Automated Calculations of Dijet Soft Functions

Guido Bell || Rudi Rahn || **Jim Talbert**

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Outline

1. ***Motivating automated soft functions***
   (a) Sudakov resummation in SCET
   (b) Log-counting and the need for two-loop soft functions
   (c) SCET_{I} vs. SCET_{II}

2. ***Universal soft functions at NLO***
   (d) Divergence structures and measurement functions
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Sudakov resummation in SCET

- Traditional approaches to resummation within QCD generically* achieve NLL accuracy ⇒ Process automated with *CAESAR (Banfi, Salam, Zanderighi* / 0407286)

- SCET allows for efficient, analytic resummations built on renormalization group techniques—higher logarithmic accuracy frequently achieved by the SCET community! Example calculations:

- **NNLL Resummations:**

  Jet Broadening: *Becher, Bell* / 1210.0580

  $t\bar{t}$ @ small p_T: *Li, Li, Shao, Yang, Zhu* / 1307.2464

  + ….

- **N^{3}LL Resummations:**

  Thrust: *Becher, Schwartz* / 0803.0342, *Abbate, Fickinger, Hoang, Mateu, Stewart* / 1006.3080

  W / Higgs @ large p_T: *Becher, Bell, Lorentzen, Marti / 1309.3245* / 1407.4111

  + ….

Resummation ingredients

- SCET permits the derivation of all-order factorization theorems:

$$d\sigma \sim H \cdot \mathcal{J} \otimes \mathcal{J} \otimes \mathcal{S}$$

- Once factorized, we resum logs using RG equations, e.g.:

$$\frac{dH(Q^2,\mu)}{d\ln\mu} = \left[2\Gamma_{cusp}\ln\left(\frac{Q^2}{\mu^2}\right) + 4\gamma_H(\alpha_s)\right] H(Q^2,\mu)$$

To achieve NNLL resummation, we need the soft anomalous dimension to two-loop accuracy

Logarithmic Accuracy	Γ_{Cusp}	$\gamma_H,\ \gamma_J,\ \gamma_S$	$C_H,\ C_J,\ C_S$
LL	1-loop	tree	tree
NLL	2-loop	1-loop	tree
NNLL	3-loop	2-loop	1-loop
N3LL	4-loop	3-loop	2-loop

SCET soft functions @ NNLO

- **e^+e^- observables:**

  Hemisphere masses: *Kelley, Schabinger, Schwartz, Zhu* / 1105.3676 & *Hornig, Lee, Stewart, Walsh, Zuberi* / 1105.4628

  Jet-mass w/ veto: *Kelley, Schwartz, Schabinger, Zhu* / 1112.3343

  Jet-broadening: *Becher, Bell* / 1210.0580

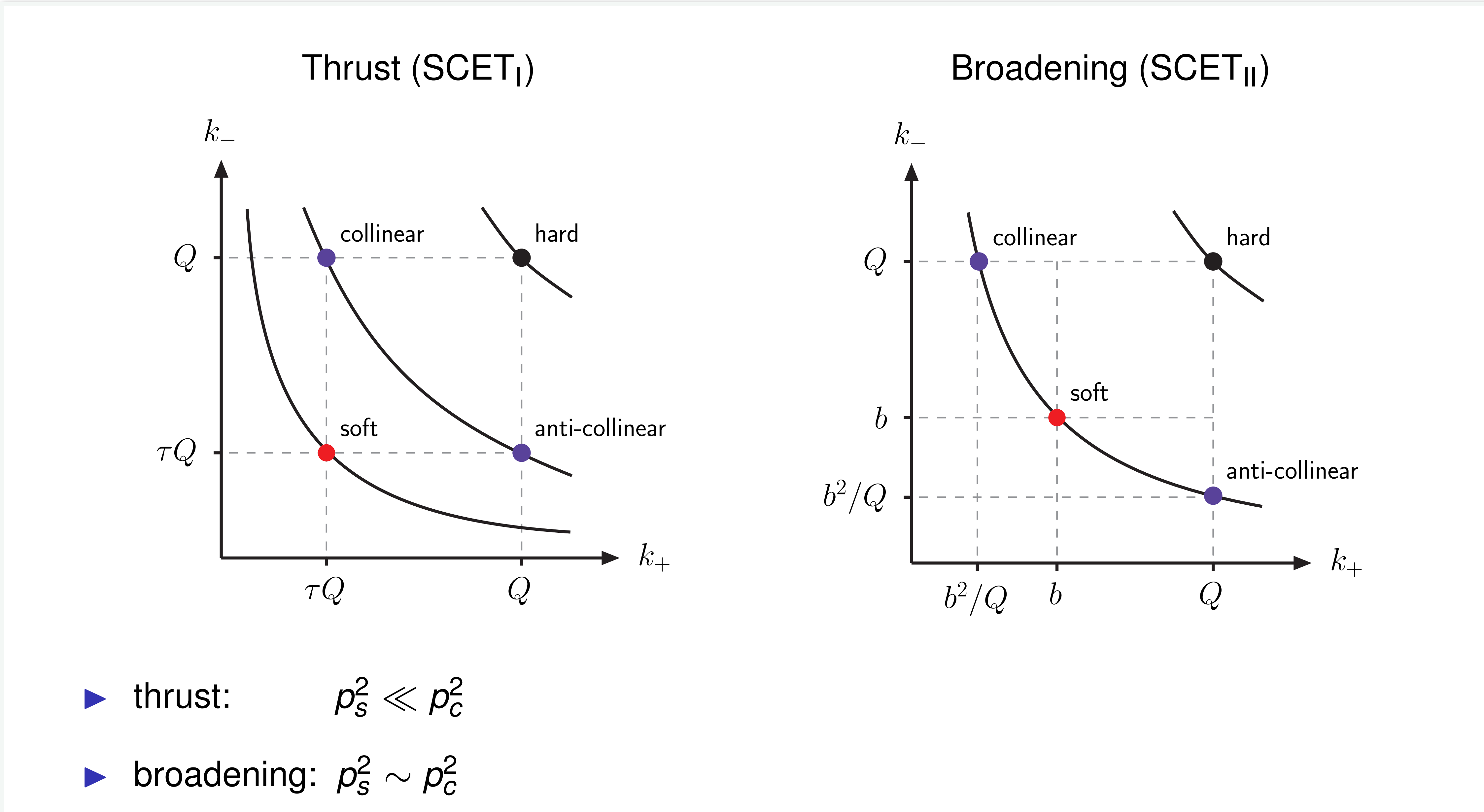
- **LHC observables:**

  Exclusive Drell-Yan: *Li, Mantry, Petriello* / 1105.5171

  W/Higgs @ large p_T: *Becher, Bell, Marti* / 1201.5572

Motivation: Can these computations be achieved more systematically?

SCET$_{\mathrm{I}}$ vs SCET$_{\mathrm{II}}$



- Additional rapidity regulator necessary for SCET$_{\mathrm{II}}$ observables
- We use phase-space regulator of *Becher, Bell* / 1112.3907

Universal dijet soft functions

- We can write down a universal dijet soft function as the vacuum matrix element of a product of Wilson lines along the direction of energetic quarks.

$$S(\omega,\mu) = \sum_{X,reg.} \mathcal{M}(\omega,\{k_i\})|\langle X|S_n^\dagger(0)S_{\bar{n}}(0)|0\rangle|^2 \qquad S_n(x) = Pexp(ig_s\int_{-\infty}^{0} n\cdot A_s(x+sn)ds)$$

- The **matrix element** of soft wilson lines is *independent of the observable*. It contains the universal (implicit) UV/IR-divergences of the function.
- The **measurement function** (M) encodes all of the information of the particular observable at hand. It is *independent of the singularity structure*. Take thrust as an example:

$$\mathcal{M}_{thrust}(\omega,\{k_i\}) = \delta(\omega - \sum_{i\in L} k_i^+ - \sum_{i\in R} k_i^-)$$

- **Idea**: isolate singularities at each order and calculate the associated coefficient numerically:

$$\bar{S}(\tau) \sim 1 + \alpha_s\{\frac{c_2}{\epsilon^2} + \frac{c_1}{\epsilon^1} + c_0\} + \mathcal{O}(\alpha_s^2)$$

- The coefficients depend on the observable. We are working in Laplace Space.

Universal soft functions: NLO

- We work in **Laplace space**, so that our functions are not distribution valued. At 1-loop the virtual corrections are scaleless in DR and we can write the NLO soft function as:

$$\bar{S}^{(1)}(\tau,\mu) = \frac{\mu^{2\epsilon}}{(2\pi)^{d-1}} \int \delta(k^2)\ \theta(k^0)\ \mathcal{R}_\alpha(\nu; k_+, k_-)\ \left(\frac{16\pi\alpha_s C_F}{k_+ k_-}\right)\ \bar{\mathcal{M}}(\tau,k)\ d^d k$$

- Where we use a symmetric version of the analytic SCET_{II} regulator (*Becher, Bell* / 1112.3907):

$$\mathcal{R}_\alpha(\nu, k_+, k_-) = \theta(k_- - k_+)(\nu/k_-)^\alpha + \theta(k_+ - k_-)(\nu/k_+)^\alpha$$

- We want to disentangle all of the UV and IR divergences. We thus split the integration region into two hemispheres and make the following physical substitutions:

$$k_- \to \frac{k_T}{\sqrt{y}} \qquad k_+ \to k_T\sqrt{y}$$

- We can now specify the measurement function *M*. We assume it can be written in terms of two dimensionless functions f & g:

$$\bar{\mathcal{M}}(\tau,k) = g(\tau k_T, y, \theta)\ exp(-\tau k_T f(y,\theta))$$

Universal soft functions: examples

$$\bar{\mathcal{M}}(\tau,k) = g(\tau k_T, y, \theta)\ exp(-\tau k_T f(y,\theta))$$

Obs.	$g(\tau k_T, y, \theta)$	$f(y,\theta)$
Thrust	1	$\sqrt{y}$
Angularities	1	$y^{(1-A)/2}$
C-Parameter	1	$\sqrt{y}/(1+y)$
Broadening	$\Gamma(1-\epsilon)\left(\frac{z\tau k_T}{4}\right)^{\epsilon}\mathcal{J}_{-\epsilon}\left(\frac{z\tau k_T}{2}\right)$	$1/2$
W/H @ large p_T	1	$\frac{1+y-2\sqrt{y}\cos\theta}{\sqrt{y}}$
Transverse Thrust	1	$\frac{1}{\lvert s\rvert}\{\sqrt{1+\frac{1}{4}\left(\frac{1}{\sqrt{y}}-\sqrt{y}\right)^2 s^2+\left(\frac{1}{\sqrt{y}}-\sqrt{y}\right)cs\cos\theta-s^2\cos^2\theta}$ $-\lvert c\cos\theta+\frac{1}{2}\left(\frac{1}{\sqrt{y}}-\sqrt{y}\right)s\rvert\}$

Universal soft functions: NLO master formula

- We switch to a dimensionless variable (x) and extract the scaling of the observables in the collinear limit y ⇒ 0:

$$\tau k_T f(y,\theta) \to x \qquad f(y,\theta) \to y^{\frac{n}{2}} \hat{f}(y,\theta)$$

- We are now in a position to write a master formula for the calculation of NLO dijet soft functions:

$$\bar{S}^{(1)}(\tau,\mu) \sim \int_{-1}^{1} \sin^{-1-2\epsilon}\theta \; d\cos\theta \int_0^\infty dx \int_0^1 dy \; x^{-1-2\epsilon-\alpha} \; y^{-1+n\epsilon+(n-1)\alpha/2} \; \hat{g}(x,y,\theta) \; [\hat{f}(y,\theta)]^{2\epsilon+\alpha} \; e^{-x}$$

- We are in a position to apply a subtraction technique to extract the singularities. Consider a simple 1-D example:

$$\int_0^1 dx \; x^{-1-n\epsilon} f(x) = \int_0^1 dx \; x^{-1-n\epsilon}\{f(x) - f(0) + f(0)\}$$

divergent

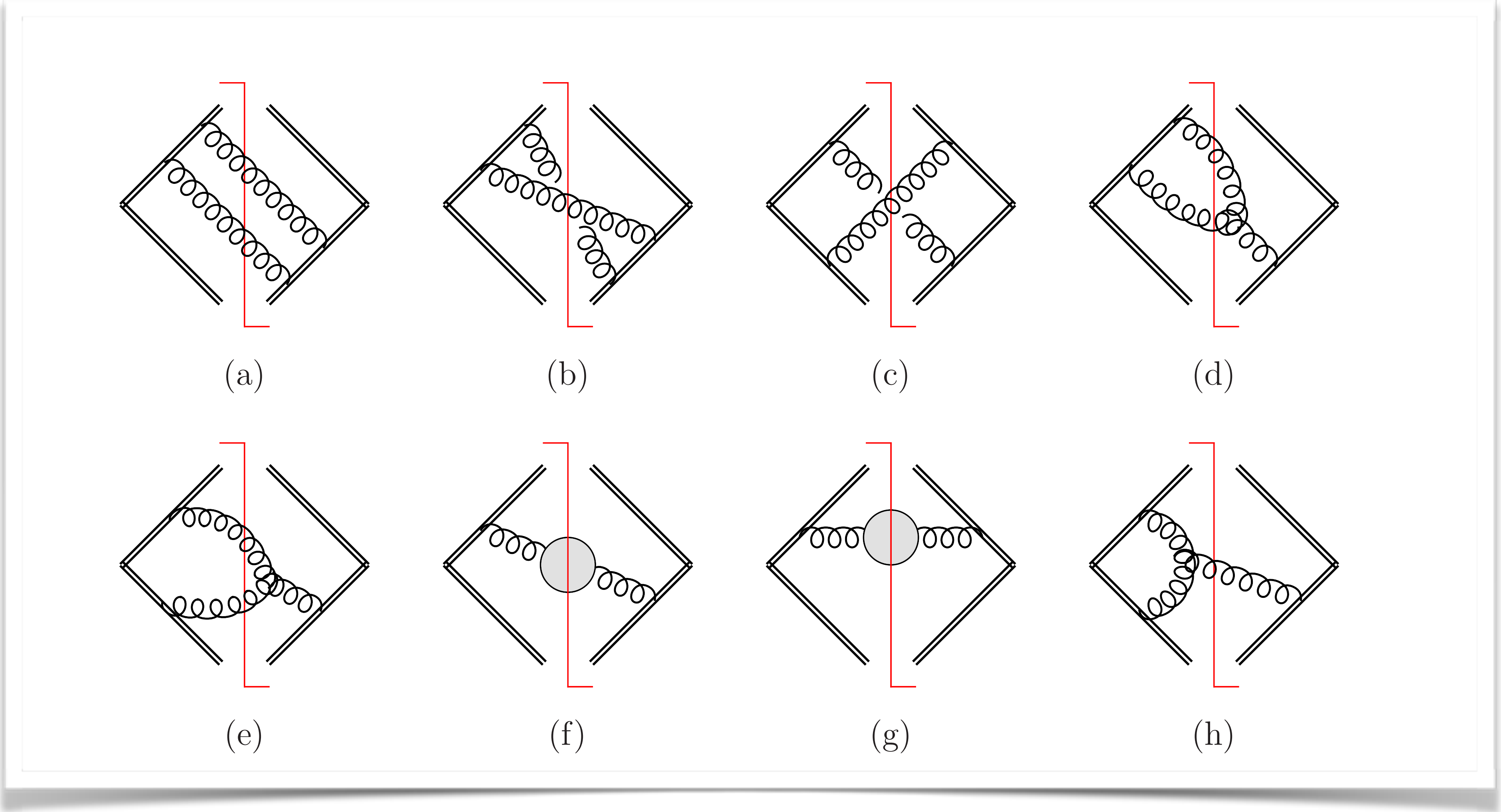
- finite / $O(x)$
- expand in ϵ
- integrate numerically

$\sim -\dfrac{1}{n\epsilon}$

- singularity isolated

- Note that n=0 corresponds to a $\mathrm{SCET_{II}}$ observable.

NNLO diagrams



- Three color structures are present: $C_F^2, \quad C_F C_A, \quad C_F T_F n_f$
- We use analytic results for the C_F^2 terms and the one-particle cuts.

** All results presented are preliminary!! **

NNLO soft functions

- Consider the double real emission (drop additional regulator):

$$\bar{S}^{(2)}_{RR}(\tau) = \frac{\mu^{4\epsilon}}{(2\pi)^{2d-2}} \int d^d k \; \delta(k^2) \; \theta(k^0) \int d^d l \; \delta(l^2) \; \theta(l^0) \; |\mathcal{A}(k,l)|^2 \; \bar{\mathcal{M}}(\tau, k, l)$$

- Decompose into light-cone coordinates and perform trivial integrations:

$$\bar{S}^{(2)}_{RR}(\tau) \sim \Omega_{d-3}\Omega_{d-4} \int_0^\infty dk_+ \int_0^\infty dk_- \int_0^\infty dl_+ \int_0^\infty dl_- \int_{-1}^1 d\cos\theta_k \;\; \sin^{d-5}\theta_k$$
$$\times \int_{-1}^1 d\cos\theta_l \;\; \sin^{d-5}\theta_l \int_{-1}^1 d\cos\theta_1 \;\; \sin^{d-6}\theta_1 \;\; (k_+k_-l_+l_-)^{-\epsilon} |\mathcal{A}(k,l)|^2 \;\; \bar{\mathcal{M}}(\tau, k, l)$$

- Consider, e.g., the $C_F T_F n_f$ color structure:

$$|\mathcal{A}(k,l)|^2 = 128\pi^2\alpha_s^2 C_F T_F n_f \frac{2k\cdot l(k_- + l_-)(k_+ + l_+) - (k_-l_+ - k_+l_-)^2}{(k_- + l_-)^2(k_+ + l_+)^2(2k\cdot l)^2}$$

- It is clear the singularity structure is non-trivial, and that the singularities are overlapping…

Sector decomposition

- Consider a simple integral over a unit hypercube with ‘overlapping singularities’ (singular as x,y simultaneously tend to 0):

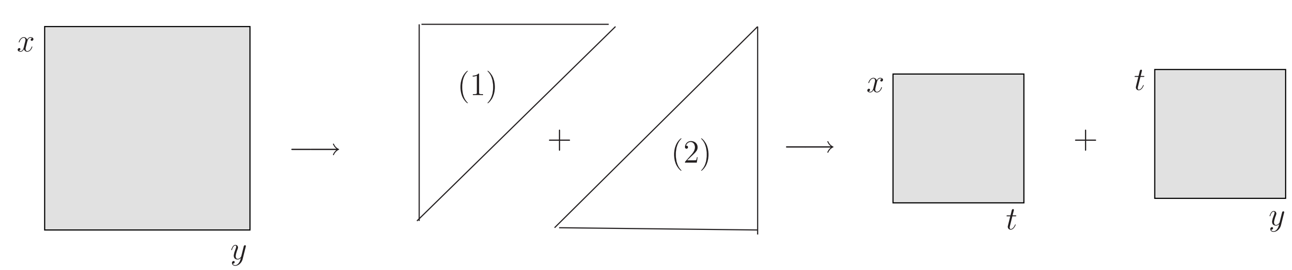
$$I = \int_0^1 dx \int_0^1 dy (x+y)^{-2+\epsilon}$$

- We want to factorize such singularities. Split the hypercube with two sectors (x>y) and (y>x):

$$I = I_1 + I_2 = \int_0^1 dx \int_0^x dy (x+y)^{-2+\epsilon} + \int_0^1 dy \int_0^y dx (x+y)^{-2+\epsilon}$$

- Now substitute y = xt in first sector and x = yt in second:

$$I_1 = \int_0^1 dx \int_0^1 dt \; x^{-1+\epsilon} (1+t)^{-2+\epsilon}$$
$$I_2 = \int_0^1 dy \int_0^1 dt \; y^{-1+\epsilon} (1+t)^{-2+\epsilon}$$

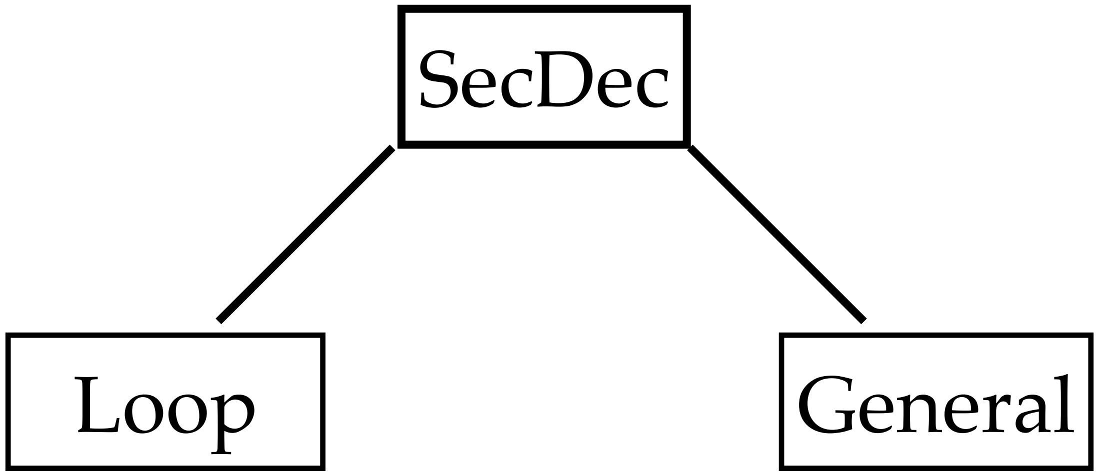


Automation: *SecDec*

https://secdec.hepforge.org

Borowka, Heinrich, Jones, Kerner, Schlenk, Zirke

- A tool is already on the market that exploits the sector decomposition algorithm: ***SecDec***
- "A program to evaluate dimensionally regularized parameter integrals numerically"



- We utilize the 'general' mode of the program. Simple interface to our NLO and NNLO master formulas (✔), multiple numerical integrators for crosschecks (✔)
- Active collaboration with *SecDec* team to implement special features of our algorithm, e.g. 'epsilon-dependent' functions and additional regulator for $\mathrm{SCET_{II}}$.
- Currently limited to $\mathrm{SCET_{I}}$ observables, though additional rapidity regulator in development.

NNLO parameterization

- We thus need to find an appropriate phase space parameterization that exposes the divergence structure and is amenable to sector decomposition (*SecDec*):

$$p_- = k_- + l_- \qquad a = \sqrt{\frac{k_- l_+}{k_+ l_-}} = e^{-(\eta_k - \eta_l)}$$

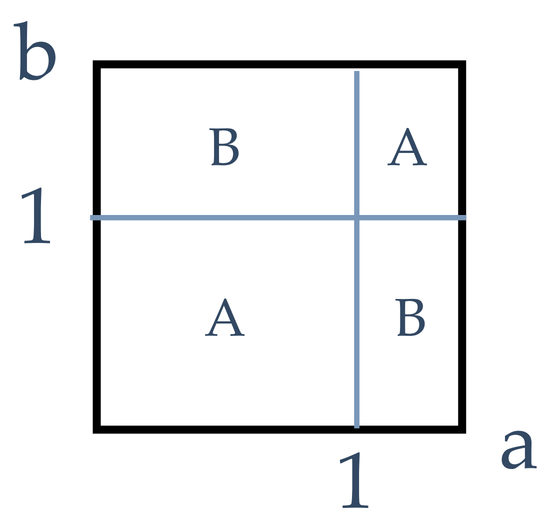
$$p_+ = k_+ + l_+ \qquad b = \sqrt{\frac{k_- k_+}{l_- l_+}} = \frac{k_T}{l_T}$$

- We further write the total momentum components in terms of p_T and y (as in NLO case):

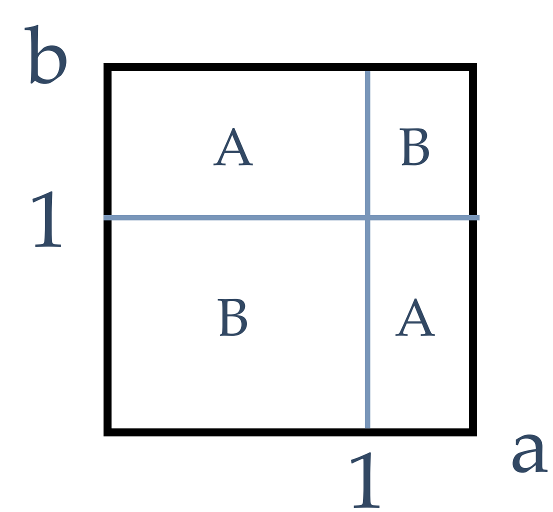
$$p_- \to \frac{p_T}{\sqrt{y}} \qquad p_+ \to p_T\sqrt{y}$$

- Finally, we map onto the unit hyper-cube:

(y<1)



(y>1)



Thrust

$$\mathcal{M}_{thrust}(\omega, \{k_i\}) = \delta(\omega - \sum_{i\in L} k_i^+ - \sum_{i \in R} k_i^-)$$

- We use *SecDec* to calculate the double emission contribution. To obtain the renormalized soft function we have to add the counterterms, which are known analytically at the required order.

- We show the cancellation of the divergences for thrust, setting $\ln(\mu\bar{\tau}) \to 0$

$$\tilde{S}^{(2)}_{ren} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\{C_A C_F \left(\frac{0}{\epsilon^4} - \frac{5.07333\times 10^{-9}}{\epsilon^3} + \frac{1.07523\times 10^{-6}}{\epsilon^2} + \frac{.0000102661}{\epsilon}\right) + C_F T_F n_f \left(-\frac{1.40667\times 10^{-8}}{\epsilon^3} + \frac{6.83778\times 10^{-8}}{\epsilon^2} - \frac{1.44697\times 10^{-8}}{\epsilon}\right)\} + \tilde{S}^{(2)}_0$$

- We thus also have an indication of our numerical precision…

- For the finite portion, we find (setting again $\ln(\mu\bar{\tau}) \to 0$):

$$\tilde{S}^{(2)}_0 = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(48.7045 C_F^2 - 56.4992 C_A C_F + 43.3902 C_F T_F n_f\right)$$

- Versus the analytic expression calculated by *Kelley, Schabinger, Schwartz, Zhu* / 1105.3676 (see also *Monni, Gehrmann, Luisoni* / 1105.4560):

$$\tilde{S}^{(2)}_0 = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(48.7045 C_F^2 - 56.4990 C_A C_F + 43.3905 C_F T_F n_f\right)$$

C-parameter

- *C-parameter* measurement function:

$$\mathcal{M}_C(\omega, \{k_i\}) = \delta(\omega - \sum_i \frac{k_+^i k_-^i}{k_+^i + k_-^i})$$

- For *C-parameter*, we obtain:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(5.41162 C_F^2 - 57.9754 C_A C_F + 43.8179 C_F T_F n_f\right)$$

- Where *Hoang, Kolodrubetz, Mateu, Stewart* / 1411.6633 extracted (using EVENT2) the following:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left((4.49 \pm 1.5) C_F^2 - (58.16 \pm .26) C_A C_F + (43.74 \pm .06) C_F T_F n_f\right)$$

- We find similar numerical precision in the subtractions.

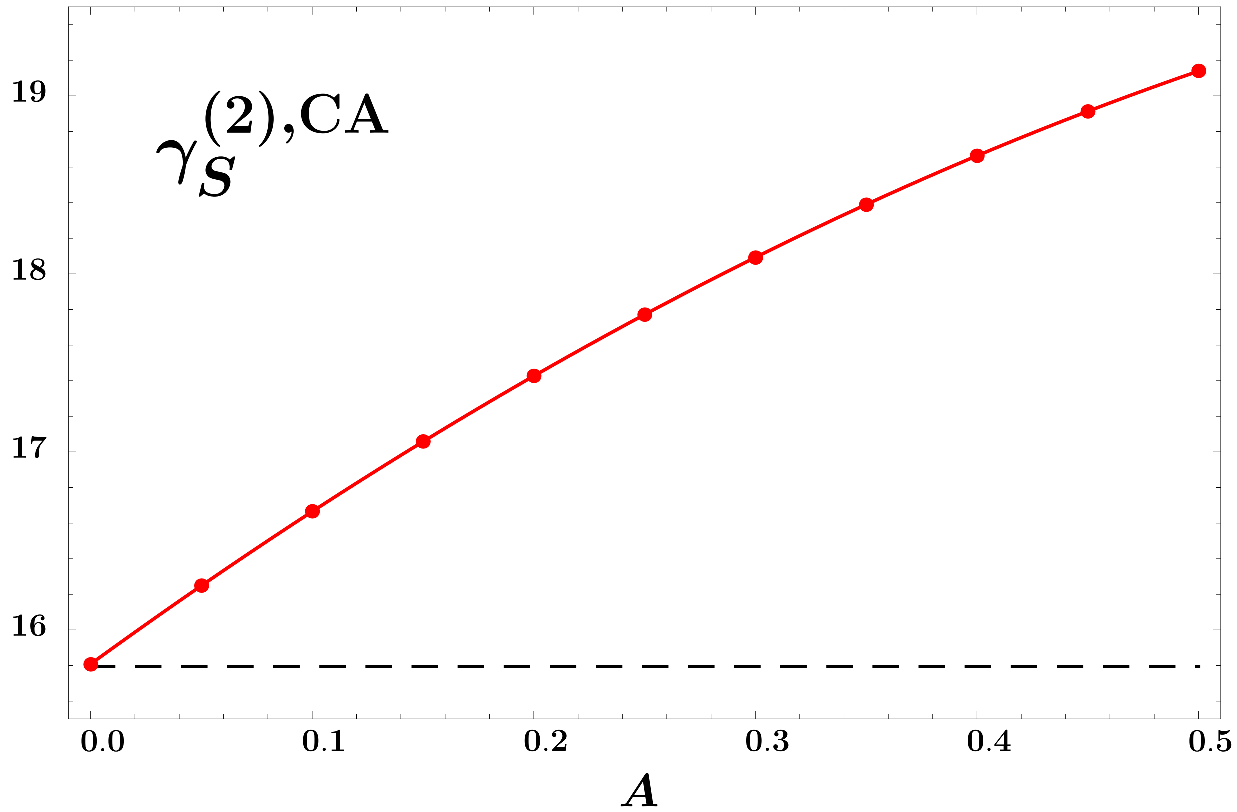
Angularities

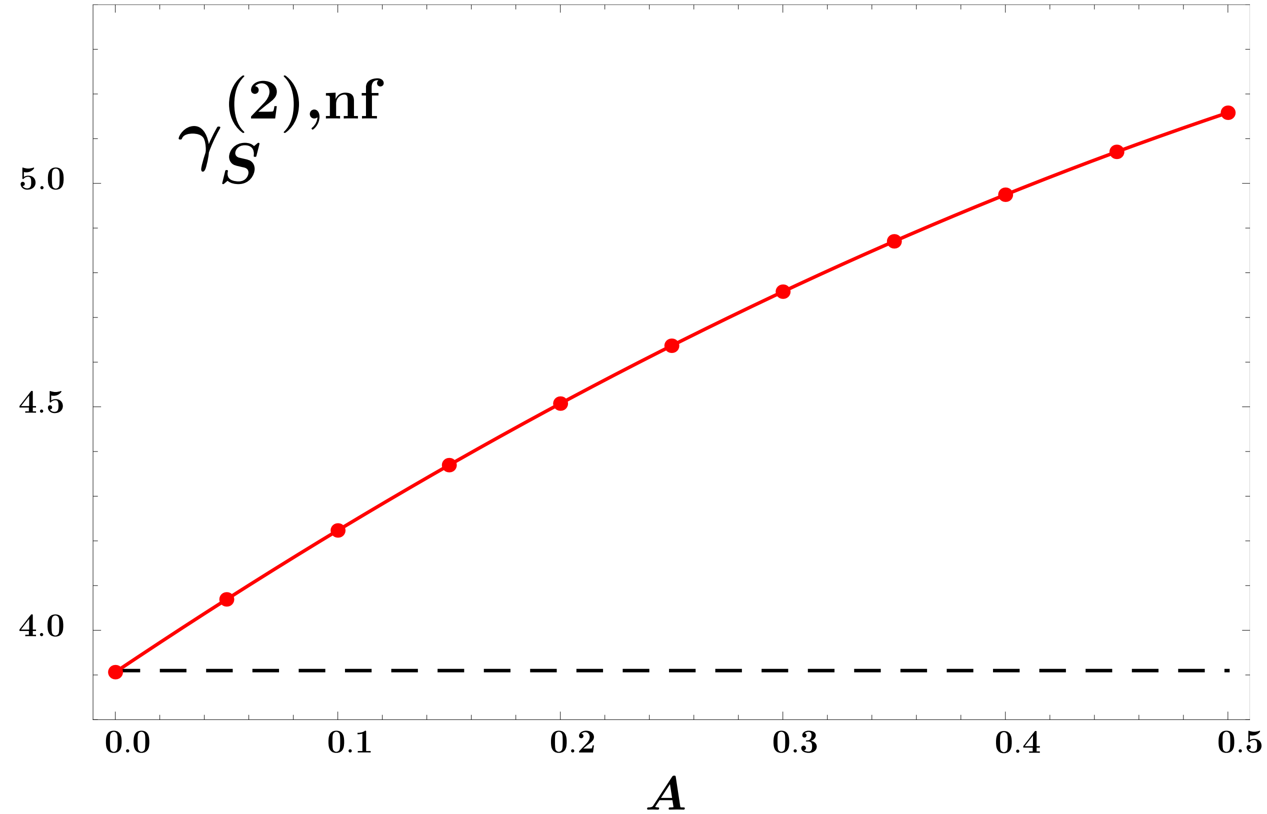
- *Angularities* measurement function:

$$\mathcal{M}_{Ang}(\omega,\{k_i\}) = \delta(\omega - \sum_{i\in L} k_{i,+}^{1-A/2}\, k_{i,-}^{A/2} - \sum_{i\in R} k_{i,+}^{A/2}\, k_{i,-}^{1-A/2})$$

- The two-loop soft anomalous dimension is not known. We define in Laplace space:

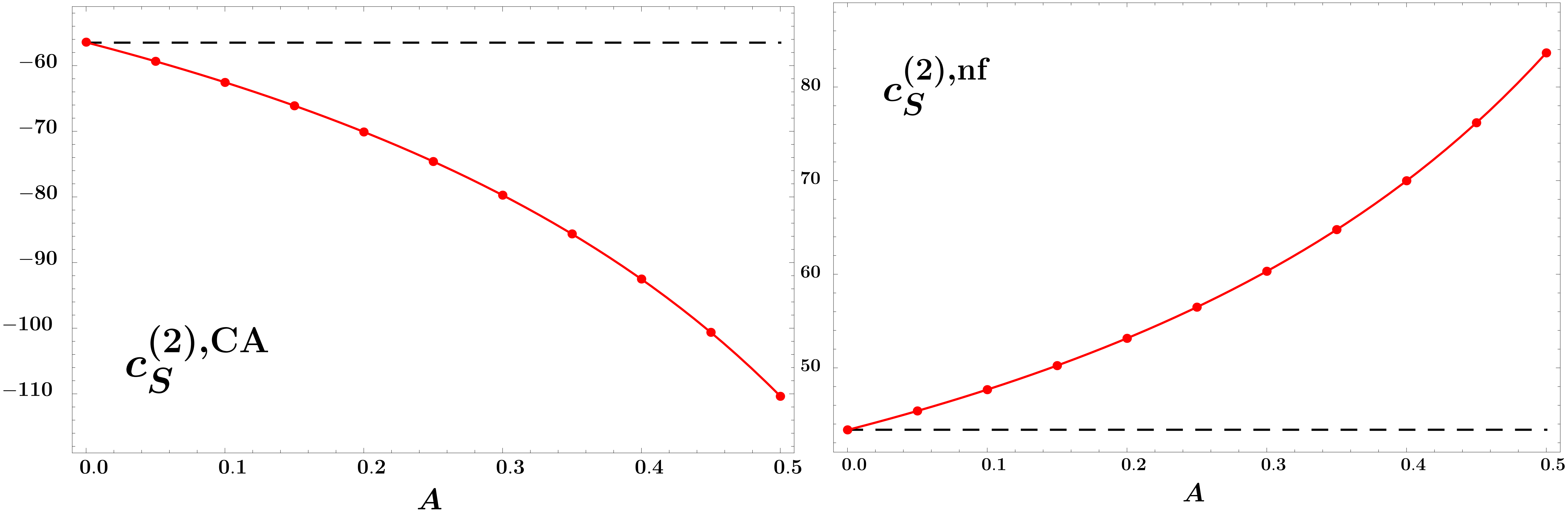
$$\frac{d\tilde{S}(\tau)}{d\ln\mu} = -\frac{1}{(1-A)}\left[4\Gamma_{cusp}\ln(\mu\bar{\tau}) - 2\gamma_S\right]\tilde{S}(\tau)$$





Angularities

$$\mathcal{M}_{Ang}(\omega, \{k_i\}) = \delta(\omega - \sum_{i \in L} k_{i,+}^{1-A/2}\ k_{i,-}^{A/2} - \sum_{i \in R} k_{i,+}^{A/2}\ k_{i,-}^{1-A/2})$$



Threshold Drell-Yan

- *Drell-Yan production @ threshold:*

$$\mathcal{M}_{DY}(\omega, \{k_i\}) = \delta(\omega - \sum_i k_+^i + k_-^i)$$

- For *Drell-Yan*, we obtain:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(5.41162 C_F^2 + 6.81281 C_A C_F - 10.6857 C_F T_F n_f\right)$$

- Whereas analytic expression calculated by *Belitsky* / 9808389 is:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(5.41162 C_F^2 + 6.81287 C_A C_F - 10.6857 C_F T_F n_f\right)$$

- Again, similar precision found for pole cancellation.

W/H @ large p_T

- Generically then the soft function depends on two initial state (1,2) and one final state (J) Wilson lines:

$$S(\omega) = \sum_X \delta(\omega - n_J \cdot p_X)\ |\langle X|S_1 S_2 S_J|0\rangle|^2$$

- However, due to a rescaling invariance of light-cone vectors and color conservation, the diagrams that contribute @ NNLO only involve attachments to the initial state Wilson Lines S_1 and S_2.

- Hence, up to NNLO, we encounter the same dijet matrix element as before.

- However, there is also now an angular dependence in the measurement function, giving six-dimensional integrals…

W/H @ large p_T

- *W/H production @ large p_T:*

$$\mathcal{M}_{W/H}(\omega,\{k_i\}) = \delta(\omega - \sum_i (k_i^+ + k_i^- - 2k_i^T \cos\theta_i))$$

- We have similar color structures with the following definitions:

$$C_s = \begin{cases} C_F - C_A/2 & q\bar{q} \to g \\ C_A/2 & qg \to q \ \ and \ \ gg \to g \end{cases}$$

- For *W/H production @ large p_T*, we obtain:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(48.7045 C_s^2 + (****) C_A C_s - 25.2824 C_s n_f T_F\right)$$

- Whereas *Becher, Bell, Marti* / 1201.5572 calculate:

$$\tilde{S}_0^{(2)} = \frac{\alpha_s^2(\mu)}{(4\pi)^2}\left(48.7045 C_s^2 - 2.6501 C_A C_s - 25.3073 C_s n_f T_F\right)$$

Conclusions and future work

- We have presented an automated algorithm to compute dijet soft functions for a wide class of observables in SCET

- Our master formulas coupled with *SecDec* can quickly and easily produce predictions for a wide class of $\mathrm{SCET_I}$ soft functions at one and two-loops. We are currently working with the developers to implement a few additional features in *SecDec.*

- This is an important ingredient for NNLL resummations in SCET…

- Next steps: $\mathrm{SCET_{II}}$ observables, n-jet soft functions and a public code…

Thanks!