cong H \times S \otimes J^{1} \otimes J^{2}$$

5. $$\int d\tau \frac{d\sigma}{d\tau} e^{-\nu\tau} \cong H \frac{\widetilde{S}(\nu)\widetilde{J}^1(\nu)\widetilde{J}^2(\nu)}{\widetilde{J}^1_{\rm eik}(\nu)\widetilde{J}^2_{\rm eik}(\nu)}$$

- **Removing the cutoffs**

—Step 1, remove Λ.

taking lambda to infinity will not introduce new types of singularities

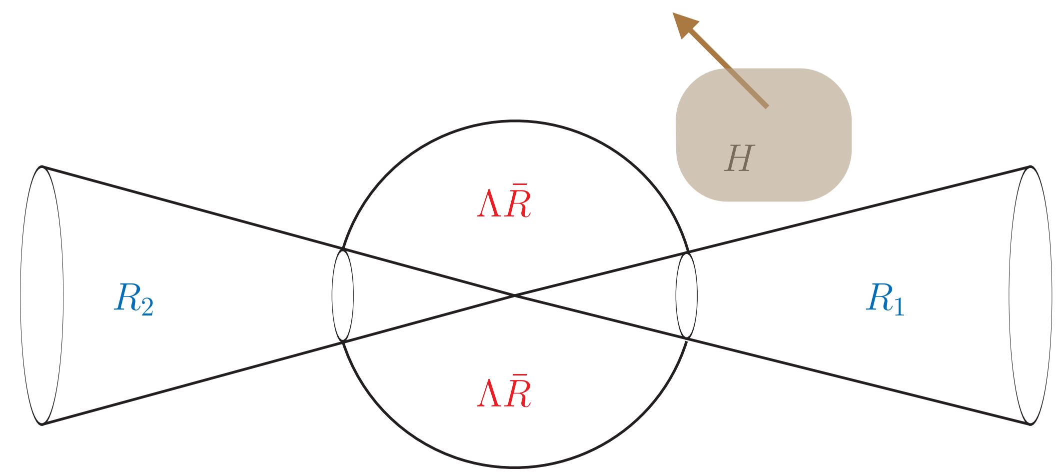
$$S^{\overline{R}} = S^{(\Lambda=\infty)\overline{R}} \quad \cong \quad S^{\Lambda\overline{R}}$$

leading τ

$$\frac{d\sigma}{d\tau} \cong H \times S^{\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

up to $\mathcal{O}(R)$

No soft or collinear



$$S^{\overline{R}}(\tau) = \delta(\tau) + C_F\frac{\alpha_s}{\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\left\{\delta(\tau)\left[-\frac{1}{\varepsilon^2} - \frac{7\pi^2}{12}\right]\right.$$
$$\left. - 2\delta(\tau)\left[-\frac{1}{\varepsilon}\ln\omega - \ln\omega\ln R + \ln^2\omega + \mathcal{O}(R)\right] - \left[\frac{2}{\tau}\ln R\right]_+\right\}$$

- **Removing cut-offs**

  — Step 2 , remove R in the soft sector

  - removing R introduces new IR singularities, which requires subtraction

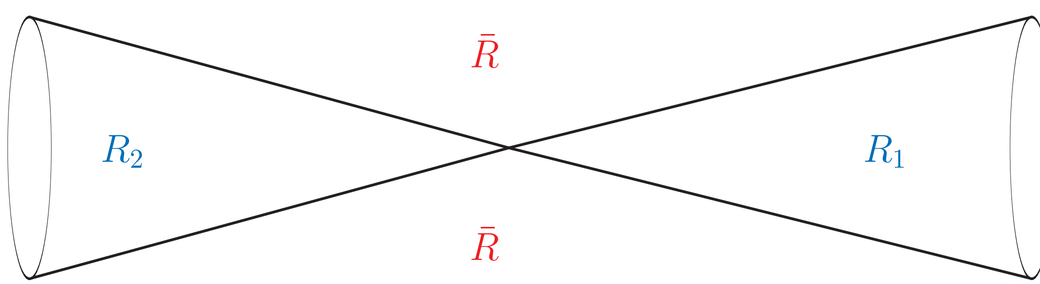
Applying factorization theorem, take $\mathcal{O} = Y_1^\dagger Y_2$

$$\langle X_1, X_2; X_s | Y_1^\dagger Y_2 |0\rangle \cong \langle X_1, X_2; X_s | \frac{Y_1^\dagger W_1}{\langle 0|Y_j^\dagger W_j|0\rangle} Y_1^\dagger Y_2 \frac{W_2^\dagger Y_2}{\langle 0|W_2^\dagger Y_2|0\rangle} |0\rangle \quad (1+\mathcal{O}(\lambda_c))$$

$$\mathcal{C}_{Y_1^\dagger Y_2} = 1$$

$$S \cong S^{\overline{R}} \otimes J_{\text{eik}}^{R_1} \otimes J_{\text{eik}}^{R_2}$$

up to $\mathcal{O}(R)$



$$\sum_{X_1} \int d\Pi_{X_1} \left| \frac{\langle X_1 | Y_1^\dagger W_1 |0\rangle}{\langle 0| Y_1^\dagger W_1 |0\rangle} \right|^2 \delta(\tau - p_{X_1}^+)$$

- **Removing the cutoffs**

  – Step 2 , remove R in the soft sector

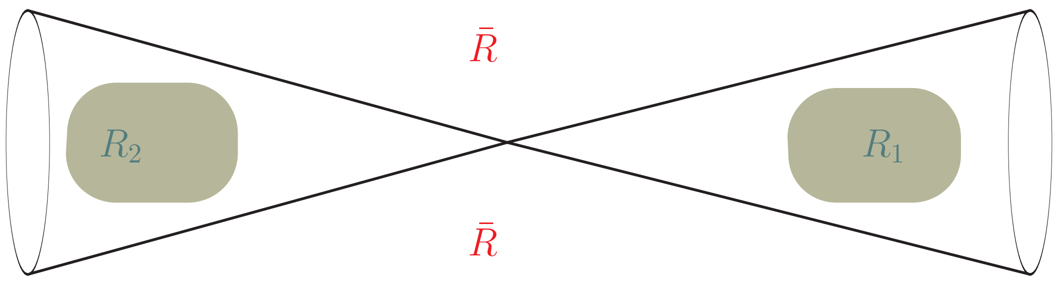
$$\frac{d\sigma}{d\tau} \cong H \times S^{\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

up to $\mathcal{O}(R)$

convolve with $J_{\text{eik}}^{R_j}$:

$$S \cong S^{\overline{R}} \otimes J_{\text{eik}}^{R_1} \otimes J_{\text{eik}}^{R_2}$$

$$\frac{d\sigma}{d\tau} \otimes J_{\text{eik}}^{R_1} \otimes J_{\text{eik}}^{R_2} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$



- **Removing the cutoffs**

— Step 3 , remove R in the collinear sector

Applying factorization theorem, take

$$\langle X_j ; X_s | \bar{\psi} W_j |0\rangle \cong \langle X_j ; X_s | \frac{\bar{\psi} W_j}{\langle 0 | Y_j^\dagger W_j | 0\rangle} Y_j^\dagger W_j |0\rangle \quad (1+\lambda_s)$$

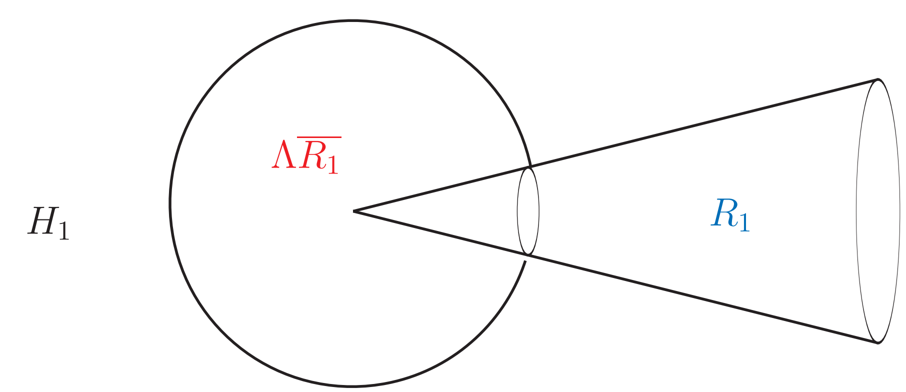
$$\mathcal{C}_{\bar{\psi} W} = 1$$

$$J^j \cong J^{R_j} \otimes J_{\mathrm{eik}}^{\Lambda \overline{R_j}}$$

up to $\mathcal{O}(\Lambda)$

leading τ

$$J^j \cong J^{R_j} \otimes J_{\mathrm{eik}}^{\overline{R_j}}$$



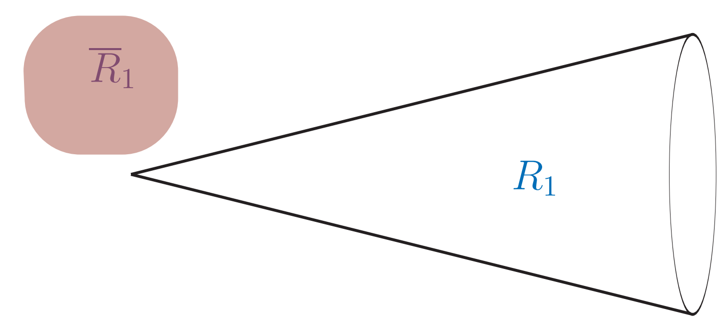
- **Removing the cutoffs**

  - Step 3 , remove R in the collinear sector

$$\frac{d\sigma}{d\tau} \otimes J_{\rm eik}^{R_1} \otimes J_{\rm eik}^{R_2} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$

convolve with $J_{\rm eik}^{\overline{R_j}}$:

$$J^j \cong J^{R_j} \otimes J_{\rm eik}^{\overline{R_j}}$$



$$\frac{d\sigma}{d\tau} \otimes J_{\rm eik}^{1} \otimes J_{\rm eik}^{2} \cong H \times S \otimes J^{1} \otimes J^{2}$$

- **Removing the cutoffs**

  - Step 3 , remove R in the collinear sector

R dependence match on both sides

$$\frac{d\sigma}{d\tau} \otimes J_{\text{eik}}^{R_1} \otimes J_{\text{eik}}^{R_2} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$

$$J^j \cong J^{R_j} \otimes J_{\text{eik}}^{\overline{R_j}}$$

hold exactly at leading τ

$$\frac{d\sigma}{d\tau} \otimes J_{\text{eik}}^{1} \otimes J_{\text{eik}}^{2} \cong H \times S \otimes J^{1} \otimes J^{2}$$

no R dependence on either side

$$\int d\tau \frac{d\sigma}{d\tau} e^{-\nu\tau} \cong H \frac{\widetilde{S}(\nu)\widetilde{J}^1(\nu)\widetilde{J}^2(\nu)}{\widetilde{J}_{\text{eik}}^1(\nu)\widetilde{J}_{\text{eik}}^2(\nu)}$$

Laplace transform

- Steps to remove the cut-offs

1. $$\frac{d\sigma}{d\tau} \cong H \times S^{\Lambda\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

2. $$\frac{d\sigma}{d\tau} \cong H \times S^{\overline{R}} \otimes J^{R_1} \otimes J^{R_2}$$

3. $$\frac{d\sigma}{d\tau} \otimes J_{\text{eik}}^{R_1} \otimes J_{\text{eik}}^{R_2} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$

4. $$\frac{d\sigma}{d\tau} \otimes J_{\text{eik}}^{1} \otimes J_{\text{eik}}^{2} \cong H \times S \otimes J^{1} \otimes J^{2}$$

5. $$\int d\tau \frac{d\sigma}{d\tau} e^{-\nu\tau} \cong H \, \frac{\widetilde{S}(\nu)\widetilde{J}^1(\nu)\widetilde{J}^2(\nu)}{\widetilde{J}_{\text{eik}}^1(\nu)\widetilde{J}_{\text{eik}}^2(\nu)}$$

Discussions

$$J^j(\tau) = \delta(\tau) + \frac{\alpha_s C_F}{2\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\left\{\delta(\tau)\left(\frac{2}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{7}{2} - \frac{\pi^2}{6}\right) + \delta(\tau)\left(-\frac{2}{\varepsilon}\ln\omega + 2\ln\omega\ln\delta_j + \ln^2\omega\right) - \left(2\ln\delta_j + \frac{3}{2}\right)\left[\frac{1}{\tau}\right]_+\right\}$$

UV-IR pole

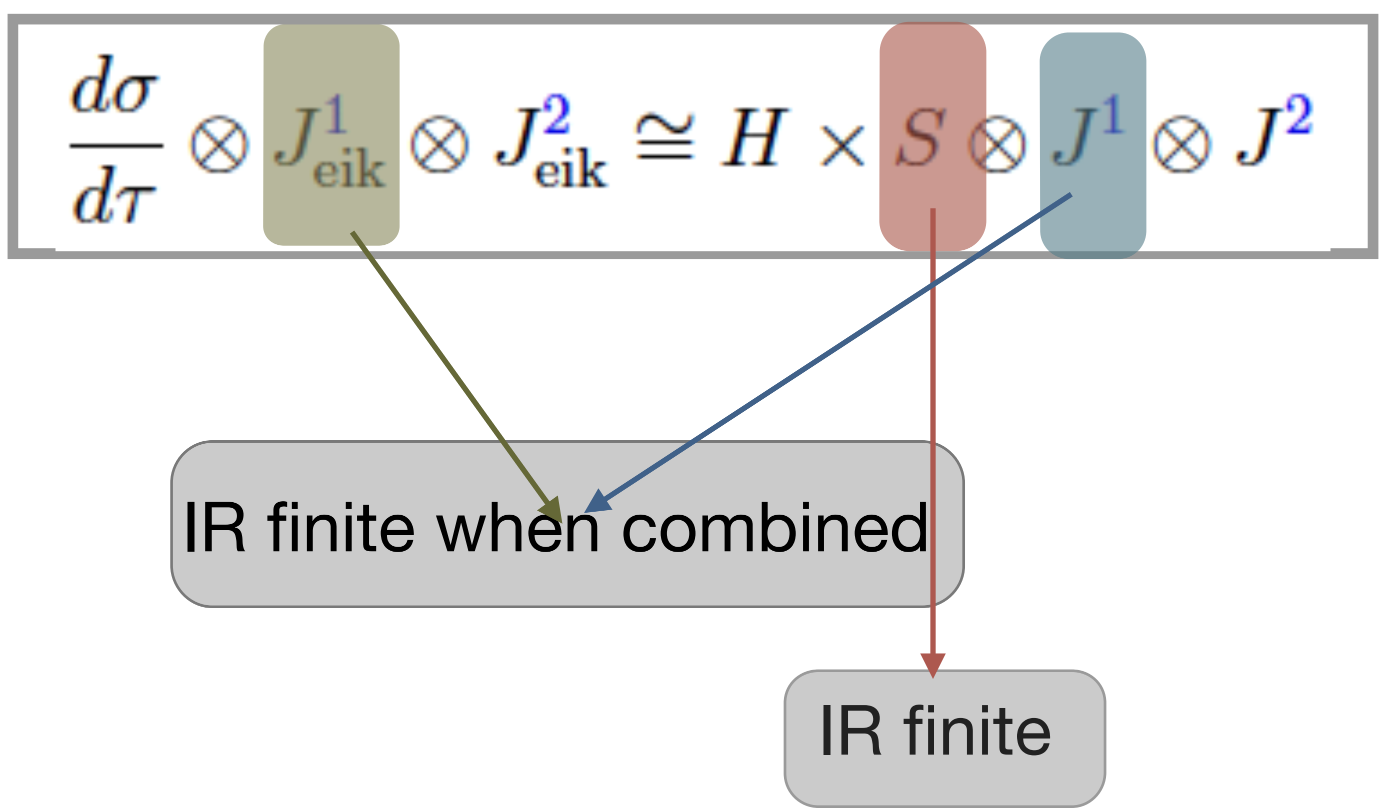
overlapping soft-collinear singularity

$$J^j_{\rm eik}(\tau) = \delta(\tau) + \frac{\alpha_s C_F}{2\pi}\left(\frac{\mu^2}{Q^2}\right)^{\varepsilon}\left\{\delta(\tau)\left[\frac{\pi^2}{3} - \frac{2}{\varepsilon}\ln\omega + 2\ln\omega\ln\delta_j + \ln^2\omega\right] + \left(\frac{2}{\varepsilon} - 2\ln\delta_j\right)\left[\frac{1}{\tau}\right]_+ - 2\left[\frac{\ln\tau}{\tau}\right]_+\right\}$$

Discussions

- Factorization with naive inclusive jet and soft functions over-counts the soft-collinear region and adds UV divergences to the phase-space integrals.

- The over-counting can be completely compensated by the eikonal jet functions.



Discussions

Generalization to other observables

our derivation is not restricted to observables whose measurement function is linear in each sector. For observables that do not satisfies linearity, integrals will not be a simple convolution.

Since the did not require $\lambda_s = \lambda_c$, if instead one take $\lambda_s^2 = \lambda_c = \lambda$ can be generalized to SCET II observables.

Discussions

- **R dependence / rapidity divergence**

$$\frac{d\sigma}{d\tau} \otimes J^{R_1}_{\text{eik}} \otimes J^{R_2}_{\text{eik}} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$

$$S \cong S^{\overline{R}} \otimes J^{R_1}_{\text{eik}} \otimes J^{R_2}_{\text{eik}}$$

R dependence match exactly at leading τ

Rapidity divergence in t_j direction cancel

$$\frac{d\sigma}{d\tau} \otimes J^{1}_{\text{eik}} \otimes J^{2}_{\text{eik}} \cong H \times S \otimes J^{1} \otimes J^{2}$$

Discussions

- **R dependence / rapidity divergence**

$$\frac{d\sigma}{d\tau} \otimes J_{\rm eik}^{R_1} \otimes J_{\rm eik}^{R_2} \cong H \times S \otimes J^{R_1} \otimes J^{R_2}$$

$$S \cong S^{\overline{R}} \otimes J_{\rm eik}^{R_1} \otimes J_{\rm eik}^{R_2}$$

R dependence match at leading τ, up to $\mathcal{O}(R)$

Rapidity divergence ($R \to 0$) cancel

$$\frac{d\sigma}{d\tau} \otimes J_{\rm eik}^{1} \otimes J_{\rm eik}^{2} \cong H \times S \otimes J^{1} \otimes J^{2}$$

Thank you