The 3-loop QCD cusp anomalous dimension

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based on
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with

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Outline

(1) Infrared divergences in gauge theories and Wilson lines

(2) Calculation of 3-loop integrals

(3) 3-loop result

(4) Conjectures

Reminder Wilson loops

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\int F_{\mu\nu}F^{\mu\nu}\,, \qquad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} + ig[A^{\mu}, A^{\nu}]$$

$$A^{\mu} = \sum_{a=1}^{N^2-1} A^{\mu}_{a}\, t^{a}_{ij}$$ gauge group SU(N)

Wilson loops:

required for gauge invariance of non-local objects

$\bar{\Psi}(x_2)$

$W_C[x_1, x_2]$

$\Psi(x_1)$

$$\mathcal{O} = \Psi(x_1)W_C[x_1, x_2]\bar{\Psi}(x_2)$$

$$W_C[x_1, x_2] = P e^{\int_C dx_\mu A^\mu}$$

P: path ordering

contain local operators

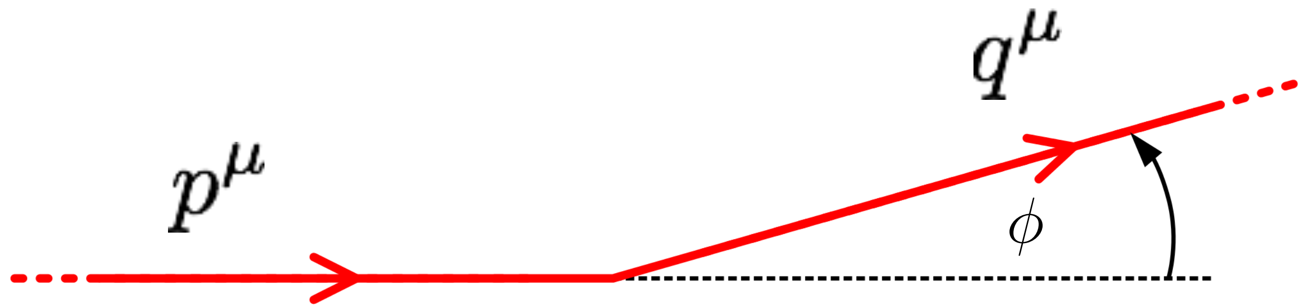
$$\sim 1 + \sigma^{\mu\nu}F_{\mu\nu} + \ldots$$

gauge dynamics - Wilson loops of arbitrary shapes

Cusp anomalous dimension

Definition: Wilson loop with cusp

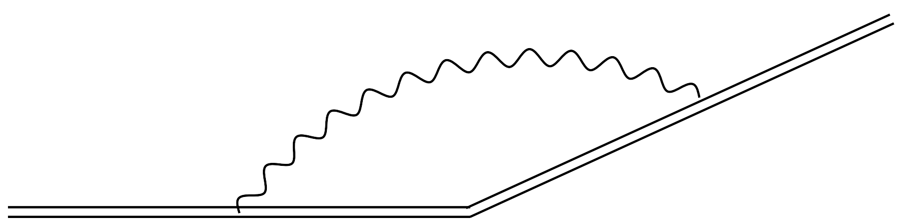
$$\cos(\phi) = \frac{p \cdot q}{\sqrt{p^2 q^2}}$$



$\Gamma_{\text{cusp}}(\phi)$ governs UV divergences at cusp

[Polyakov (1980);
Brandt, Sato, Neri (1981)]

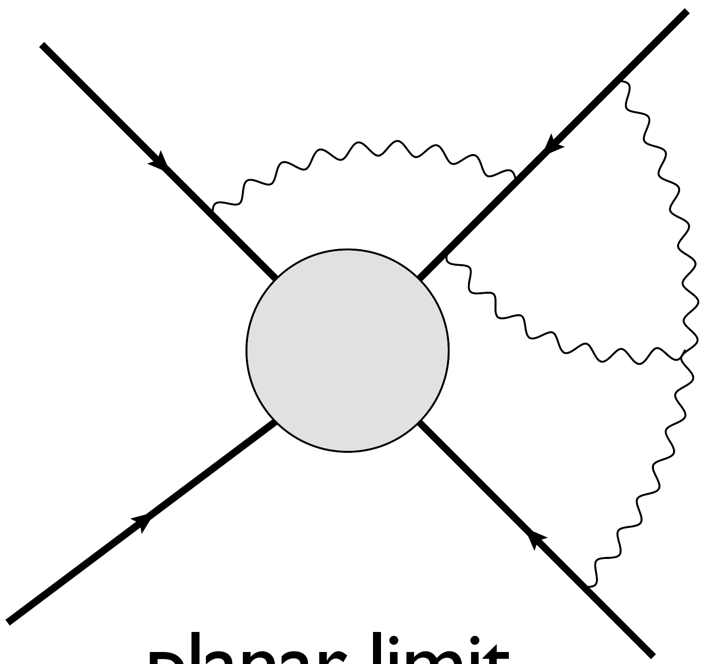
[Korchemsky, Radyushkin (1987)]



$$\langle W \rangle \sim e^{-|\ln \frac{\mu_{UV}}{\mu_{IR}}| \, \Gamma_{\text{cusp}}}$$

Physical relevance of $\Gamma_{\rm cusp}$

Describes infrared structure of scattering amplitudes



planar limit

$$\mathcal{A} \sim e^{-|\log \mu_{IR}|\, \Gamma_{\rm cusp}}$$

- predicts form of IR divergences of a generic 3-loop scattering amplitude

- can be used for resummation to increase theoretical precision, e.g. in top quark physics

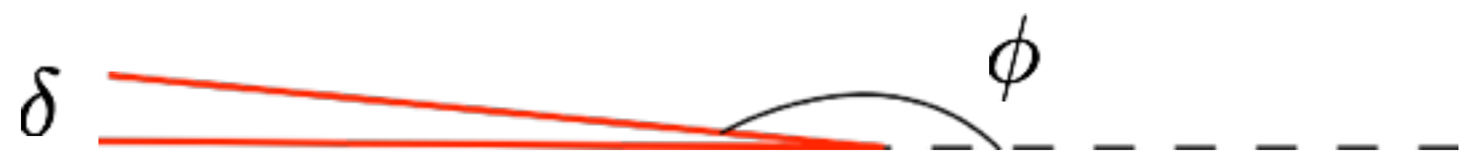
[see e.g. Czakon, Mitov, Sterman (2009)]

Limits and relations of $\Gamma_{\rm cusp}(\phi)$

- vanishes at zero angle (straight line) $\quad \Gamma_{\rm cusp}(\phi=0,\lambda) \;=\; 0$

- quark-antiquark potential $\quad \delta = \pi - \phi \qquad \delta \ll 1$

$$\Gamma_{\rm cusp} \sim \frac{C}{\delta}$$



[Kilian, Mannel, Ohl (1993)]

up to terms proportional to beta function

- light-like limit $\quad x = e^{i\phi} \qquad x \to 0$

$$\lim_{x\to 0} \Gamma_{\rm cusp} = -K \log x + \mathcal{O}(x^0)$$

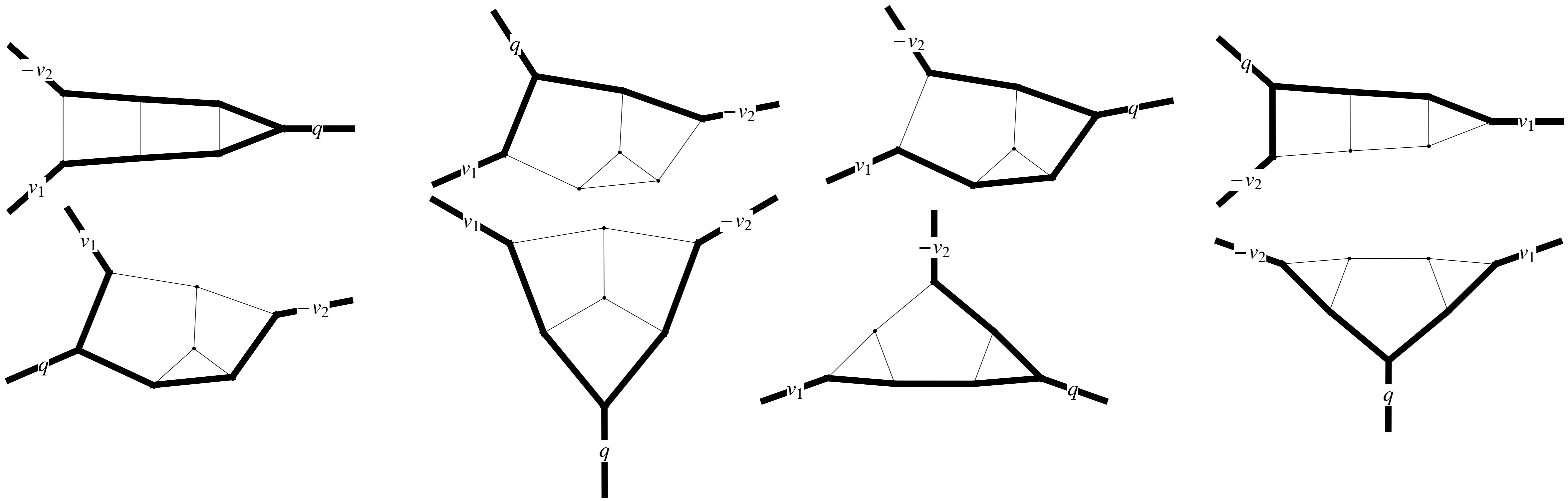
[Korchemsky (1989);
Korchemsky, Marchesini (1993)]

K light-like cusp anomalous dimension

also governs anomalous dimension of large spin operators

Master integrals

- abelian eikonal exponentiation: need only planar integrals



- 71 master integrals $\vec{f}(x;\epsilon)$ $D = 4 - 2\epsilon$ $x = e^{i\phi}$
- differential equations in suitable basis

[method: see JMH, PRL 110 (1013) 25]

$$\partial_x \vec{f}(x;\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+1} \right] \vec{f}(x;\epsilon)$$

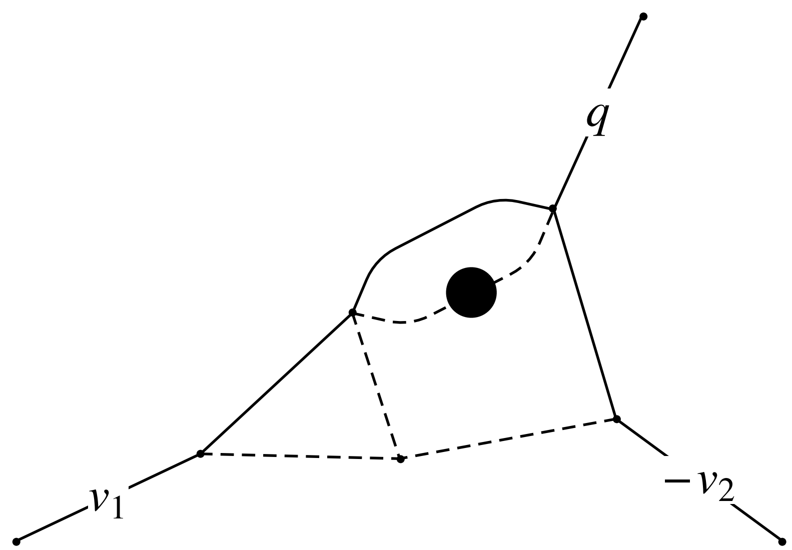
a, b, c constant 71x71 matrices

- boundary conditions trivially from $x = 1$

one integral: [Chetyrkin, Grozin, NP B666 (2003)]

- solution in terms of harmonic polylogarithms

Example



$$f_{44} = \epsilon^5 \frac{1-x^2}{x} G_{1,0,1,0,1,0,1,1,2,0,1,0}$$

$$x = e^{i\phi}$$

$$f_{44} = \epsilon^4 \Big[-\frac{1}{6}\pi^2 H_{0,0}(x) - \frac{2}{3}\pi^2 H_{1,0}(x) - 4H_{0,-1,0,0}(x) + 2H_{0,0,-1,0}(x) + 2H_{0,1,0,0}(x) - 4H_{1,0,0,0}(x) + 4\zeta_3 H_0(x) - \frac{17\pi^4}{360} \Big] + \mathcal{O}(\epsilon^5)$$

- all basis integrals are pure functions of uniform weight
- numerical checks with FIESTA
- confirmed previously known `N=4 SYM` result

Calculation at three loops

(1) compute proper vertex function

(2) take into account renormalization of Lagrangian

(3) compute vertex renormalization

(4) extract Gamma cusp $\Gamma_{\rm cusp} = \frac{\partial}{\partial \log \mu} \log Z$

Checks:

- expected divergence structure

$$\log Z = -\frac{1}{2\epsilon}\left(\frac{\alpha_s}{\pi}\right)\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{\beta_0}{16\epsilon^2}\Gamma^{(1)} - \frac{1}{4\epsilon}\Gamma^{(2)}\right] + \left(\frac{\alpha_s}{\pi}\right)^3\left[-\frac{\beta_0^2\Gamma^{(1)}}{96\epsilon^3} + \frac{\beta_1\Gamma^{(1)} + 4\beta_0\Gamma^{(2)}}{96\epsilon^2} - \frac{\Gamma^{(3)}}{6\epsilon}\right] .$$

- reproduce HQET wavefunction renormalization

[Grozin (2001), Chetyrkin, Grozin (2003)]

- dependence of gauge parameter disappears from $\Gamma_{\rm cusp}$

Result (1)

$$\Gamma_{\text{cusp}}(\alpha_s, x) = \sum_{k\geq 1} \left(\frac{\alpha_s}{\pi}\right)^k \Gamma^{(k)}_{\text{cusp}}(x) \qquad \alpha_s = \frac{g^2_{\text{YM}}}{4\pi}$$

One loop $\qquad \Gamma^{(1)}_{\text{cusp}} = C_F\, \tilde{A}_1$ [Polyakov (1980)]

Two loops [Korchemsky, Radyushkin (1987)][nf: Braun, Beneke, 1995]

$$\Gamma^{(2)}_{\text{cusp}} = \frac{1}{2} C_F C_A \left[\tilde{A}_3 + \tilde{A}_2\right] + \left(\frac{67}{36} C_F C_A - \frac{5}{9} C_F T_F n_f\right) \tilde{A}_1$$

with

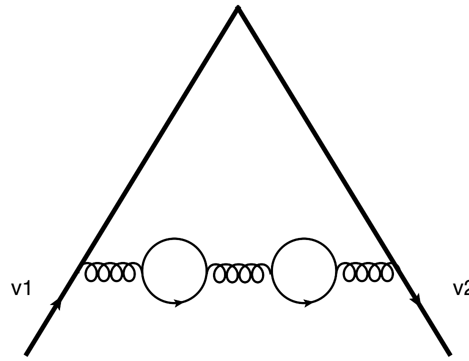
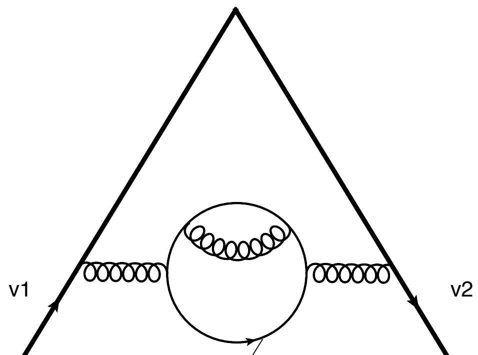
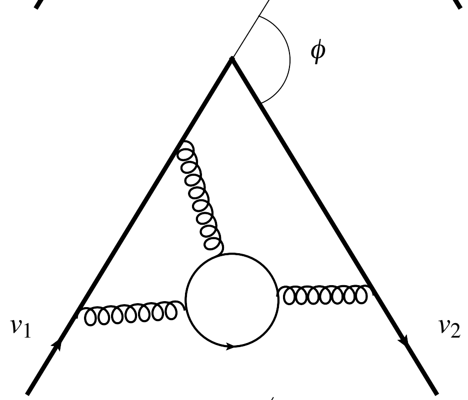
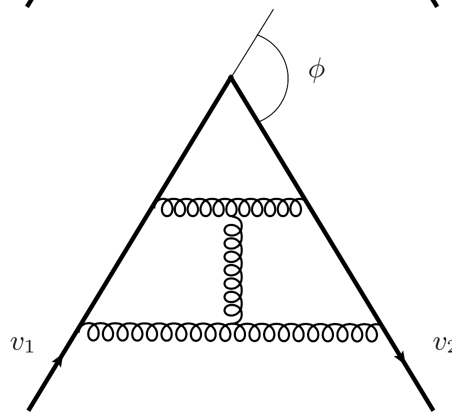
$$A_1(x) = \xi\, \frac{1}{2} H_1(y)\,, \qquad A_2(x) = \left[\frac{\pi^2}{3} + \frac{1}{2} H_{1,1}(y)\right] + \xi \left[-H_{0,1}(y) - \frac{1}{2} H_{1,1}(y)\right]\,,$$

$$A_3(x) = \xi \left[-\frac{\pi^2}{6} H_1(y) - \frac{1}{4} H_{1,1,1}(y)\right] + \xi^2 \left[\frac{1}{2} H_{1,0,1}(y) + \frac{1}{4} H_{1,1,1}(y)\right]\,,$$

and $\quad \xi = \dfrac{1+x^2}{1-x^2} \qquad y = 1 - x^2 \qquad \tilde{A} = A(x) - A(1)$

in agreement with the literature

Three-loop result

color structure	sample diagrams	contribution to $\Gamma_{\text{cusp}}^{(3)}$
$C_F (T_F n_f)^2$		$c_2 = -\frac{1}{27}\tilde{A}_1$ [Braun, Beneke, 1995]
$C_F^2 T_F n_f$		$c_3 = \left(\zeta_3 - \frac{55}{48}\right)\tilde{A}_1$
$C_F C_A T_F n_f$		$c_4 = -\frac{5}{9}\left[\tilde{A}_3 + \tilde{A}_2\right] - \frac{1}{6}\left(7\zeta_3 + \frac{209}{36}\right)\tilde{A}_1$
$C_F C_A^2$		$c_1 = \frac{1}{4}\left[\tilde{A}_5 + \tilde{A}_4 + \tilde{B}_5 + \tilde{B}_3\right] + \frac{67}{36}\tilde{A}_3 + \frac{29}{18}\tilde{A}_2 + \left(\frac{245}{96} + \frac{11}{24}\zeta_3\right)\tilde{A}_1$

Result (3)

$$A_1(x) = \xi\,\frac{1}{2}H_1(y)\,, \qquad A_2(x) = \left[\frac{\pi^2}{3} + \frac{1}{2}H_{1,1}(y)\right] + \xi\left[-H_{0,1}(y) - \frac{1}{2}H_{1,1}(y)\right]\,,$$

$$A_3(x) = \xi\left[-\frac{\pi^2}{6}H_1(y) - \frac{1}{4}H_{1,1,1}(y)\right] + \xi^2\left[\frac{1}{2}H_{1,0,1}(y) + \frac{1}{4}H_{1,1,1}(y)\right]\,,$$

$$\begin{aligned}A_4(x) = &\left[-\frac{\pi^2}{6}H_{1,1}(y) - \frac{1}{4}H_{1,1,1,1}(y)\right] + \\ &+ \xi\left[\frac{\pi^2}{3}H_{0,1}(y) + \frac{\pi^2}{6}H_{1,1}(y) + 2H_{1,1,0,1}(y) + \frac{3}{2}H_{0,1,1,1}(y) + \frac{7}{4}H_{1,1,1,1}(y) + 3\zeta_3 H_1(y)\right] \\ &+ \xi^2\left[-2H_{1,0,0,1}(y) - 2H_{0,1,0,1}(y) - 2H_{1,1,0,1}(y) - H_{1,0,1,1}(y) - H_{0,1,1,1}(y) - \frac{3}{2}H_{1,1,1,1}(y)\right]\,,\end{aligned}$$

$$\begin{aligned}A_5(x) = &\,\xi\left[\frac{\pi^4}{12}H_1(y) + \frac{\pi^2}{4}H_{1,1,1}(y) + \frac{5}{8}H_{1,1,1,1,1}(y)\right] + \xi^2\Bigg[-\frac{\pi^2}{6}H_{1,0,1}(y) - \frac{\pi^2}{3}H_{0,1,1}(y) - \frac{\pi^2}{4}H_{1,1,1}(y) \\ &-H_{1,1,1,0,1}(y) - \frac{3}{4}H_{1,0,1,1,1}(y) - H_{0,1,1,1,1}(y) - \frac{11}{8}H_{1,1,1,1,1}(y) - \frac{3}{2}\zeta_3 H_{1,1}(y)\Bigg] \\ &+ \xi^3\left[H_{1,1,0,0,1}(y) + H_{1,0,1,0,1}(y) + H_{1,1,1,0,1}(y) + \frac{1}{2}H_{1,1,0,1,1}(y) + \frac{1}{2}H_{1,0,1,1,1}(y) + \frac{3}{4}H_{1,1,1,1,1}(y)\right]\,,\end{aligned}$$

$$B_3(x) = \left[-H_{1,0,1}(y) + \frac{1}{2}H_{0,1,1}(y) - \frac{1}{4}H_{1,1,1}(y)\right] + \xi\left[2H_{0,0,1}(y) + H_{1,0,1}(y) + H_{0,1,1}(y) + \frac{1}{4}H_{1,1,1}(y)\right]\,,$$

$$\begin{aligned}B_5(x) = &\frac{x}{1-x^2}\Bigg[-\frac{\pi^4}{60}H_{-1}(x) - \frac{\pi^4}{60}H_1(x) - 4H_{-1,0,-1,0,0}(x) + 4H_{-1,0,1,0,0}(x) - 4H_{1,0,-1,0,0}(x) \\ &+ 4H_{1,0,1,0,0}(x) + 4H_{-1,0,0,0,0}(x) + 4H_{1,0,0,0,0}(x) + 2\zeta_3 H_{-1,0}(x) + 2\zeta_3 H_{1,0}(x)\Bigg]\,,\end{aligned}$$

Checks of result

- light-like limit $\Gamma_{\text{cusp}}(\alpha_s, x) \overset{x\to 0}{=} K(\alpha_s)\log(1/x) + \mathcal{O}(x^0)$

$$
\begin{aligned}
K^{(1)} &= C_F\,, \\
K^{(2)} &= C_A C_F \left(\frac{67}{36} - \frac{\pi^2}{12}\right) - \frac{5}{9} n_f T_f C_F\,, \\
K^{(3)} &= C_A^2 C_F \left(\frac{245}{96} - \frac{67\pi^2}{216} + \frac{11\pi^4}{720} + \frac{11}{24}\zeta_3\right) \\
&\quad + C_A C_F n_f T_f \left(-\frac{209}{216} + \frac{5\pi^2}{54} - \frac{7}{6}\zeta_3\right) \\
&\quad + C_F^2 n_f T_f \left(\zeta_3 - \frac{55}{48}\right) - \frac{1}{27} C_F (n_f T_f)^2
\end{aligned}
$$

in agreement with

[nf^2: Beneke, Braun (1995)]
[Vogt (2001); Berger (2002)]
[Moch, Vermeaseren, Vogt (2004)]

- quark-antiquark limit $\phi = \pi - \delta, \delta \to 0$

$$\Gamma_{\text{cusp}} \sim \frac{V}{\delta} + \mathcal{O}(\beta)$$

in agreement with [Peter (1997), Schroeder (1999)]

A theory independent observable?

- define 'effective coupling' $a := \pi / C_F K(\alpha_s)$

$$\Omega(a, x) := \Gamma_{\text{cusp}}(\alpha_s, x)$$

$$\Omega(a, x) \stackrel{x \to 0}{=} \frac{a}{\pi} C_F \log(1/x) + \mathcal{O}(x^0)\,,$$

- we find

$$\begin{aligned}\Omega(a,x) = \; & \frac{a}{\pi} C_F \tilde{A}_1 + \left(\frac{a}{\pi}\right)^2 \frac{C_A C_F}{2} \left[\tilde{A}_3 + \tilde{A}_2 + \frac{\pi^2}{6}\tilde{A}_1\right] \\ & + \left(\frac{a}{\pi}\right)^3 \frac{C_F C_A^2}{4} \Big[\tilde{A}_5 + \tilde{A}_4 - \tilde{A}_2 + \tilde{B}_5 + \tilde{B}_3 \\ & + \frac{\pi^2}{3}\tilde{A}_3 + \frac{\pi^2}{3}\tilde{A}_2 - \frac{\pi^4}{180}\tilde{A}_1\Big] + \mathcal{O}(a^4)\,. \end{aligned} \tag{17}$$

independent of nf to three loops!

Plot of 1,2,3-loop results

$$x = e^{-\theta}$$

nf-dependence enters only through effective coupling `a`

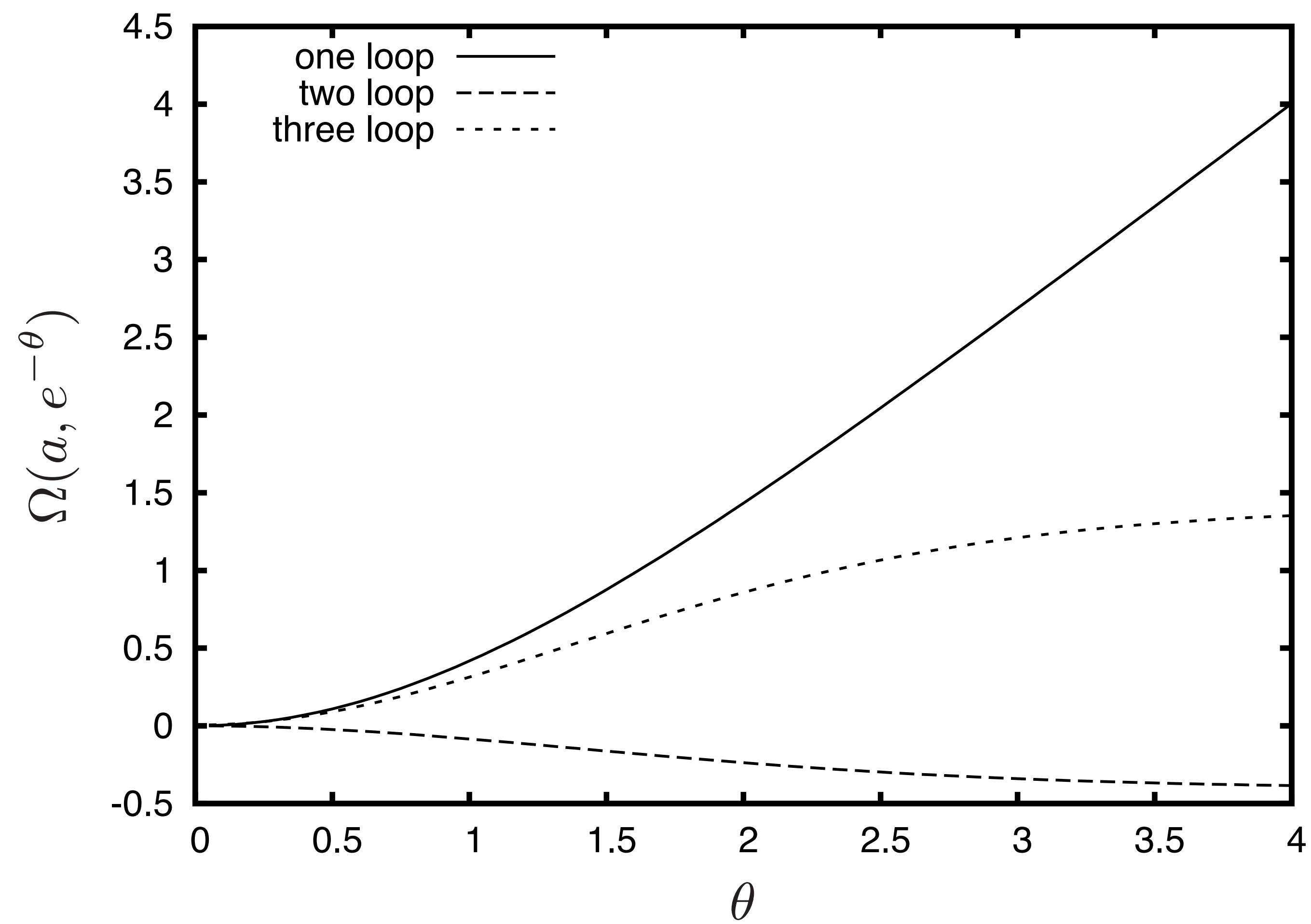


FIG. 2: θ dependence of the cusp anomalous dimension $\Omega(a, e^{-\theta})$ at one (solid), two (dashed), and three (dotted) loops.

slope at small angle

$$\Omega(a, e^{-\theta}) = C_F \left[\left(\frac{a}{\pi}\right) \frac{1}{3} + \left(\frac{a}{\pi}\right)^2 \frac{C_A}{4} \left(1 - \frac{\pi^2}{9}\right) + \left(\frac{a}{\pi}\right)^3 \frac{C_A^2}{12} \left(-\frac{5}{3} - \frac{\pi^2}{6} + \frac{\pi^4}{20} - \zeta_3\right) + \mathcal{O}(a^4) \right] \theta^2 + \mathcal{O}(\theta^4) \,. \tag{18}$$

Conjecture (1)

- assuming theory independence of Omega

allows to predict cusp anomalous dimension in N=4 SYM

we predict (in DRED bar scheme):

$$\Gamma_{\mathcal{N}=4}(\alpha_s,x) = \frac{\alpha_s}{\pi} C_F \tilde{A}_1 + \frac{C_A C_F}{2} \left(\frac{\alpha_s}{\pi}\right)^2 \left[\tilde{A}_3 + \tilde{A}_2\right] + \frac{C_F C_A^2}{4} \left(\frac{\alpha_s}{\pi}\right)^3 \left[\tilde{A}_5 + \tilde{A}_4 - \tilde{A}_2 + \tilde{B}_5 + \tilde{B}_3\right] + \mathcal{O}(\alpha_s^4)$$

test quark-antiquark limit

$$\Gamma_{\mathcal{N}=4}(\alpha_s,x) \overset{\delta\to 0}{=} -\frac{C_F \alpha_s}{\delta} \left\{1 - \left(\frac{\alpha_s}{\pi}\right) C_A + \left(\frac{\alpha_s}{\pi}\right)^2 C_A^2 \left[\frac{5}{4} + \frac{\pi^2}{4} - \frac{\pi^4}{64}\right] + \mathcal{O}(\alpha_s^3)\right\} + \mathcal{O}(\delta^0)$$

agrees perfectly with result from [Prausa, Steinhauser (2013)]

Conjecture (2)

- assume Omega stays nf-independent at four loops allows to predict e.g. first quartic Casimir terms

$$\Gamma_{\rm cusp}(\alpha_s, x) = \frac{1}{64} n_f \left(\frac{\alpha_s}{\pi}\right)^4 g(x) C_F C_4 + \dots$$

$$C_4 = d_F^{abcd} d_F^{abcd}/N_A = \frac{18 - 6N^2 + N^4}{96N^2}$$

- assumption implies

$$g(x) = g_0 \tilde{A}_1$$

- fix g_0 from known quark-antiquark limit [Smirnov^2, Steinhauser (2008)]

$$g_0 = -56.83(1)$$

- implies non-zero value for corresponding color structure in light-like cusp anomalous dimension!

Summary QCD cusp anomalous dimension

- full analytic 3-loop result
- predicts infrared divergences of all planar 3-loop scattering amplitudes
- can be used for resummation to improve theoretical predictions
e.g. in top quark physics

- nf-dependence very simple!
- leads to predictions/conjectures:
 -- e.g. N=4 SYM result (very similar to QCD answer!)
 -- quartic Casimir terms at 4 loops