Optical Stochastic Cooling Experiment at IOTA

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- Introduction to Optical Stochastic Cooling
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- Conclusions
**Principles of Optical Stochastic Cooling**

- OSC - suggested by Zolotorev, Zholents and Mikhailichenko (1994)
- OSC obeys the same principles as the microwave stochastic cooling, but exploits the superior bandwidth of optical amplifiers ~ $10^{14}$ Hz

At optimum the cooling rates of stochastic cooling are

Dimensionless damping rate: $\lambda f_0 \approx \frac{W}{N} \Leftrightarrow \lambda \approx \frac{1}{N_{\text{sample}}}$

- Potential gain in damping rates: $10^3 \div 10^4$
- Pickup and kicker must operate at the optical frequencies (same band as an opt. amplifier)
- Undulators suggested for pickups & kickers
- Slow particles do not radiate at optical frequencies
- OSC can operate only with ultra-relativistic particles
Principles of Optical Stochastic Cooling (continue)

- Radiation wave length
  \[ \lambda = \frac{\lambda_{wgl}}{2\gamma^2} \left( 1 + \gamma^2 \left( \frac{1}{2} \theta_e^2 + \theta^2 \right) \right) \]

  - helical undulator
  \[ \lambda = \frac{\lambda_{wgl}}{2\gamma^2} \left( 1 + \gamma^2 \theta_e^2 \right) \]

  - flat undulator

  Undulator parameter: \( K = \gamma \theta_e \Rightarrow \lambda_{|\theta=0} = \lambda_{wgl} \left( 1 + K^2 / 2 \right) / \left( 2\gamma^2 \right) \)

- Correction signal is proportional to longitudinal position change on the travel from pickup to kicker

- Only longitudinal kicks are effective for ultra-relativistic beam
  - \( s-x \) coupling for long. cooling
  - \( x-y \) coupling for vertical cooling

- Introduce partial slip factor: describes a long. particle displacement on the way from pickup to kicker with \( \Delta p/p \neq 0 \) & no betatron motion
  \[ \tilde{M}_{56} = M_{51}D_1 + M_{52}D'_1 + M_{56} \quad \Leftrightarrow \quad \Delta s = \tilde{M}_{56} \left( \Delta p / p \right) \]

- Cooling rates:
  \[ \lambda_x = \frac{k \xi_0}{2} \left( M_{56} - \tilde{M}_{56} \right) \]
  \[ \lambda_s = \frac{k \xi_0}{2} \tilde{M}_{56} \]

  \[ \Leftrightarrow \quad \lambda_x + \lambda_s = \frac{k \xi_0}{2} M_{56}^{pk} \]
Test of OSC in Fermilab

- First attempt to test the OSC in BATES, ~2007
  - Existing electron synchrotron
  - Did not get sufficient support

- Presently Fermilab is constructing a dual purpose small electron ring called IOTA to test:
  - Integrable optics
  - OSC

- Part of ASTA program
  - Full energy injection from SC linac

- Test in a small electron ring is a cost effective way to test the OSC
Basics of OSC: Damping Rates

- **Pickup-to-Kicker Transfer Matrix**
  
  Vertical plane is uncoupled and we omit it

\[
M^{pk} = \begin{bmatrix}
M_{11} & M_{12} & 0 & M_{16} \\
M_{21} & M_{22} & 0 & M_{26} \\
M_{51} & M_{52} & 1 & M_{56} \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad x = \begin{bmatrix}
x \\
\theta_x \\
s \\
\Delta p / p
\end{bmatrix}
\]

- **Partial slip factor (pickup-to-kicker)** describes a longitudinal particle displacement in the course of synchrotron motion

\[
\tilde{M}_{56} = M_{51} D_1 + M_{52} D_1' + M_{56}
\]

- **Linearized longitudinal kick in pickup wiggler**

\[
\frac{\delta p}{p} = k \xi_0 \Delta s = k \xi_0 \left( M_{51} x_1 + M_{52} \theta_x + M_{56} \frac{\Delta p}{p} \right)
\]

- **Cooling rates (per turn)**

\[
\lambda_x = \frac{k \xi_0}{2} \left( M_{56} - \tilde{M}_{56} \right)
\]
\[
\lambda_s = \frac{k \xi_0}{2} \tilde{M}_{56}
\]

\[\iff\]
\[
\lambda_x + \lambda_s = \frac{k \xi_0}{2} M^{pk}_{56}
\]
Basics of OSC: Cooling Range

cooling force depends on \( \Delta s \) nonlinearly

\[
\frac{\delta p}{p} = k \xi_0 \Delta s \quad \Rightarrow \quad \frac{\delta p}{p} = \xi_0 \sin(k \delta s)
\]

where

\[ k \delta s = a_x \sin(\psi_x) + a_p \sin(\psi_p) \]

and \( a_x \) & \( a_p \) are the amplitudes of longitudinal displacements in cooling chicane due to \( \perp \) and \( L \) motions measured in units of laser phase

\[
a_x = k \sqrt{\varepsilon \left( \beta_p M_{51}^2 - 2\alpha_p M_{51}M_{52} + \gamma_p M_{52}^2 \right)} , \quad \text{where} \quad \varepsilon = \beta_p \theta^2 - 2\alpha_p x \theta + \gamma_p x^2
\]

\[
a_p = k \tilde{M}_{56} \left( \Delta p / p \right)
\]

Averaging yields the form-factors for damping rates

\[
\lambda_{s,x}(a_x, a_p) = F_{s,x}(a_x, a_p) \lambda_{s,x}
\]

\[
F_x(a_x, a_p) = \frac{2}{a_x} J_0(a_p) J_1(a_x)
\]

\[
F_p(a_x, a_p) = \frac{2}{a_p} J_0(a_x) J_1(a_p)
\]

Damping requires both lengthening amplitudes \((a_x \text{ and } a_p)\) to be smaller than \( \mu_0 \approx 2.405 \)
Transfer Matrix for OSC Chicane

Chicane displaces the beam closer to its center

\[ M_{te} = \begin{pmatrix} 1 & L_d & 0 & \frac{L_d\Phi}{2} \\ 0 & 1 & 0 & \phi \\ \phi & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_d & 0 & -\frac{L_d\Phi}{2} \\ 0 & 1 & 0 & -\phi \\ -\phi & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_d & 0 & \frac{L_d\Phi}{2} \\ 0 & 1 & 0 & \phi \\ \phi & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_d & 0 & -\frac{L_d\Phi}{2} \\ 0 & 1 & 0 & -\phi \\ -\phi & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Leaving only major terms we obtain

\[ M_{ta} = \begin{bmatrix} L_t\Phi + 1 & L_t(L_t\Phi + 2) & 0 & \Phi\cdot h\cdot L_t \\ \Phi & L_t\Phi + 1 & 0 & \Phi\cdot h \\ -\Phi\cdot h & -\Phi\cdot h\cdot L_t & 1 & 2\cdot \Delta s \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \Delta s = \varphi^2 \left( L_1 + \frac{2}{3} L_d \right) \]

\[ h = \varphi \left( L_1 + L_d \right) \]

Matrix comparison:

Exact \((M_t)\) versus approximate \((M_{ta})\)

Test of Optical stochastic cooling in the IOTA ring, Valeri Lebedev, PAC-2013
OSC Chicane and Limitations on IOTA Optics

Dispersion in the chicane center

- In the first approximation, the orbit offset in the chicane ($h$), the path lengthening ($\Delta s$), the defocusing strength of $Q_d$ ($\Phi$) and dispersion in the chicane center ($D^*$) determine the entire cooling dynamics.
- $\Delta s$ is set by delay in the amplifier $\Rightarrow M_{56}$ ($\Delta s = 3$ mm is chosen, includes delay in lenses).
- Choose $(dD / ds)^* = 0 \Rightarrow D|_{s=\pm L_t} \approx D^*$
- $\Phi D^* h$ determines the ratio of decrements
  - Choose: $\lambda_x = 2 \lambda_s \Rightarrow \Phi D^* h \approx 4 \Delta s / 3$
- For the wave length of $\lambda = 2.2$ $\mu$m and momentum spread of $\sigma_p = 1.2 \cdot 10^{-4}$
  $\Rightarrow$ Cooling acceptance for longitudinal degree of freedom ($n_{\sigma p} = 3.6$)
- Thus $D^*$ determines the ratio of cooling rates and cooling acceptance in momentum

This is the first limitation which sets the wave length to be $\geq 2$ $\mu$m
OSC Chicane and Limitations on IOTA Optics (2)

Beta-function in the chicane center

- Behavior of the horizontal $\beta$-function determines the cooling range for horizontal degree of freedom
  - At optimum $\alpha^* = 0$
  - Cooling acceptance:
    \[
    \varepsilon_{\text{max}} = \frac{\mu_0^2}{k^2 \left( \beta_0 M_{51}^2 - 2\alpha_0 M_{51} M_{52} + \gamma_0 M_{52}^2 \right)} \approx \frac{\mu_0^2}{k^2 \Phi^2 h^2 \beta^*} \]

- For known rms emittance, $\varepsilon$, we can rewrite it as following
  \[
  n_{\sigma x} = \sqrt{\frac{\varepsilon_{\text{max}}}{\varepsilon}} \approx \frac{\mu_0}{k \Phi h \sqrt{\varepsilon \beta^*}} \quad \Phi D^* h = 2\Delta \frac{\lambda_x}{\lambda_x + \lambda_s} \quad n_{\sigma x} = \frac{\mu_0}{2k \Delta} \left( 1 + \frac{\lambda_s}{\lambda_x} \right) \sqrt{\frac{A_x^*}{\varepsilon}} \quad A_x^* = \frac{D^*}{\beta^*}
  \]

- Thus the cooling range, $n_{\sigma x}$, determines the dispersion invariant $A_x^*$
- Average value of $A_x$ in dipoles determines the equilibrium emittance.
  - $A_x^*$ is large and $A_x$ needs to be reduced fast to get an acceptable value of the equilibrium emittance ($\varepsilon$)
- Getting sufficiently large cooling acceptance requires long wave length of the radiation: another reason for $\lambda \geq 2 \mu m$
## Linear Beam Optics for Cooling Chicane

### Major parameters of cooling chicane

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam energy</td>
<td>100 MeV</td>
</tr>
<tr>
<td>Dipole type</td>
<td>Rbend</td>
</tr>
<tr>
<td>B of dipole</td>
<td>4.14 kG</td>
</tr>
<tr>
<td>L of dipole</td>
<td>10 cm</td>
</tr>
<tr>
<td>Orbit offset, h</td>
<td>28.4 mm</td>
</tr>
<tr>
<td>Delay, $\Delta s$</td>
<td>3 mm</td>
</tr>
<tr>
<td>GdL of Qd quad</td>
<td>720 Gs</td>
</tr>
<tr>
<td>$\beta_x^*$</td>
<td>4 cm</td>
</tr>
<tr>
<td>$D_x^*$</td>
<td>66 cm</td>
</tr>
<tr>
<td>Damping rates ratio, $\lambda_x/\lambda_s$</td>
<td>1.86</td>
</tr>
<tr>
<td>Basic wave length, $\lambda$</td>
<td>2.2 $\mu$m</td>
</tr>
<tr>
<td>Cooling range, $(\Delta p/p)_{\text{max}}$</td>
<td>$\pm 1.2 \times 10^{-3}$</td>
</tr>
<tr>
<td>Cooling acceptance, $\varepsilon_{\text{max}}$</td>
<td>0.46 $\mu$m</td>
</tr>
</tbody>
</table>

Test of Optical stochastic cooling in the IOTA ring, Valeri Lebedev, PAC-2013
Sample Lengthening on the Travel through Chicane

- Very large sample lengthening on the travel through chicane
- High accuracy of dipole field is required to prevent uncontrolled lengthening, $\Delta (BL)/(BL)_{\text{dipole}} < 10^{-3}$

Sample lengthening due to momentum spread (top) and due to betatron motion (bottom)
Non-linear Sample Lengthening

- Major contribution to the 2nd order lengthening comes from particle angle:
  \[
  \Delta s_2 = \int_{-L/2}^{L/2} \frac{\theta(s)^2}{2} ds
  \]

- Expressing it through particle phase in the chicane center (\(\mu_0\)), particle Courant-Snyder invariant (\(\varepsilon\)) and Twiss parameters =>
  \[
  I_1 = \int_0^{\mu/2} \left(1 + \alpha^2(\mu)\right) d\mu, \\
  I_2 = \int_0^{\mu/2} \left(\left(1 - \alpha^2(\mu)\right) \cos(2\mu) - 2\alpha(\mu) \sin(2\mu)\right) d\mu
  \]
  \[
  \Delta s_2 = \frac{\varepsilon}{2} \left(I_1 - I_2 \cos(2\mu_0)\right), \\
  \Rightarrow \text{maximum lengthening:} \quad \Delta s_2 = \frac{\varepsilon}{2} \left(I_1 + |I_2|\right)
  \]

- For IOTA cooling chicane we have: \(k\Delta s_2x \approx 26 \text{ rad}, k\Delta s_{2x} \approx 5 \text{ rad}\)
  for the boundary of cooling acceptance (\(\varepsilon_{\text{max}}=0.46 \text{ \(\mu\)m})

- Cooling is weakly affected if \(k\Delta s_2 \leq 1.5\)
  - Thus, in the absence of compensation we lose a factor of 4 in cooling range (\(\sqrt{26/1.5}\))
  - Effect of vertical motion is at the boundary of acceptable
  - It is the main reason why \(\lambda \geq 2 \text{ \(\mu\)m} \)
Non-linear sample lengthening due to H. betatron motion can be compensated by 2 sextupoles in the chicane

- Lengthening due to angle:
  \[ \Delta s_2 = -L_Q \theta^2, \quad L_Q = \frac{\beta^*}{2} (I_1 + |I_2|) \approx 77 \text{ cm} \]

- Shortening due to sextupoles (\( \delta \theta_S = x^2 / (2x_{os}^2) \) for defocusing sext):
  \[ \Delta s_2 = M_{s_{5,2}} \delta \theta_S \left( \frac{x^2}{2x_{os}^2} \frac{(L_S \theta)^2}{2x_{os}^2} \right) = L_S^3 \phi \theta^2, \quad \begin{cases} L_S \approx 26.5 \text{ cm} \\ \phi \approx 0.124 \text{rad} \end{cases} \]

- Comparing, we obtain the sextupole strength: \( SdL \approx -11 \text{ kG/cm} \) (defocus.)

- Vertical compensation is questionable

- Next round of simulations will follow
**IOTA Optics for OSC**

- Doublet focusing is adjusted to greatly reduce $A_x$ at the first ring dipole
- Tunes are adjusted to be near half-integer
- Geometric acceptances: $\varepsilon_x = 20\ \mu m$, $\varepsilon_x = 16\ \mu m$, $\Delta p/p = \pm 0.005$

Test of Optical stochastic cooling in the IOTA ring, Valeri Lebedev, PAC-2013
## IOTA Optics

### Main Parameters of IOTA storage ring for OSC

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumference</td>
<td>40 m</td>
</tr>
<tr>
<td>Nominal beam energy</td>
<td>100 MeV</td>
</tr>
<tr>
<td>Bending field of main dipoles</td>
<td>4.8 kG</td>
</tr>
<tr>
<td>Tunes, $Q_x / Q_y$</td>
<td>5.464/4.454</td>
</tr>
<tr>
<td>Natural chromaticities, $\xi_x / \xi_y$</td>
<td>-19 / -23</td>
</tr>
<tr>
<td>Chromaticities with OSC sextupoles</td>
<td>-81 / 29*</td>
</tr>
<tr>
<td>$\perp$ emittance, $\varepsilon_{SR}/2 = \varepsilon_x = \varepsilon_y$, rms</td>
<td>8.6 nm</td>
</tr>
<tr>
<td>Rms momentum spread, $\sigma_p$</td>
<td>1.29·10⁻⁴</td>
</tr>
<tr>
<td>SR damping times (ampl.), $\tau_s/ (\tau_x=\tau_y)$</td>
<td>1.5 / 1.4 s</td>
</tr>
<tr>
<td>Cooling ranges* (before OSC), $n_{sx}/n_{so}$</td>
<td>6.9 / 3.4</td>
</tr>
</tbody>
</table>

*For hor. plane it is defined as $\sqrt{\varepsilon_{max}/\varepsilon_x}$. The 2nd order lengthening is neglected. Expected that in the hor. plane it will not be a problem after compensation with sextupoles. The 2nd order lengthening limits the vertical cooling range to $n_{sy} \approx 4$.*

• Energy is reduced 150$\rightarrow$100 MeV to reduce $\varepsilon$, $\sigma_p$ and undulator period and length

• Operation on coupling resonance reduces horizontal emittance and introduces vertical damping

• Tunes are chosen to maximize dynamic aperture limitation by OSC sextupoles

* Chromaticities need to be compensated to be $|\xi|\leq20$
**Dynamic Aperture Limitation by Sextupoles of OSC Insert**

- Introduce dimensionless variables
  \[ \tilde{\theta} = \beta^2 \frac{\theta + \alpha x / \beta}{x_{0S}} , \quad \tilde{x} = \frac{\beta x}{x_{0S}} \quad \text{where} \quad x_{0S}^2 = \frac{pc}{e(SL)} \]

- Then the following transforms drive particle motion
  \[
  \left[ \begin{array}{c}
  \tilde{x}' \\
  \tilde{\theta}'
  \end{array} \right] = \left[ \begin{array}{cc}
  \cos \mu & \sin \mu \\
  -\sin \mu & \cos \mu
  \end{array} \right] \left[ \begin{array}{c}
  \tilde{x} \\
  \tilde{\theta}
  \end{array} \right] , \quad \tilde{\theta}' = \tilde{\theta} + \frac{\tilde{x}^2}{x_{0S}^2}
  \]

- In vicinity of 3\textsuperscript{rd} order resonance:
  \[ \tilde{x}_b \approx 25 \left[ \nu \right] \approx 1 \Rightarrow \varepsilon_b \approx \frac{625 x_{0S}^4}{\beta^3} \left( \left[ \nu \right] \approx \frac{1}{3} \right)^2 \]

- Far from the resonance the stability boundary can be estimated from the phase space distortion =>
  \[ \tilde{x}_b \approx 3 \Rightarrow \varepsilon_b \approx \frac{9 x_{0S}^4}{\beta^3} \]

- Transition happens at detuning \( \Delta \nu \approx 0.1 \)
**Dynamic Aperture Limitation by Sextupoles of OSC Insert(2)**

- Phase advance between OSC sextupoles $\Delta Q_x=0.451$
  - Although it is close to half integer it does not help with cancellation of sextupole effect

- Operation closer to half-integer resonance improves dynamic aperture
- For estimate we use
  \[
  \varepsilon_{bx} \approx \frac{9 x_{0S}^4}{\beta_x^3}, \quad x_{0S}^2 = \frac{p c}{e (SL)}
  \]
  \[\Rightarrow \varepsilon_{bx}=14 \, \mu m, \, n_{ax} \approx 40\] (compare to $\varepsilon_{x_{geom}}=20 \, \mu m$)
- Looks like aperture limitation by OSC sextupoles looks OK
- Orbit stability within sextupoles <100 $\mu m$
- Detailed simulations are required

Test of Optical stochastic cooling in the IOTA ring, Valeri Lebedev, PAC-2013
**Undulators**

- Undulator period was chosen so that $\lambda_{\theta=0}=2.2$ $\mu$m

<table>
<thead>
<tr>
<th>Radiation wavelength at zero angle</th>
<th>2.2 $\mu$m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Undulator parameter, $K$</td>
<td>0.8</td>
</tr>
<tr>
<td>Undulator period</td>
<td>12.9 cm</td>
</tr>
<tr>
<td>Number of periods, $m$</td>
<td>6</td>
</tr>
<tr>
<td>Total undulator length, $L_w$</td>
<td>0.77 m</td>
</tr>
<tr>
<td>Peak magnetic field</td>
<td>664 G</td>
</tr>
<tr>
<td>Distance between centers of undulators</td>
<td>3.3 m</td>
</tr>
<tr>
<td>Energy loss per undulator per pass</td>
<td>22 meV</td>
</tr>
<tr>
<td>Average power per undulator for $N_e=10^6$</td>
<td>26 nW</td>
</tr>
<tr>
<td>Radiation size in 2$^{nd}$ undulator, HWHM</td>
<td>0.35 mm</td>
</tr>
</tbody>
</table>

Rms beam sizes in absence of OSC, $\sigma_x=0.25$ mm - in undulators
Cooling Rates

- Passive OSC increases the SR damping rates by about one order of magnitude
- Optical amplifier with 20 dB gain could increase the damping rate by factor of ~3
  - Factor of 3 will be lost due to smaller bandwidth
  - Detailed design is pending

Main parameters of passive OSC

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Band</td>
<td>2.2 - 3.3 μm</td>
</tr>
<tr>
<td>Angular acceptance</td>
<td>4.1 mrad</td>
</tr>
<tr>
<td>Optics system radius</td>
<td>7 mm</td>
</tr>
<tr>
<td>Damp. rates ($\chi=y/s$)</td>
<td>6.5/7.6 s⁻¹</td>
</tr>
</tbody>
</table>
Focusing of Beam Radiation to OA and Kicker

- Two possibilities
  - For passive OSC: four lens system with complete suppression of depth of field
  - Two lens system (F=8 cm, radius - 3.5 mm)
    - Reasonable compromise between 4 major requirements
    - The spot size in OA to be sufficiently small: $r<60 \mu m$
      $\Rightarrow$ diffraction limited size in OA: HWHM=12 $\mu m$ or total size $r\approx30 \mu m$
      $\Rightarrow$ size due to beam convergence/divergence at OA input/exit $\approx50 \mu m$
Other Limitations

- Touschek lifetime and multiple IBS limit the number of particles in the bunch, $N_e \sim 10^6$
- Scattering on the residual gas results in short lifetime in the conditions of small cooling acceptance
- Quantum effects play little role in the OSC cooling

Quantum Mechanical Treatment of Transit-Time Optical Stochastic Cooling of Muons

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(Dated: April 9, 2009)

Quantum theory of Optical Stochastic Cooling

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Conclusions

- Optical stochastic cooling looks as a promising technique for future hadron colliders
- Experimental study of OSC in Fermilab is in its initial phase
  - It is aimed to validate cooling principles and to demonstrate cooling with and without optical amplifier
    - Even in the absence of amplification (passive system, $G = 1$) the OSC damping exceeds SR damping by about an order of magnitude
- The beam intensity ranges from a single electron to the bunch population limited by operation at the optimum gain ($10^8$-$10^9$)
  - Single electron cooling - localization of electron wave function and essence of quantum mechanics
    - Quantum noise for passive cooling
  - Cooling at the optimal gain (ultimate cooling) gets us to otherwise hidden details of OSC, in particular, to signal suppression
Backup Slides
Basics of OSC – Radiation from Undulator

- Liénard-Wiechert potentials and E-field of moving charge in wave zone

\[
\begin{align*}
\varphi(r,t) &= \frac{e}{(R - \beta \cdot R)} \bigg|_{t=R/c} \\
A(r,t) &= \frac{ev}{(R - \beta \cdot R)} \bigg|_{t=R/c} \\
E(r,t) &= \frac{e}{c^2} \frac{(R - \beta \cdot R)(a \cdot R) - aR(R - \beta \cdot R)}{(R - \beta \cdot R)^3} \bigg|_{t=R/c}
\end{align*}
\]

where \( a \equiv \frac{dv}{dt} \)

- Radiation of ultra-relativistic particle is concentrated in \( 1/\gamma \) angle
- Undulator parameter:

\[
K \equiv \gamma \theta_e = \frac{\lambda_{wgl} eB_0}{2\pi mc^2}
\]

- For \( K \geq 1 \) the radiation is mainly radiated into higher harmonics
Basics of OSC – Radiation Focusing to Kicker Undulator

- Modified Kirchhoff formula

\[ E(r) = \frac{\omega}{2\pi ic} \int_{s} \frac{E(r') e^{i\omega|r-r'|}}{|r-r'|} ds' \]

\[ \Rightarrow \quad E(r) = \frac{1}{2\pi ic} \int_{s} \frac{\omega(r')e^{i\omega|r-r'|}E(r')}{|r-r'|} ds' \]

- Effect of higher harmonics
  - Higher harmonics are normally located outside window of optical lens transparency and are absorbed in the lens material

Dependences of retarded time \((t_p)\) and \(E_x\) on time for helical undulator

- Only first harmonic is retained in the calculations presented below
Basics of OSC – Longitudinal Kick for $K<<1$

For $K<<1$ refocused radiation of pickup undulator has the same structure as radiation from kicker undulator. They are added coherently:

$$E = E_1 + E_2 e^{i\phi} \rightarrow 2 \cos\left(\frac{\phi}{2}\right) E_1 e^{i\phi/2}$$

⇒ Energy loss after passing 2 undulators

$$\Delta U \propto \left|E^2\right| = 4 \cos\left(\frac{\phi}{2}\right)^2 \left|E_1^2\right| = 2 \left(1 + \cos\phi\right) \left|E_1^2\right| = 2 \left(1 + \cos\left(kM_{56} \frac{\Delta p}{p}\right)\right) \left|E_1^2\right|$$

Large derivative of energy loss on momentum amplifies damping rates and creates a possibility to achieve damping without optical amplifier

♦ SR damping: $$\lambda_{\parallel, SR} \approx \frac{2\Delta U_{SR}}{pc} f_0$$

♦ OSC:

$$\lambda_{\parallel, OSC} \approx f_0 \frac{2\Delta U_{wgl}}{pc} \left(GkM_{56}\right)_{\frac{kM_{56}(\Delta p/p)_{\text{max}} = \pi}} \rightarrow f_0 \frac{2\Delta U_{wgl}}{pc} \left(\frac{G}{(\Delta p / p)_{\text{max}}}\right)$$

where $G$ - optical amplifier gain, $(\Delta p / p)_{\text{max}}$ - cooling system acceptance

⇒ $$\lambda_{\parallel, OSC} \propto B^2 L \propto K^2 L$$ - but cooling efficiency drops with $K$ increase above $\sim 1$
Basics of OSC – Longitudinal Kick for $K<<1$(continue)

- Radiation wavelength depends on $\theta$ as

$$\lambda = \frac{\lambda}{2\gamma^2} \left(1 + \gamma^2 \theta^2\right)$$

Limitation of system bandwidth by (1) optical amplifier band or (2) subtended angle reduce damping rate

$$\lambda_{||,SR} = \lambda_{||,SR0} F(\gamma \theta_m), \quad F(x) = 1 - \frac{1}{\left(1 + x^2\right)^3}$$

For narrow band: $\Delta U_{wgl} = \Delta U_{wgl0} \left(\frac{3\Delta \omega}{\omega}\right)$, $\frac{3\Delta \omega}{\omega} << 1$

where $\Delta U_{wgl0} = \frac{e^4 B^2 \gamma^2 L}{3m^2 c^4} \begin{cases} 1, & \text{Flat wiggler} \\ 2, & \text{Helical wiggler} \end{cases}$ the energy radiated in one undulator
Basics of OSC – Radiation from Flat Undulator

For arbitrary undulator parameter we have

\[
\Delta U_{OSC,F} = \frac{1}{2} \frac{4e^4 B_0^2 \gamma^2 L}{3m^2 c^4} GF_f \left( K, \gamma \theta_{\text{max}} \right) F_u \left( \kappa_u \right)
\]

\[
F_u \left( \kappa_u \right) = J_0 \left( \kappa_u \right) - J_1 \left( \kappa_u \right), \quad \kappa_u = K^2 / \left( 4 \left( 1 + K^2 / 2 \right) \right)
\]

Fitting results of numerical integration yields:

\[
F_h \left( K, \infty \right) \approx \frac{1}{1 + 1.07K^2 + 0.11K^3 + 0.36K^4}, \quad K \equiv \gamma \theta_e \leq 4
\]

**Dependence of wavelength on \( \theta \):**

\[
\lambda \approx \frac{\lambda_{wgl}}{2 \gamma^2} \left( 1 + \gamma^2 \left( \theta^2 + \frac{\theta_e^2}{2} \right) \right)
\]

**Flat undulator is “more effective” than the helical one**

**For the same \( K \) and \( \lambda_{wgl} \), flat undulator generates shorter wave lengths**

For both cases of the flat and helical undulators and for fixed \( B \) a decrease of \( \lambda_{wgl} \) and, consequently, \( \lambda \) yields kick increase

- but wavelength is limited by both beam optics and light focusing
**Basics of OSC – Correction of the Depth of Field**

- It was implied above that the radiation coming out of the pickup undulator is focused on the particle during its trip through the kicker undulator.
  - It can be achieved with lens located at infinity
    
    \[
    \frac{1}{2F + \Delta s} + \frac{1}{2F - \Delta s} = \frac{1}{F} \quad \Rightarrow \quad \frac{1}{F - \Delta s^2 / 4F} = \frac{1}{F} \quad \Rightarrow \quad \frac{F}{\infty} \rightarrow \frac{1}{F} = \frac{1}{F}
    \]
  - but this arrangement cannot be used in practice
- A 3-lens telescope can address the problem within limited space

\[
\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_1^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_2^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -F_1^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & L \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\]

Test of Optical stochastic cooling in the IOTA ring, Valeri Lebedev, PAC-2013