

MATTER EFFECT FROM NON-STANDARD INTERACTIONS OF THE NEUTRINO

nu@Fermilab, July 23, 2015

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Matter Effect on Neutrino Oscillation

Neutrinos of long-baseline neutrino oscillation experiments and atmospheric neutrinos coming from below necessarily traverse Earth's interior

- Matter effects must be taken into account to extract vacuum parameters
- Matter effects depend on ALL interactions between the matter fermions and neutrinos
- Can matter effects be used to explore those interactions and constrain/discover new physics?

Interactions \leftrightarrow Matter Effects

- Numerical calculations **obscure the physics**.
- **Exact analytical expressions exist** for the oscillation probabilities in constant density matter, but are **too complicated** to be of practical use
- **Non-Standard Interactions** (NSI's) of the neutrino further complicate the situation → What are their signatures?
- **Simple analytical approximations** to the oscillation probabilities in matter, with/without NSI's, would go a long way in helping us analyze the physics potential of various experiments

Analytical Approximations:

1. Absorb matter effects into the “running” of effective mass-squared differences, mixing angles, and CP phase and approximate their a -dependence
 - S. Toshev, Phys. Lett. B185 (1987) 177
 - S. T. Petcov & S. Toshev, Phys. Lett. B187 (1987) 120
 - P. I. Krastev & S. T. Petcov, Phys. Lett. B205 (1988) 84
 - H. Zaglauer & K. Schwarzer, Z. Phys. C40 (1988) 273, etc.
2. Approximate oscillation probabilities directly
 - A. Cervera et al., Nucl. Phys. B593 (2001) 731
 - M. Freund, Phys. Rev. D64 (2001) 053003
 - E. K. Akhmedov et al., JHEP 0404 (2004) 078
 - K. Asano & H. Minakata, JHEP 1106 (2011) 022, etc.

SM Case (No NSI's)

Neutrino Oscillation in Matter : SM case

Effective Hamiltonian in the flavor eigenbasis

$$H = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a = 2\sqrt{2}G_F N_e E$$

In the mass eigenbasis in vacuum, the in-matter Hamiltonian is no longer diagonal

$$U^\dagger H U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + a U^\dagger \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U$$

Must find the new mass eigenbasis in matter

$$\tilde{U}^\dagger H \tilde{U} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Formula of Cervera et al.

- A. Cervera, A. Donini, M. B. Gavela, J. J. Gomez Cádenas, P. Hernández, O. Mena, and S. Rigolin, Nuclear Physics B **579** (2000) 17-55.

$$\begin{aligned} P(\nu_\mu \rightarrow \nu_e) \approx & \sin^2 2\theta_{13} \sin^2 2\theta_{23} \frac{\sin^2[(1-A)\Delta]}{(1-A)^2} \\ & - \alpha \sin 2\theta_{13} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin \delta \sin \Delta \frac{\sin(A\Delta)}{A} \frac{\sin[(1-A)\Delta]}{(1-A)} \\ & + \alpha \sin 2\theta_{13} \cos \theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos \delta \cos \Delta \frac{\sin(A\Delta)}{A} \frac{\sin[(1-A)\Delta]}{(1-A)} \\ & + \alpha^2 \cos^2 \theta_{23} \sin^2 2\theta_{12} \frac{\sin^2(A\Delta)}{A^2}, \end{aligned}$$

$$\alpha = \frac{\delta m_{21}^2}{\delta m_{31}^2} \approx 0.03, \quad A = \frac{a}{\delta m_{31}^2}, \quad \Delta = \frac{\delta m_{31}^2}{4E} L, \quad \sin 2\theta_{13} \approx 0.3$$

Asano-Minakata

- K. Asano and H. Minakata, JHEP 1106 (2011) 022 [arXiv:1103.4387[hep-ph]]

$$P_{AM}(\nu_\mu \rightarrow \nu_e) \approx P_C(\nu_\mu \rightarrow \nu_e)$$

$$\begin{aligned} & -4 \sin^2 \theta_{23} \left[\sin^4 \theta_{13} \left(\frac{1+A}{1-A} \right)^2 - 2\alpha \sin^2 \theta_{13} \sin^2 \theta_{12} \left(\frac{A}{1-A} \right) \right] \frac{\sin^2 [(1-A)\Delta]}{(1-A)^2} \\ & + 4 \sin^2 \theta_{23} \left[2 \sin^4 \theta_{13} \left(\frac{A}{1-A} \right) - \alpha \sin^2 \theta_{13} \sin^2 \theta_{12} \right] \frac{\Delta \sin [2(1-A)\Delta]}{(1-A)^2}, \end{aligned}$$

$$\sin^4 \theta_{13} \approx 0.0005, \quad \alpha \sin^2 \theta_{13} \approx 0.0007$$

Our “Running” Parameter Approach : SM case

S. K. Agarwalla, Y. Kao, & TT, JHEP04 (2014) 047 [arXiv:1302.6773]

- Use the expressions for the vacuum oscillation probabilities as is, but make the following replacements:

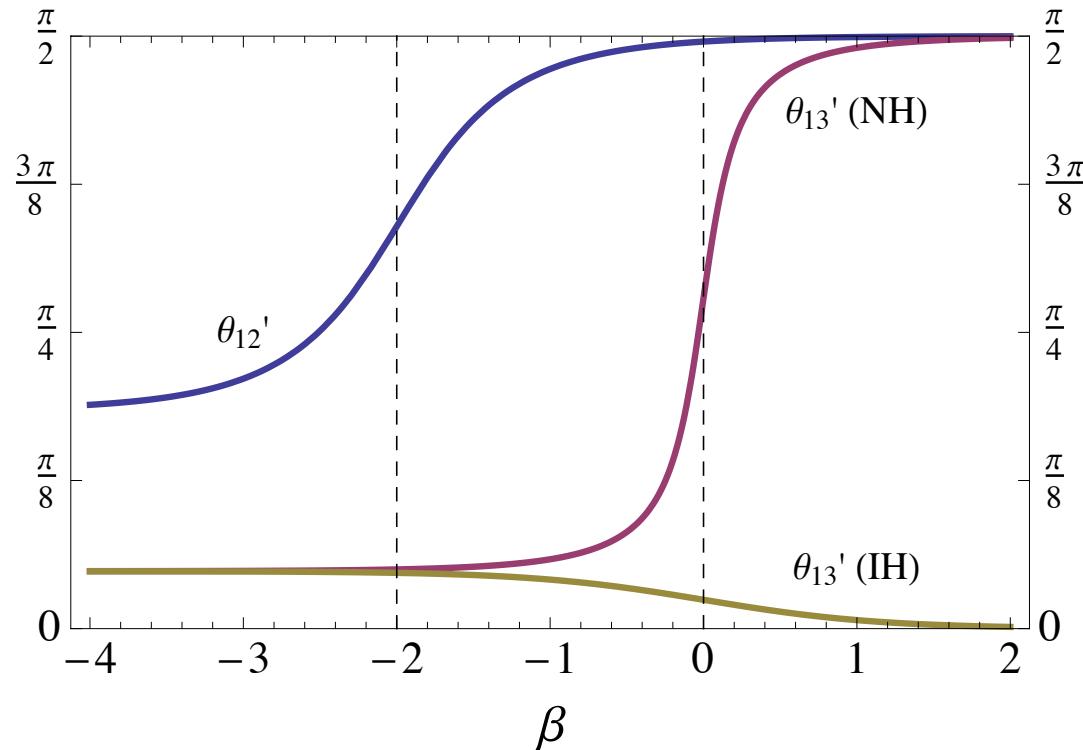
$$\theta_{12} \rightarrow \theta'_{12}, \quad \theta_{13} \rightarrow \theta'_{13}, \quad \delta m^2_{jk} \rightarrow \lambda_j - \lambda_k$$

where

$$\tan 2\theta'_{12} = \frac{(\delta m^2_{21} / c^2_{13}) \sin 2\theta_{12}}{(\delta m^2_{21} / c^2_{13}) \cos 2\theta_{12} - a}, \quad \tan 2\theta'_{13} = \frac{(\delta m^2_{31} - \delta m^2_{21} s^2_{12}) \sin 2\theta_{13}}{(\delta m^2_{31} - \delta m^2_{21} s^2_{12}) \cos 2\theta_{13} - a},$$

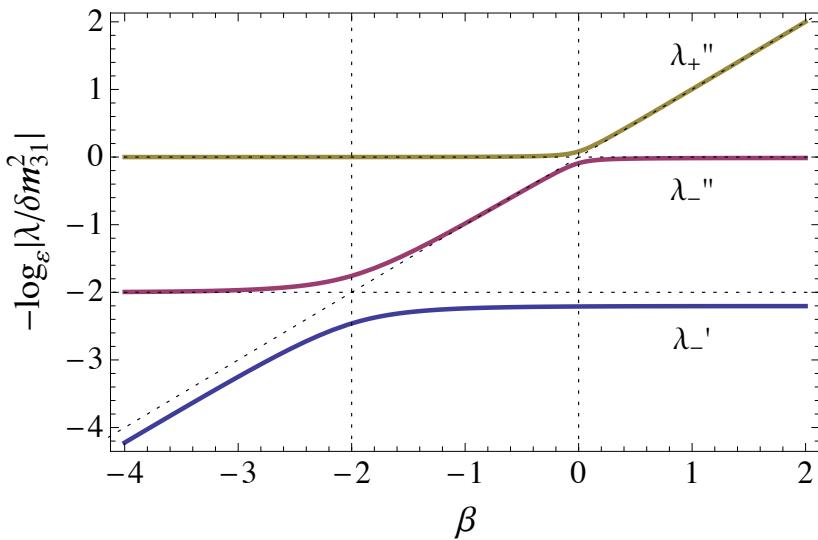
$$\begin{aligned} \lambda_1 &= \lambda'_- & \lambda'_\pm &= \frac{(\delta m^2_{21} + ac^2_{13}) \pm \sqrt{(\delta m^2_{21} - ac^2_{13})^2 + 4ac^2_{13}s^2_{12}\delta m^2_{21}}}{2} \\ \lambda_2 &= \lambda''_+ & \lambda''_\pm &= \frac{\left[\lambda'_+ + (\delta m^2_{31} + as^2_{13}) \right] \pm \sqrt{\left[\lambda'_+ - (\delta m^2_{31} + as^2_{13}) \right]^2 + 4a^2s'^2_{12}c^2_{13}s^2_{13}}}{2} \\ \lambda_3 &= \lambda''_\pm \end{aligned}$$

a -dependence of effective mixing angles

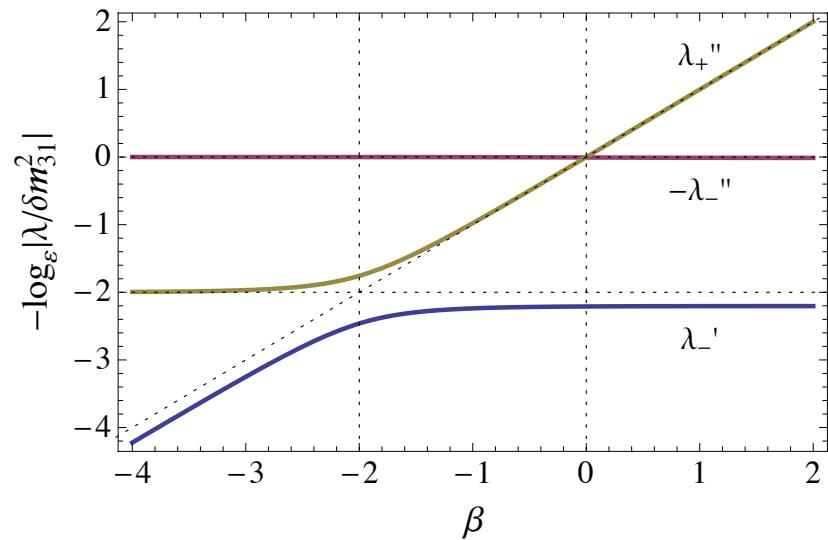


$$\frac{a}{|\delta m_{31}^2|} = \varepsilon^{-\beta}, \quad \varepsilon = \sqrt{\frac{\delta m_{21}^2}{|\delta m_{31}^2|}} \approx 0.17$$

a -dependence of effective mass-squares

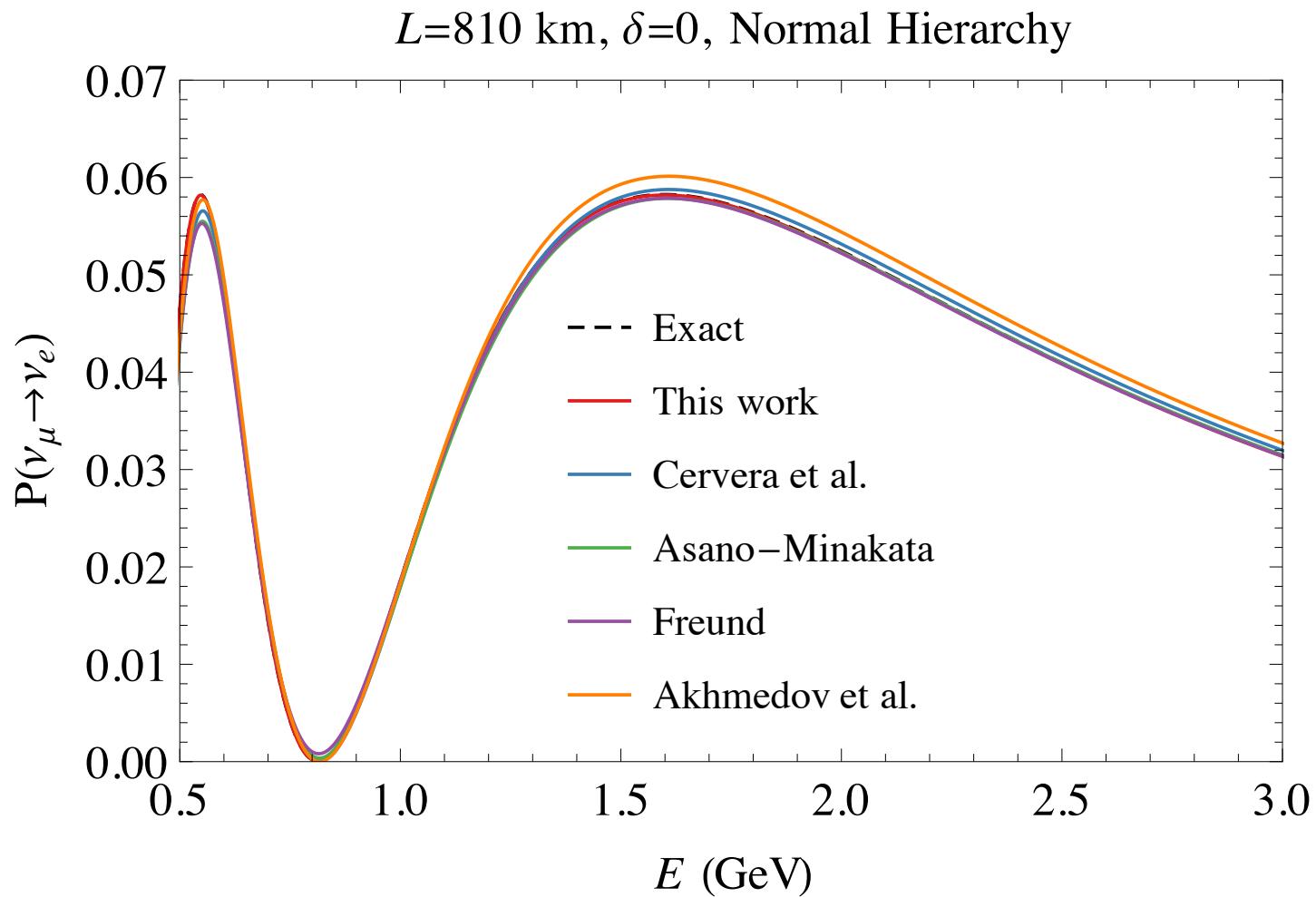


Normal Hierarchy

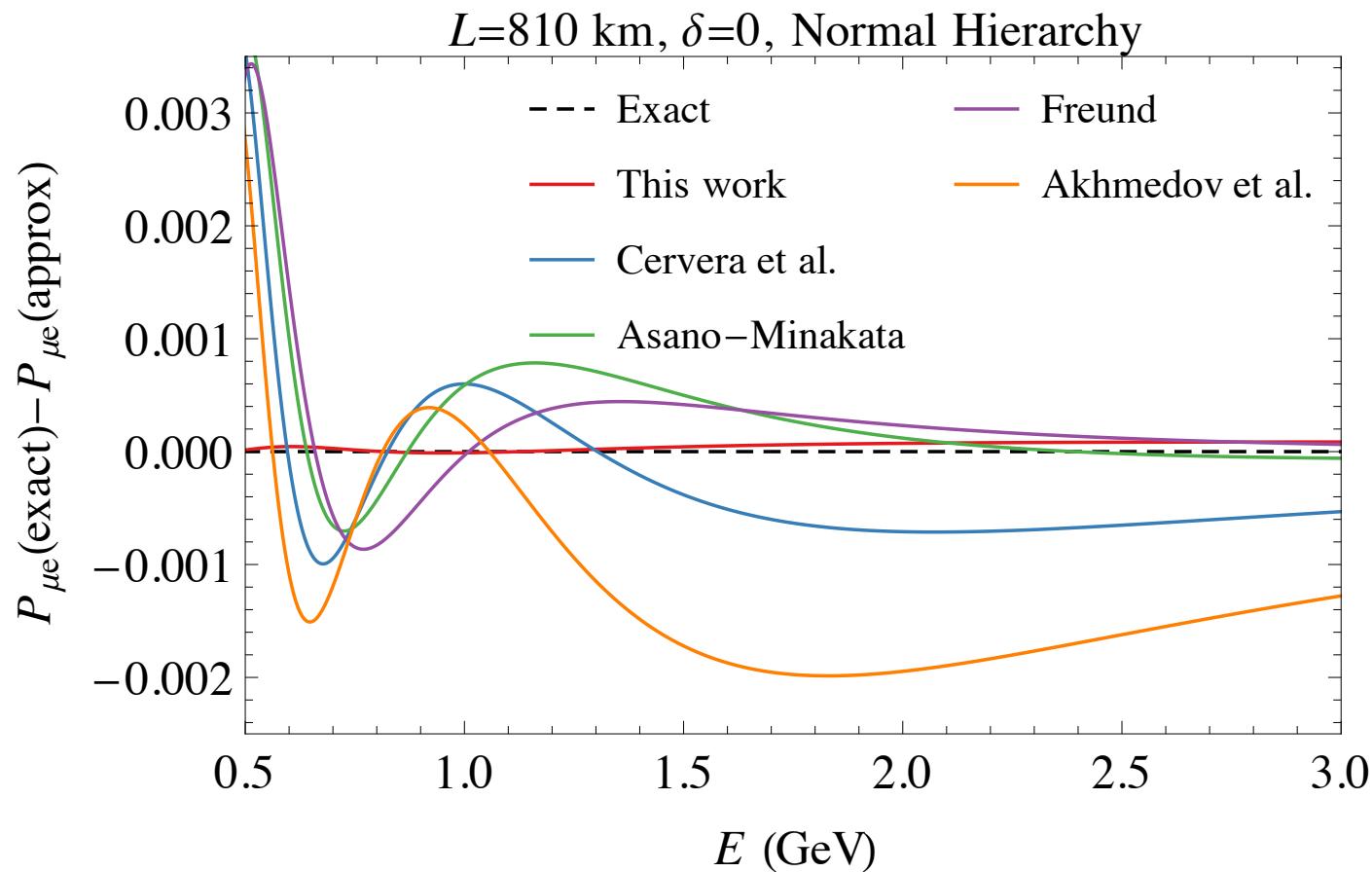


Inverted Hierarchy

L=810 km (Fermilab→NOvA)

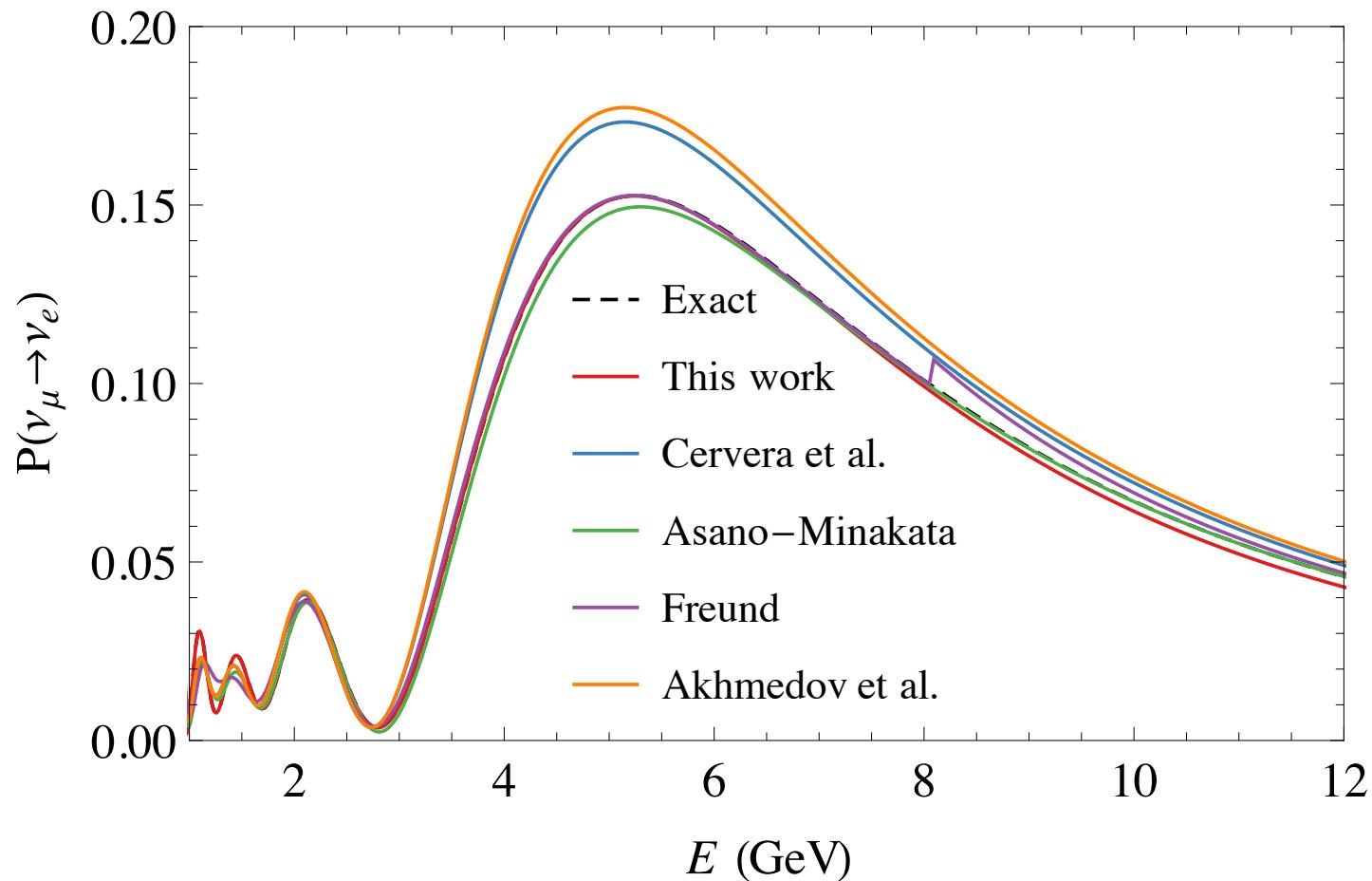


$L=810$ km (Fermilab \rightarrow NOvA)

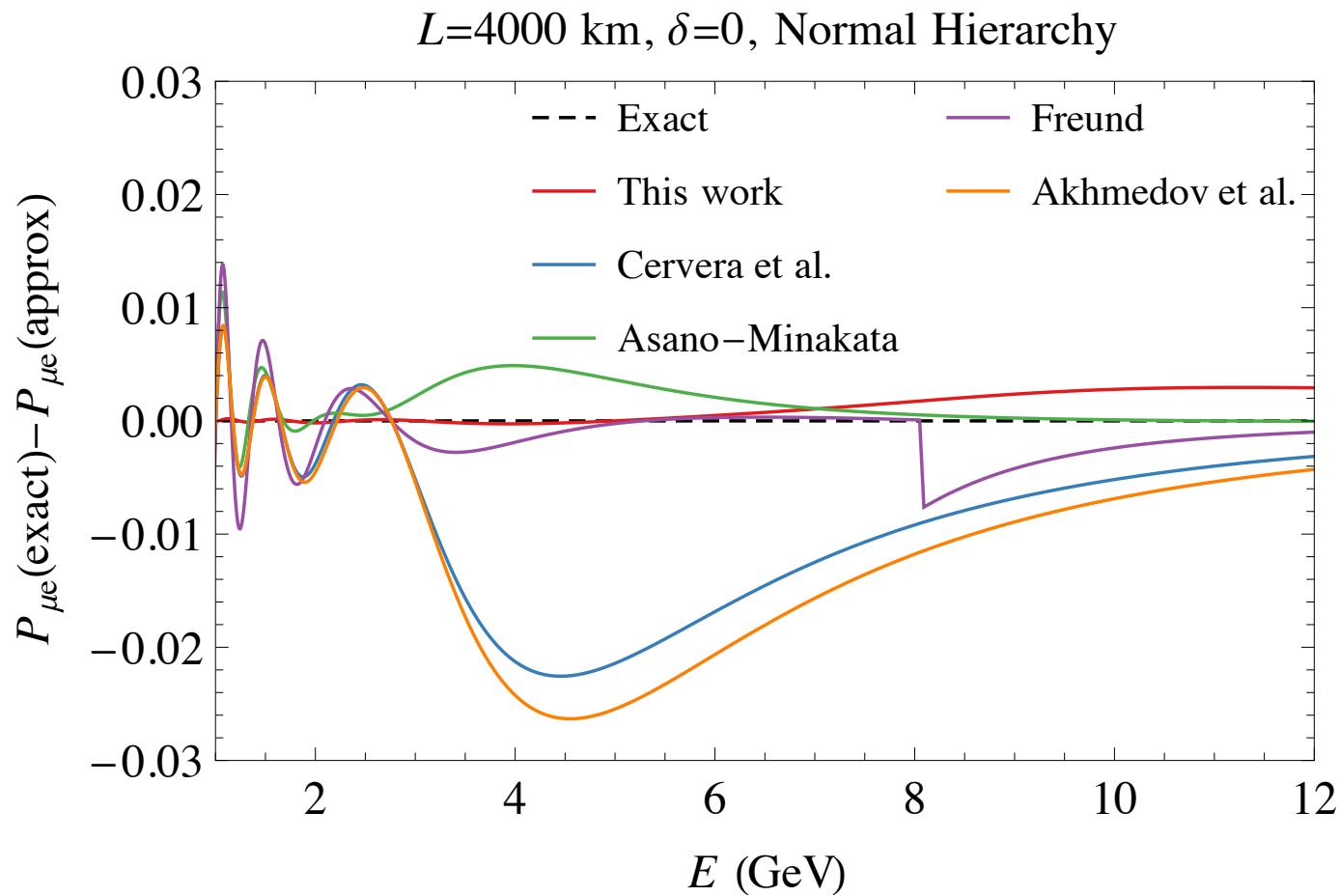


L=4000 km

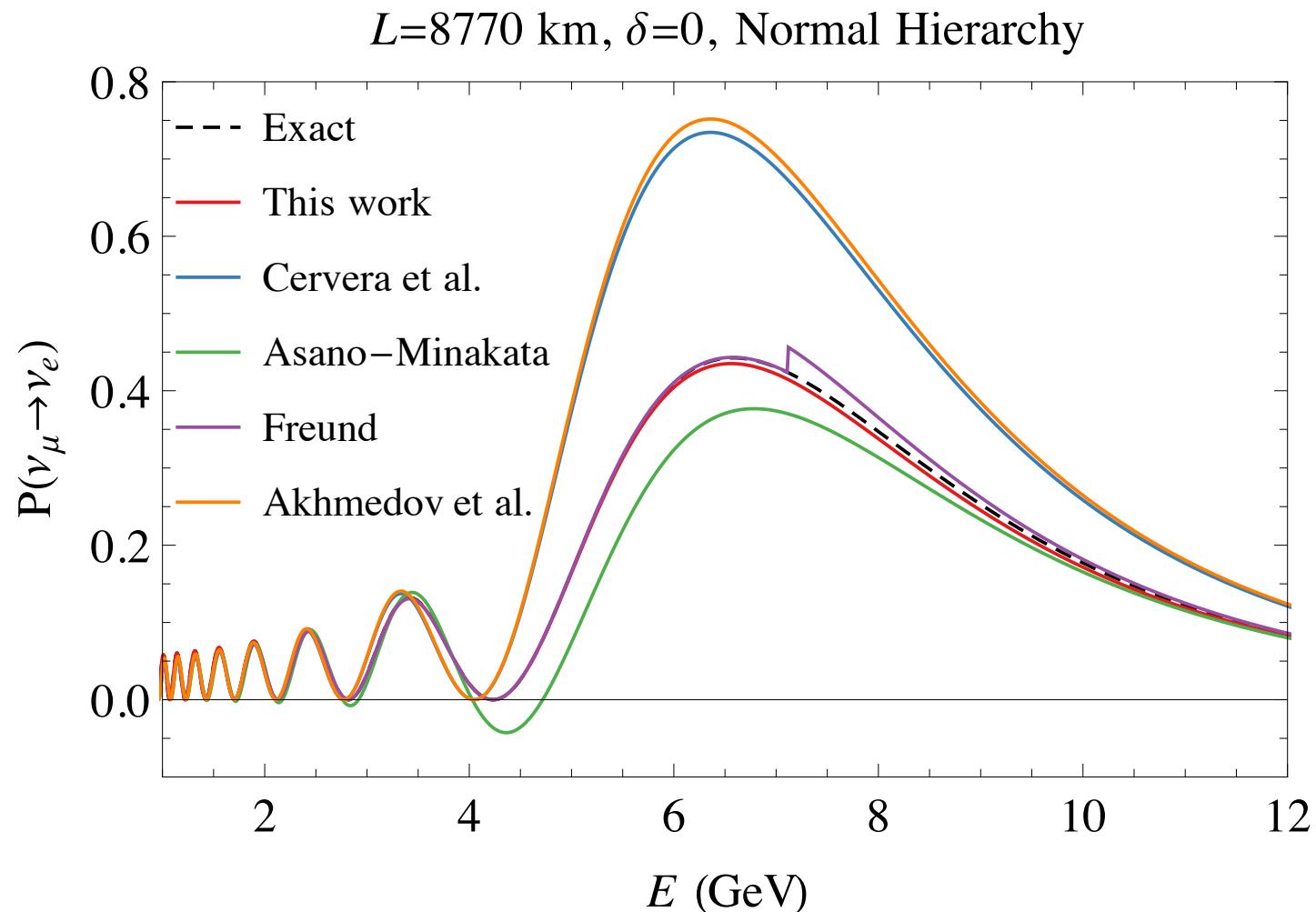
$L=4000 \text{ km}, \delta=0, \text{Normal Hierarchy}$



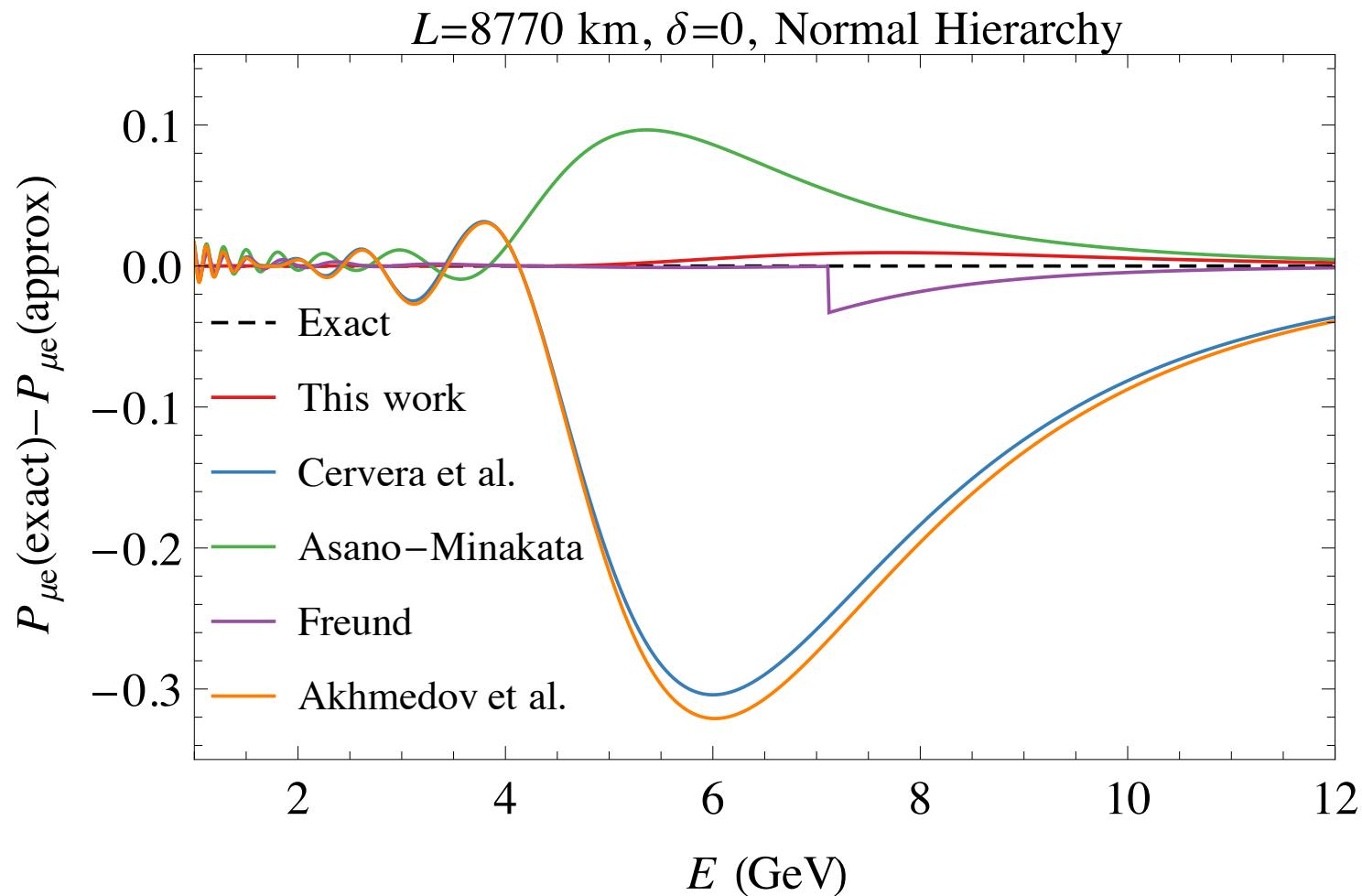
L=4000 km



L=8770 km (CERN→HyperK)



L=8770 km (CERN→HyperK)



Application : Mass Hierarchy Dependence

$$\begin{aligned}
& P(\nu_e \rightarrow \nu_e) \\
&= 1 - 4|\tilde{U}_{e2}|^2 \left(1 - |\tilde{U}_{e2}|^2 \right) \sin^2 \frac{\tilde{\Delta}_{21}}{2} - 4|\tilde{U}_{e3}|^2 \left(1 - |\tilde{U}_{e3}|^2 \right) \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + 2|\tilde{U}_{e2}|^2 |\tilde{U}_{e3}|^2 \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) \\
&= 1 - 4c'_{13}^2 s'_{12}^2 \left(1 - c'_{13}^2 s'_{12}^2 \right) \sin^2 \frac{\tilde{\Delta}_{21}}{2} - \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
&\quad + s'_{12}^2 \sin^2(2\theta'_{13}) \left(2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \frac{1}{2} \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) \\
&\xrightarrow{s'_{12} \approx 1} 1 - \sin^2(2\theta'_{13}) \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} + \sin^2 \frac{\tilde{\Delta}_{31}}{2} - 2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} - \frac{1}{2} \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) \\
&= 1 - \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{32}}{2}
\end{aligned}$$

Application : Mass Hierarchy Dependence

Similarly:

$$\begin{aligned}
 P(\nu_e \rightarrow \nu_\mu) &= 4 \left| \tilde{U}_{e2} \right|^2 \left| \tilde{U}_{\mu 2} \right|^2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4 \left| \tilde{U}_{e3} \right|^2 \left| \tilde{U}_{\mu 3} \right|^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
 &+ 2 \Re \left(\tilde{U}_{e3}^* \tilde{U}_{\mu 3} \tilde{U}_{e2} \tilde{U}_{\mu 2}^* \right) \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) + 4 \tilde{J}_{(e,\mu)} \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin \tilde{\Delta}_{31} - \sin^2 \frac{\tilde{\Delta}_{31}}{2} \sin \tilde{\Delta}_{21} \right) \\
 &\xrightarrow{s'_{12} \approx 1} s_{23}^2 \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{32}}{2}
 \end{aligned}$$

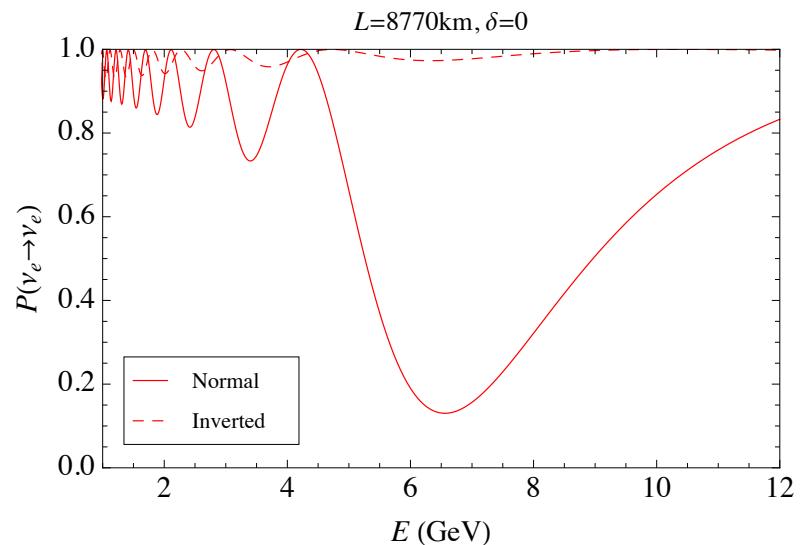
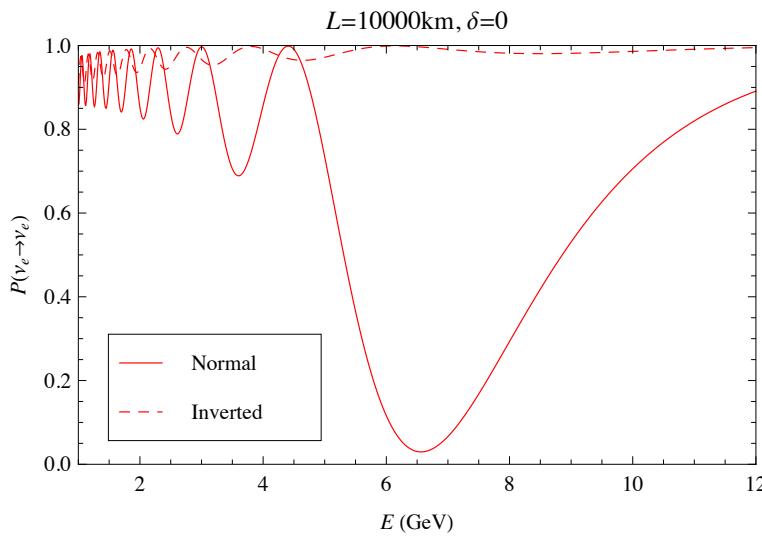
$$\begin{aligned}
 P(\nu_e \rightarrow \nu_\tau) &= 4 \left| \tilde{U}_{e2} \right|^2 \left| \tilde{U}_{\tau 2} \right|^2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4 \left| \tilde{U}_{e3} \right|^2 \left| \tilde{U}_{\tau 3} \right|^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2} \\
 &+ 2 \Re \left(\tilde{U}_{e3}^* \tilde{U}_{\tau 3} \tilde{U}_{e2} \tilde{U}_{\tau 2}^* \right) \left(4 \sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin^2 \frac{\tilde{\Delta}_{31}}{2} + \sin \tilde{\Delta}_{21} \sin \tilde{\Delta}_{31} \right) + 4 \tilde{J}_{(e,\tau)} \left(\sin^2 \frac{\tilde{\Delta}_{21}}{2} \sin \tilde{\Delta}_{31} - \sin^2 \frac{\tilde{\Delta}_{31}}{2} \sin \tilde{\Delta}_{21} \right) \\
 &\xrightarrow{s'_{12} \approx 1} c_{23}^2 \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{32}}{2}
 \end{aligned}$$

Application : Mass Hierarchy Dependence

Demand the oscillation term is maximized for normal hierarchy:

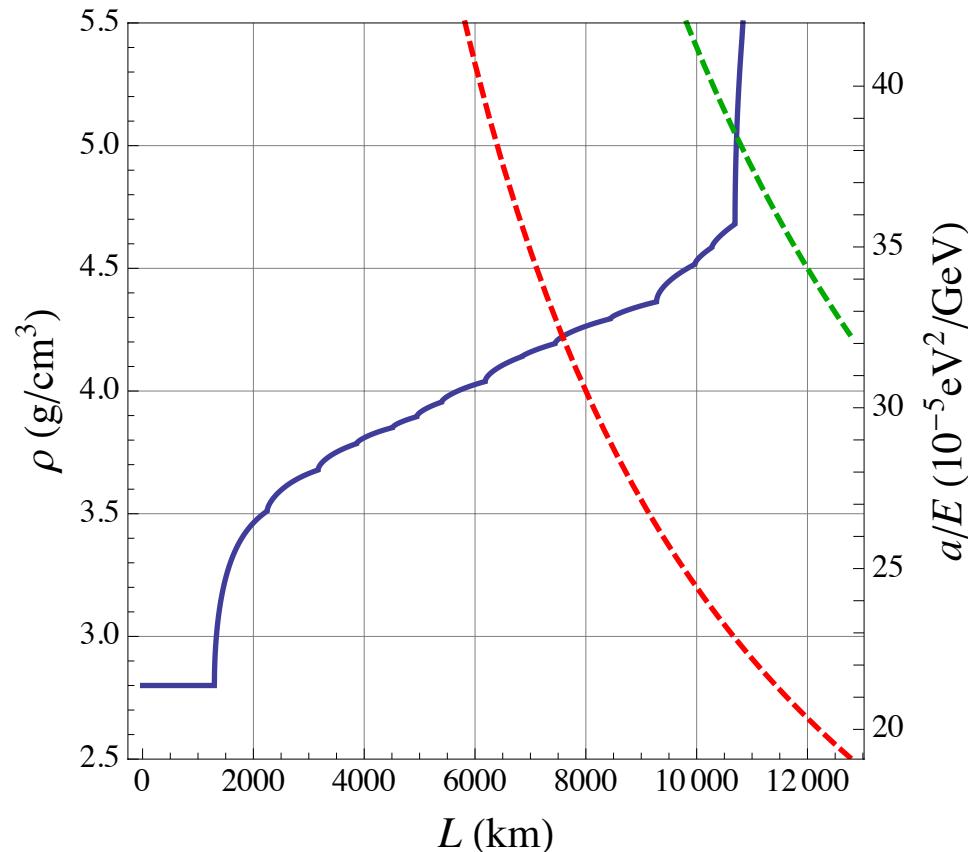
$$2\theta'_{13} \approx \frac{\pi}{2} \rightarrow a \approx \delta m_{31}^2, \quad \frac{\tilde{\Delta}_{32}}{2} \approx \frac{s_{13}\delta m_{31}^2}{2E} L \approx \frac{\pi}{2}$$

This is satisfied at $\rho L \approx 54000 \text{ km} \cdot \text{g/cm}^3$

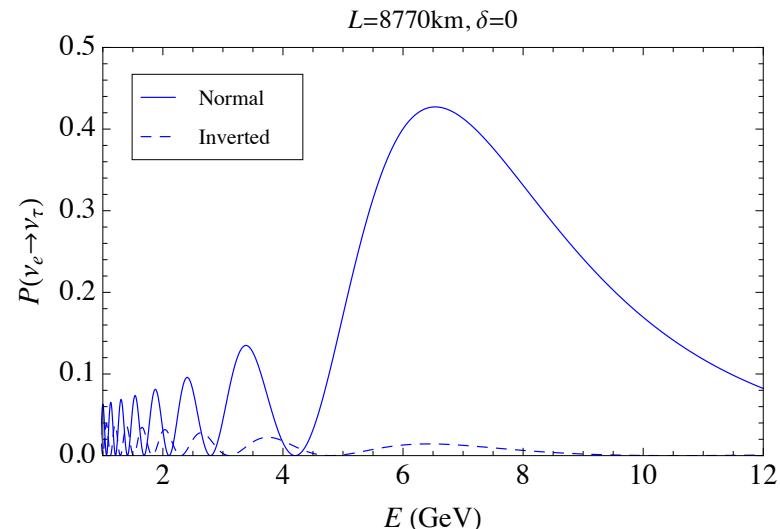
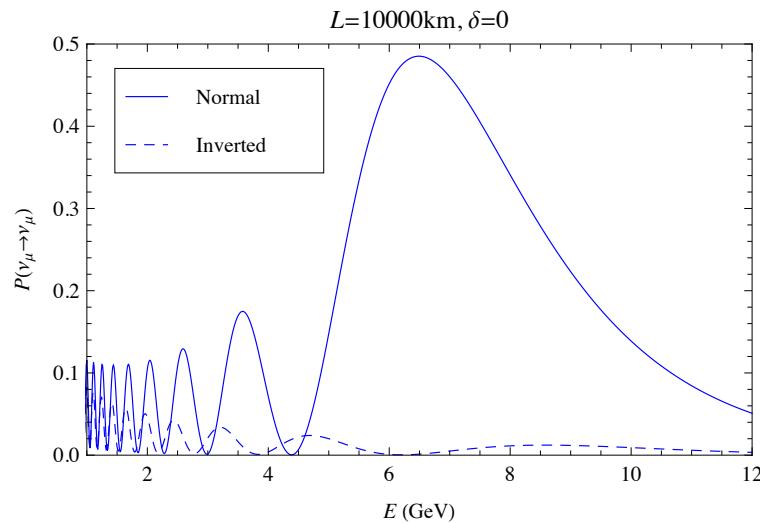
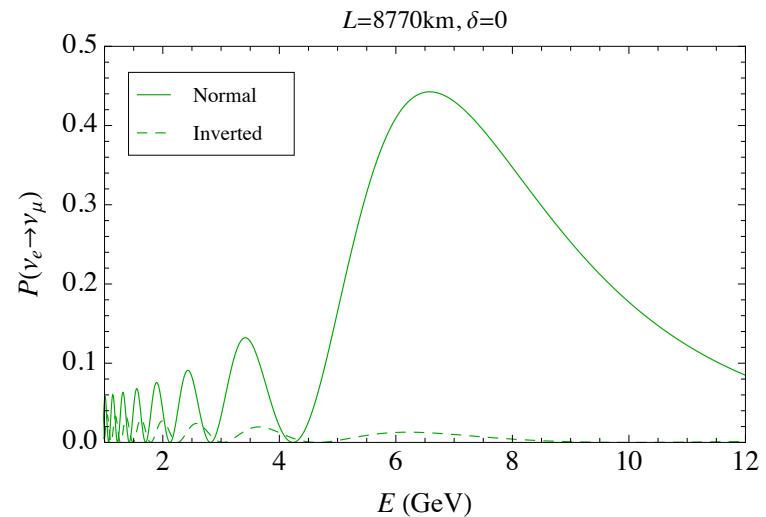
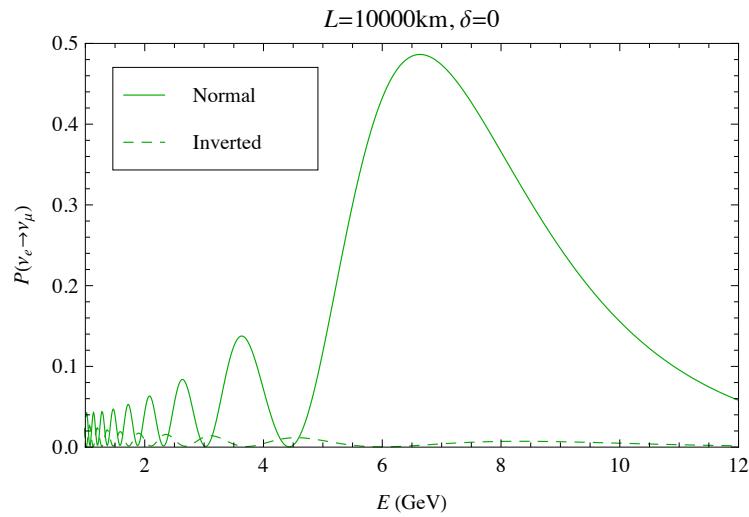


Matter Effect Parameter a

$$a = 2\sqrt{2}G_F N_e E = 7.63 \times 10^{-5} (\text{eV}^2) \left(\frac{\rho}{\text{g/cm}^3} \right) \left(\frac{E}{\text{GeV}} \right)$$



Application : Mass Hierarchy Dependence



With NSI's

Non-Standard Interactions of the neutrino

Standard Model (SM) W-exchange interaction:

$$L = -2\sqrt{2}G_F (\bar{\nu}_e \gamma P_L \nu_e)(\bar{e} \gamma P_L e)$$

Non-Standard Interactions (NSI's) of the neutrino:

$$L = -2\sqrt{2}G_F \sum_{\alpha,\beta,f} \varepsilon_{\alpha\beta}^{fC} (\bar{\nu}_\alpha \gamma_\mu P_L \nu_\beta)(\bar{f} \gamma^\mu P_C f) \quad \varepsilon_{\beta\alpha}^{fC} = (\varepsilon_{\alpha\beta}^{fC})^*$$
$$\alpha, \beta = e, \mu, \tau, \quad f = e, u, d, \quad C = L, R$$

These NSI's can be generated in a variety of new physics models.

Neutrino Oscillation in Matter : with NSI's

$$H = U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + a \begin{bmatrix} 1 + \varepsilon_{ee} & \varepsilon_{e\mu} & \varepsilon_{e\tau} \\ \varepsilon_{e\mu}^* & \varepsilon_{\mu\mu} & \varepsilon_{\mu\tau} \\ \varepsilon_{e\tau}^* & \varepsilon_{\mu\tau}^* & \varepsilon_{\tau\tau} \end{bmatrix} = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger,$$

where

$$\varepsilon_{\alpha\beta} = \sum_f \varepsilon_{\alpha\beta}^{fV} \frac{N_f}{N_e} = \sum_f (\varepsilon_{\alpha\beta}^{fL} + \varepsilon_{\alpha\beta}^{fR}) \frac{N_f}{N_e}$$

On the Earth

$$\varepsilon_{\alpha\beta} \approx \varepsilon_{\alpha\beta}^{eV} + 3\varepsilon_{\alpha\beta}^{uV} + 3\varepsilon_{\alpha\beta}^{dV}$$

Can the oscillation probabilities be approximated by simple functions of a in the presence/absence of the NSI's?

Case 1: Flavor-Diagonal NSI's

S. K. Agarwalla, Y. Kao, D. Saha, & TT, arXiv:1506.08464

$$\begin{aligned}
 H_\eta &= U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + a \begin{bmatrix} 1 + \varepsilon_{ee} & 0 & 0 \\ 0 & \varepsilon_{\mu\mu} & 0 \\ 0 & 0 & \varepsilon_{\tau\tau} \end{bmatrix} \\
 &= U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + a \underbrace{\begin{pmatrix} 1 + \varepsilon_{ee} - \frac{\varepsilon_{\mu\mu} + \varepsilon_{\tau\tau}}{2} \\ \xi \end{pmatrix}}_{\hat{a}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}}{2}\right) & 0 \\ 0 & 0 & -\left(\frac{\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}}{2}\right) \end{bmatrix} + \dots \\
 &= \underbrace{U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger}_{H_{\hat{a}}} + \hat{a} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \hat{a} \eta \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{M_\eta}, \quad \eta = \left(\frac{\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}}{2}\right) \ll 1.
 \end{aligned}$$

Case 2: Flavor-Non-Diagonal NSI's

S. K. Agarwalla, Y. Kao, C. Sun, & TT, arXiv:1507.?????

$$\begin{aligned}
 H_{\varepsilon\omega} &= U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \varepsilon_{\mu\tau} \\ 0 & \varepsilon_{\mu\tau}^* & 0 \end{bmatrix} \\
 &= \underbrace{U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger}_{H_a} + a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a |\varepsilon_{\mu\tau}| \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{i\omega} \\ 0 & e^{-i\omega} & 0 \end{bmatrix}}_{M_{\varepsilon\omega}},
 \end{aligned}$$

Existing Bounds @90% C.L. (1.64σ) :

- M. Maltoni, Nucl.Phys.Proc.Suppl. 114 (2003) 191 [hep-ph/0210111]

$$-0.05 < \varepsilon_{\mu\tau} < 0.03, \quad |\eta| = \left| \frac{\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}}{2} \right| < 0.04 \quad (\text{Solar \& Atmos})$$

- M. Gonzalez-Garcia, M. Maltoni, & J. Salvado, JHEP 1105 (2011) 075 [arXiv:1103.4365]

$$|\varepsilon_{\mu\tau}| < 0.035, \quad |\eta| = \left| \frac{\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}}{2} \right| < 0.055 \quad (\text{Atmos \& MINOS})$$

- A. Esmaili & A. Y. Smirnov, JHEP 1306 (2013) 026 [arXiv:1304.1042]

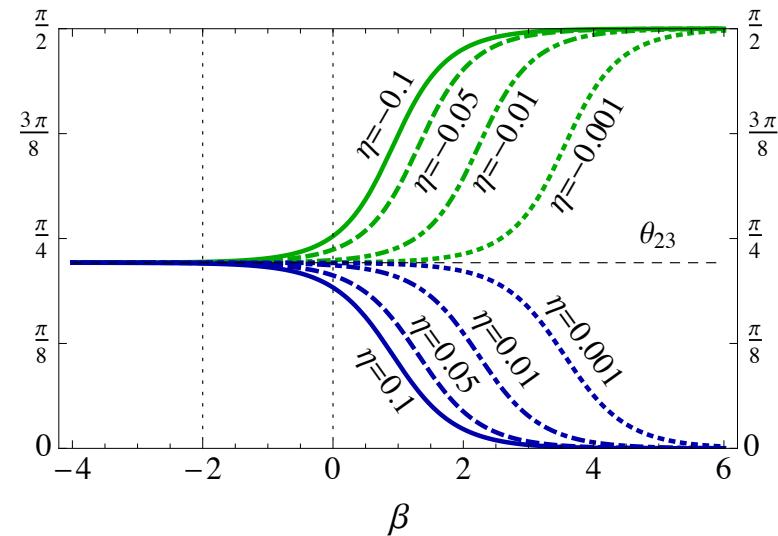
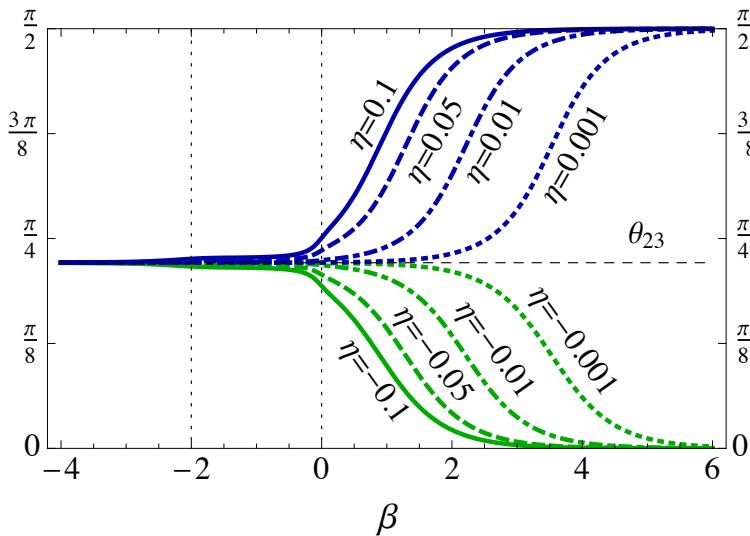
$$-0.0061 < \varepsilon_{\mu\tau} < 0.0056, \quad |\eta| = \left| \frac{\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}}{2} \right| < 0.02 \quad (\text{IceCube DeepCore})$$

Case 1 : “Running” Effective Mixing Angles

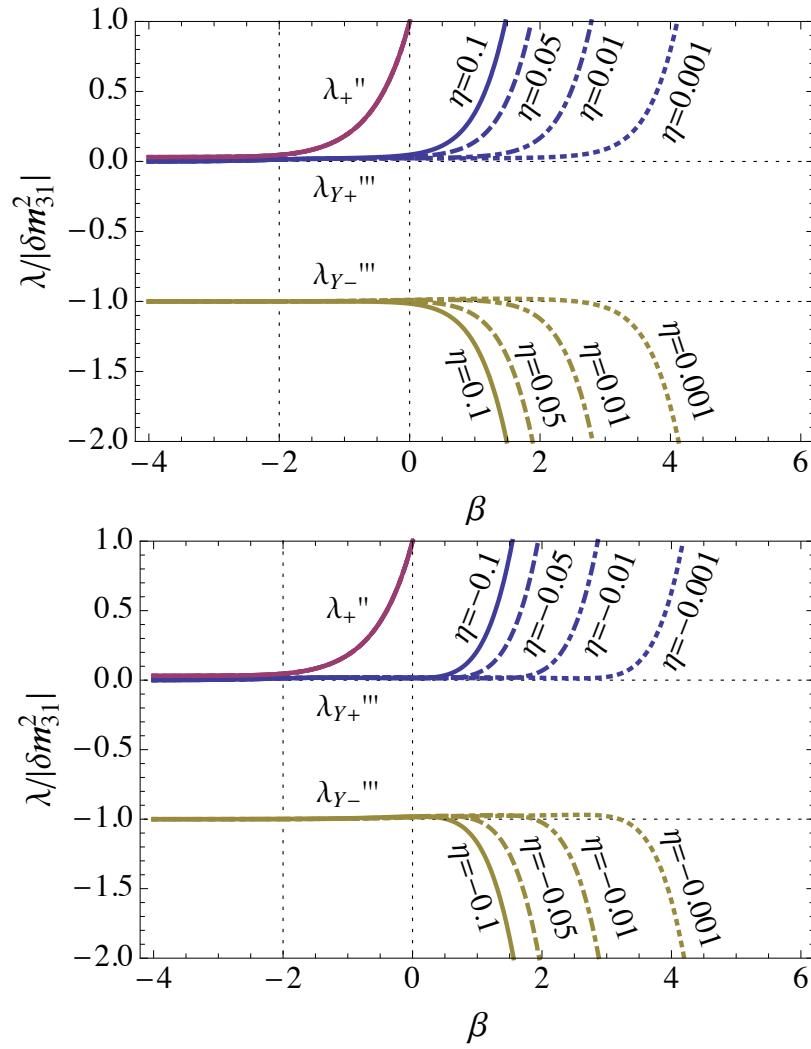
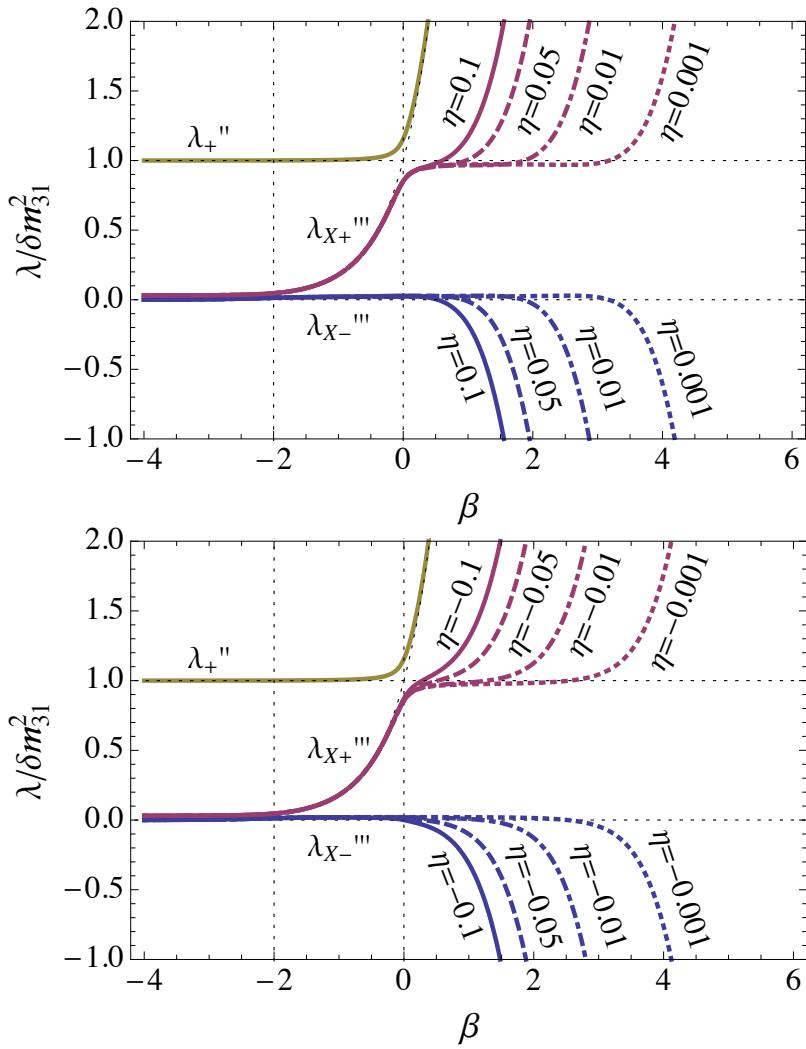
$$\tan 2\theta'_{12} = \frac{(\delta m_{21}^2 / c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) \cos 2\theta_{12} - \hat{a}},$$

$$\tan 2\theta'_{13} = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - \hat{a}},$$

$$\tan 2\theta'_{23} = \frac{[\delta m_{31}^2 c_{13}^2 - \delta m_{21}^2 (c_{12}^2 - s_{12}^2 s_{13}^2)] \sin 2\theta_{23}}{[\delta m_{31}^2 c_{13}^2 - \delta m_{21}^2 (c_{12}^2 - s_{12}^2 s_{13}^2)] \cos 2\theta_{23} - 2\hat{a}\eta}$$



Case 1: “Running” Effective Mass-squares :

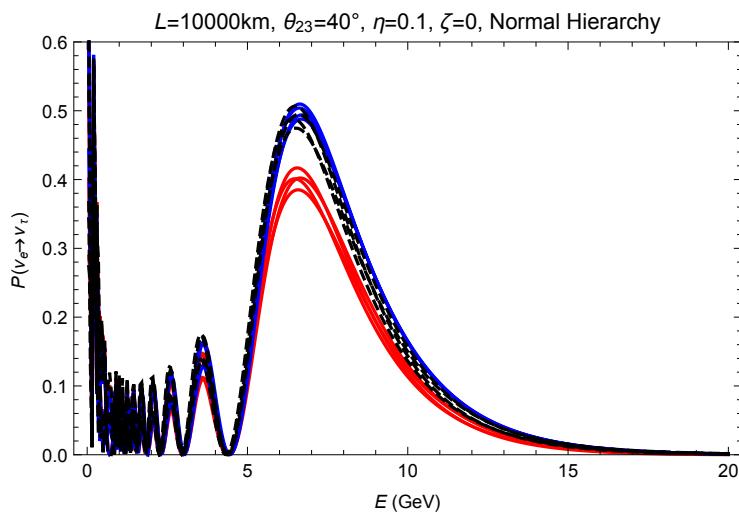


Case 1 : Applications (NH) :

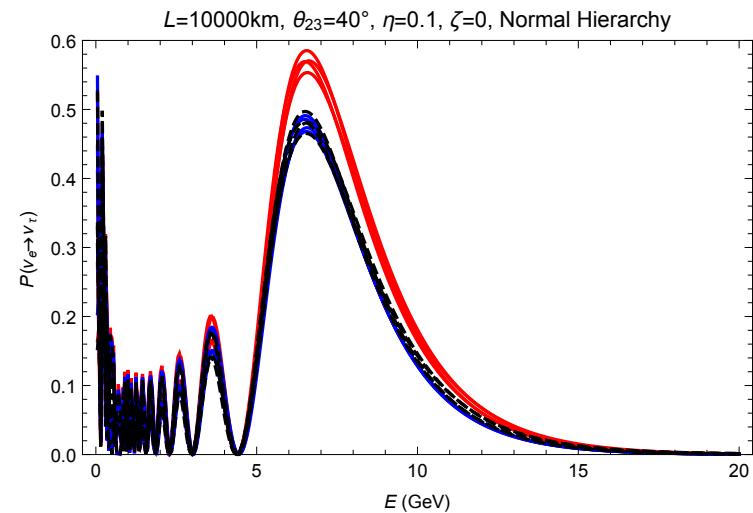
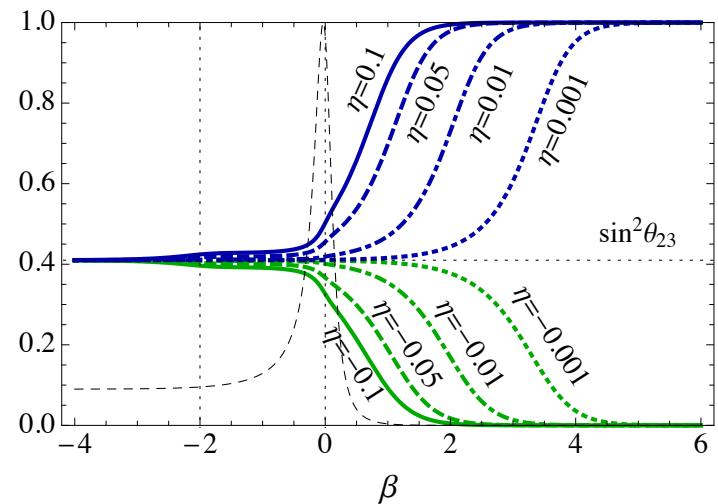
$$\tilde{P}(\nu_e \rightarrow \nu_e) \approx 1 - \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{32}}{2}$$

$$\tilde{P}(\nu_e \rightarrow \nu_\mu) \approx s'^2_{23} \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{32}}{2}$$

$$\tilde{P}(\nu_e \rightarrow \nu_\tau) \approx c'^2_{23} \sin^2(2\theta'_{13}) \sin^2 \frac{\tilde{\Delta}_{32}}{2}$$



$$s'^2_{23} \approx s^2_{23} + \eta \text{ at oscillation peak}$$

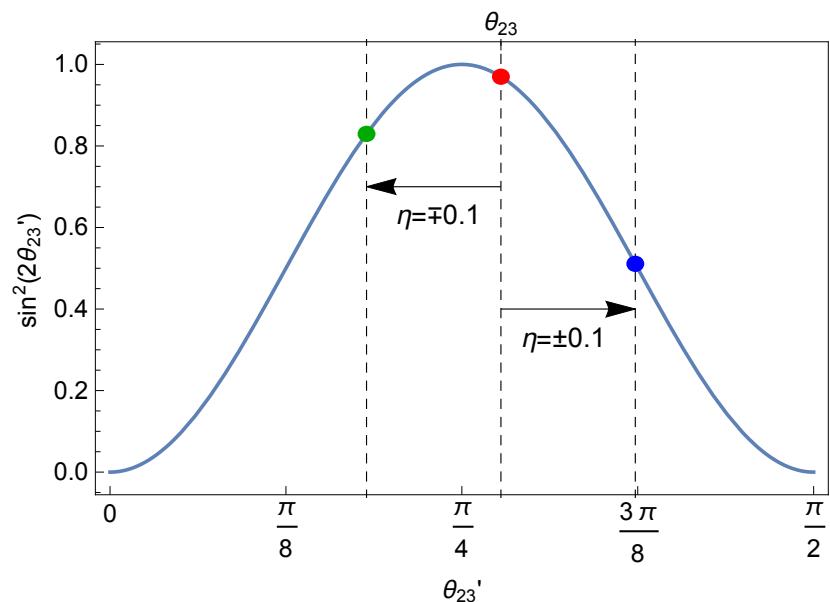
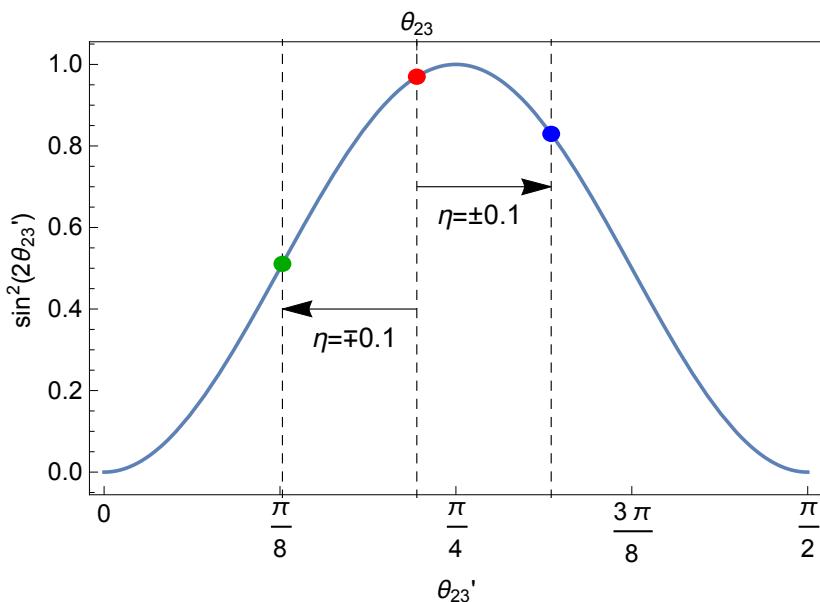
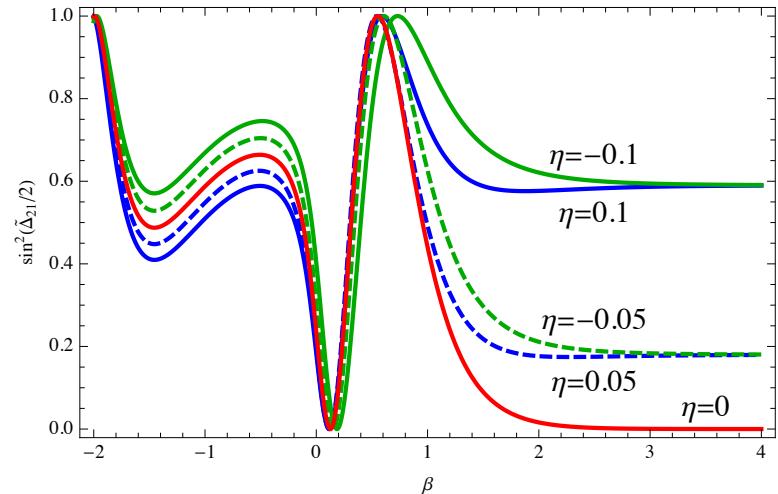


Case 1 : Applications (NH) :

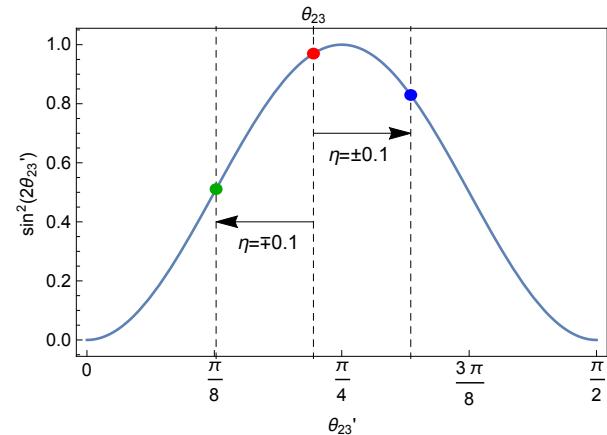
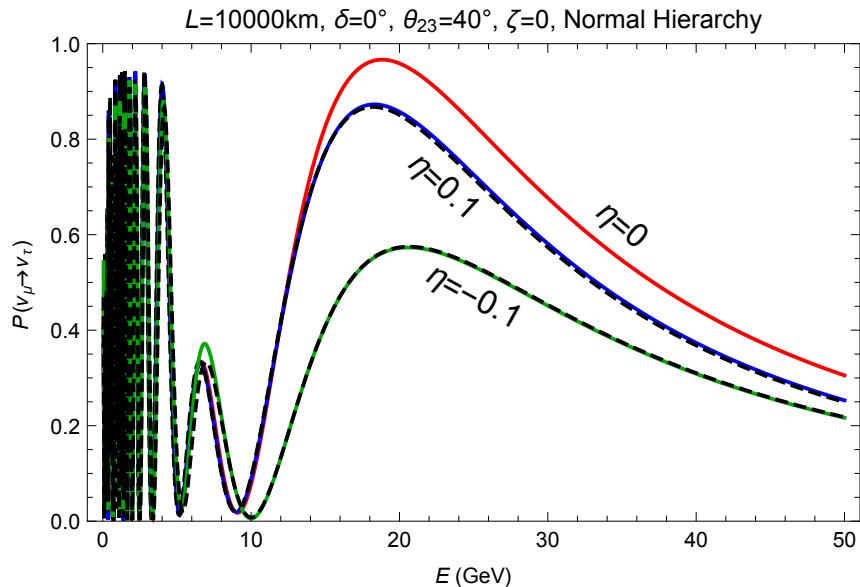
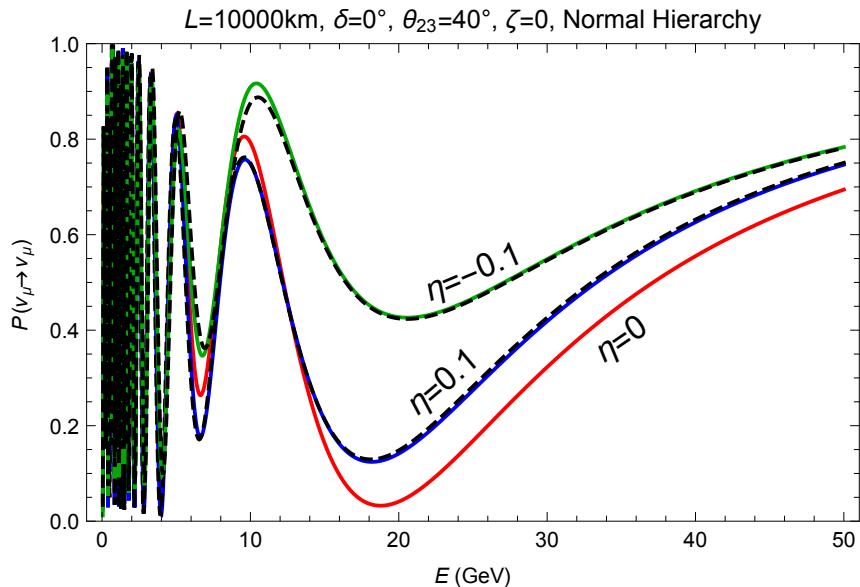
$$s'^2_{23} \approx s^2_{23} + 3\eta \text{ at oscillation peak}$$

$$\tilde{P}(\nu_\mu \rightarrow \nu_\mu) \approx 1 - \sin^2(2\theta'_{23}) \sin^2 \frac{\tilde{\Delta}_{21}}{2}$$

$$\tilde{P}(\nu_\mu \rightarrow \nu_\tau) \approx \sin^2(2\theta'_{23}) \sin^2 \frac{\tilde{\Delta}_{21}}{2}$$



Case 1 : Applications:



Case 2 : “Running” Effective Mixing Angles

$\omega = 0$ case:

$$\tan 2\theta'_{12} = \frac{(\delta m_{21}^2 / c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) \cos 2\theta_{12} - \hat{a}},$$

$$\tan 2\theta'_{13} = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - \hat{a}},$$

$$\tan 2\theta'_{23} = \frac{[\delta m_{31}^2 c_{13}^2 - \delta m_{21}^2 (c_{12}^2 - s_{12}^2 s_{13}^2)] \sin 2\theta_{23} + 2a |\varepsilon_{\mu\tau}|}{[\delta m_{31}^2 c_{13}^2 - \delta m_{21}^2 (c_{12}^2 - s_{12}^2 s_{13}^2)] \cos 2\theta_{23}}$$

Case 2 : “Running” Effective Mixing Angles

$\omega \neq 0$ case:

$$\tan 2\theta'_{12} = \frac{(\delta m_{21}^2 / c_{13}^2) \sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) \cos 2\theta_{12} - \hat{a}}, \quad \tan 2\theta'_{13} = \frac{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{21}^2 s_{12}^2) \cos 2\theta_{13} - \hat{a}},$$

$$c'_{23} = \sqrt{c_{23}^2 c_\chi^2 - 2 c_{23} s_{23} c_\chi s_\chi \cos \Omega + s_{23}^2 s_\chi^2}$$

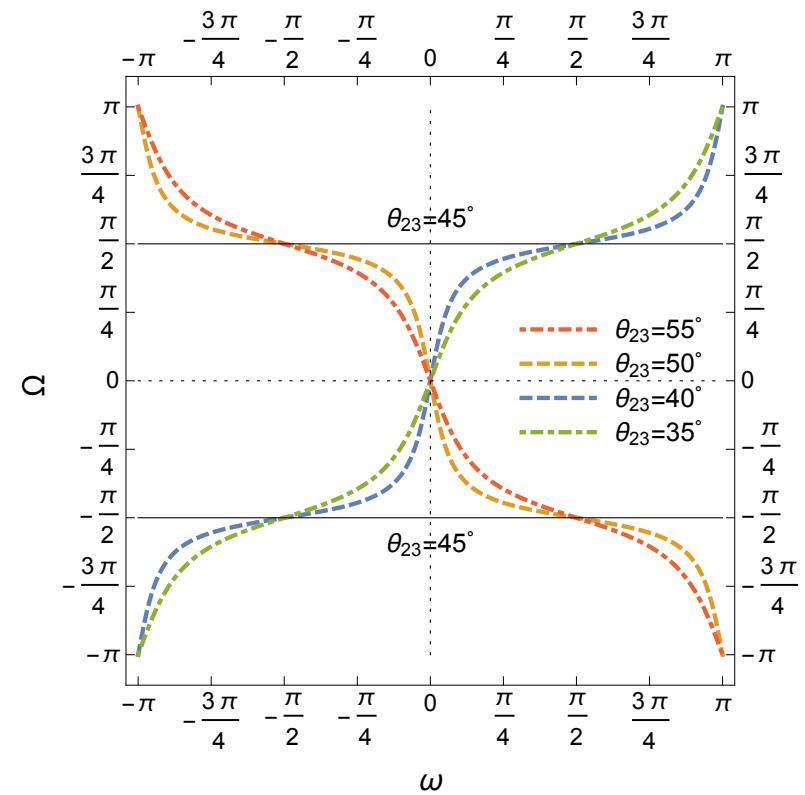
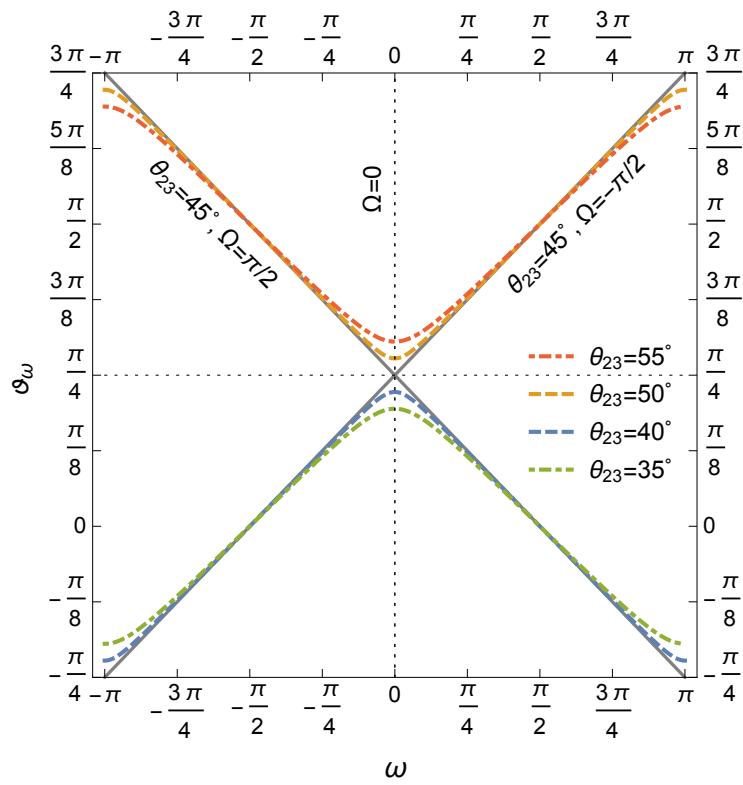
$$\delta' = \delta + \Omega + \arctan \left[- \frac{\sin(2\theta_{23}) \sin \Omega}{\sin(2\chi) \cos(2\theta_{23}) + \cos(2\chi) \sin(2\theta_{23}) \cos \Omega} \right]$$

$$\tan 2\chi = \frac{2a |\varepsilon_{\mu\tau}| \cos 2\vartheta_\omega}{[\delta m_{31}^2 c_{13}^2 - \delta m_{21}^2 (c_{12}^2 - s_{12}^2 s_{13}^2)] + 2a |\varepsilon_{\mu\tau}| \sin 2\vartheta_\omega}$$

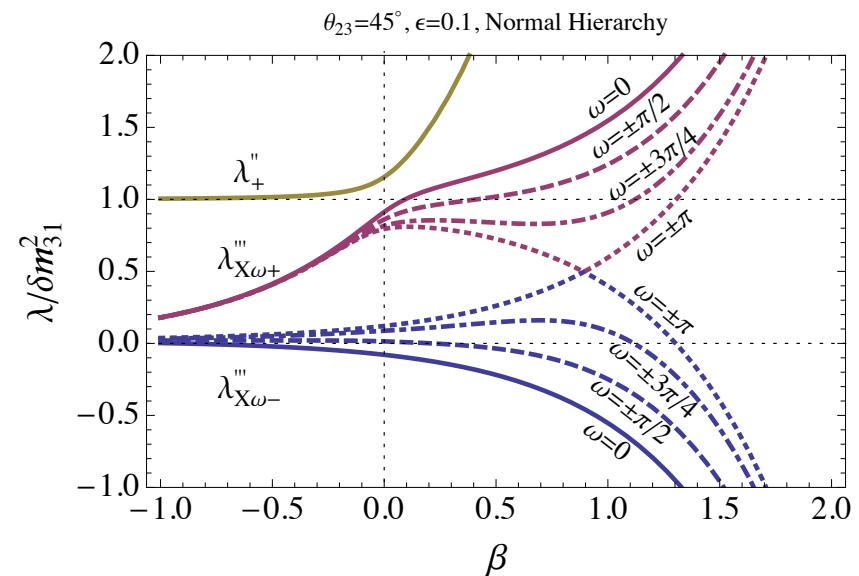
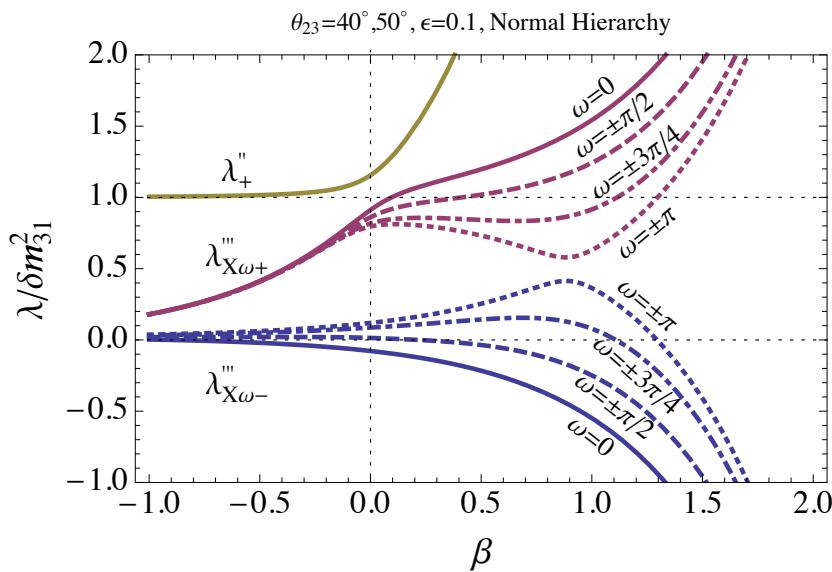
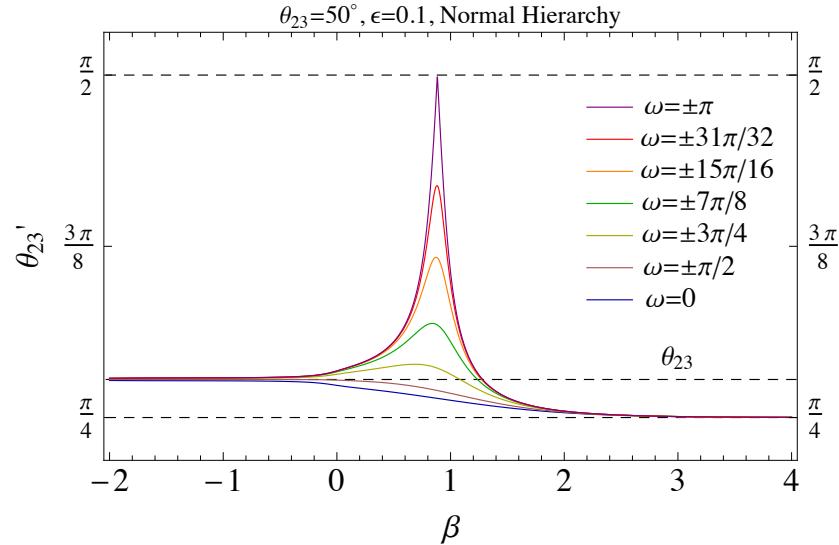
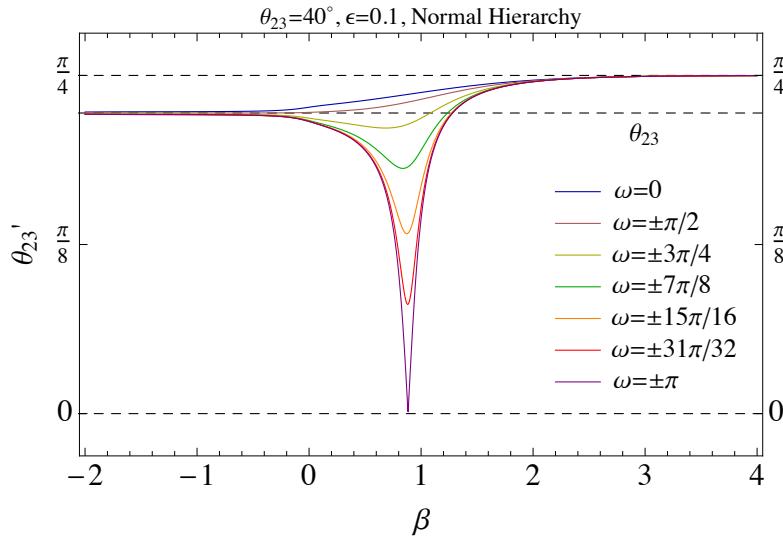
$$\sin(2\vartheta_\omega) = \sin(2\theta_{23}) c_\omega, \quad e^{i\Omega} \cos(2\vartheta_\omega) = \cos(2\theta_{23}) c_\omega + i s_\omega.$$

Case 2: Intermediate Parameters :

$$\sin(2\theta_\omega) = \sin(2\theta_{23})c_\omega, \quad e^{i\Omega} \cos(2\theta_\omega) = \cos(2\theta_{23})c_\omega + i s_\omega.$$

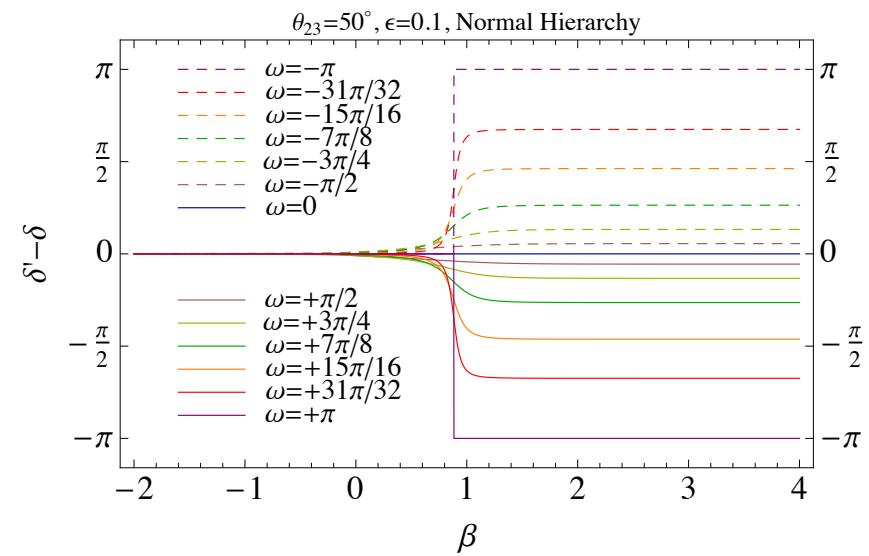
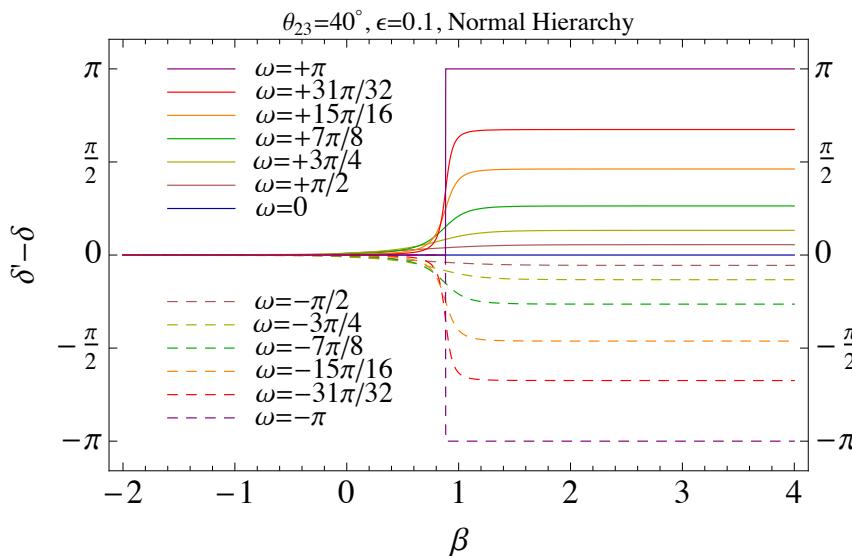


Case 2: Normal Hierarchy Case :

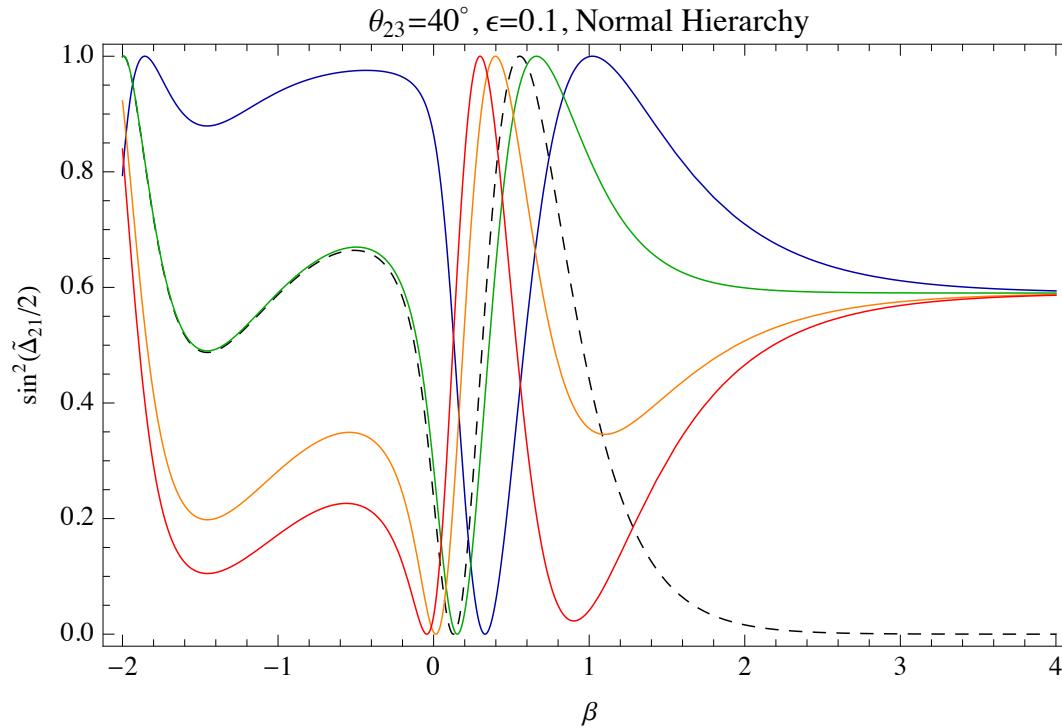


Case 2 : Running of δ' :

$$\delta' = \delta + \Omega + \arctan \left[-\frac{\sin(2\theta_{23}) \sin \Omega}{\sin(2\chi_\omega) \cos(2\theta_{23}) + \cos(2\chi_\omega) \sin(2\theta_{23}) \cos \Omega} \right]$$



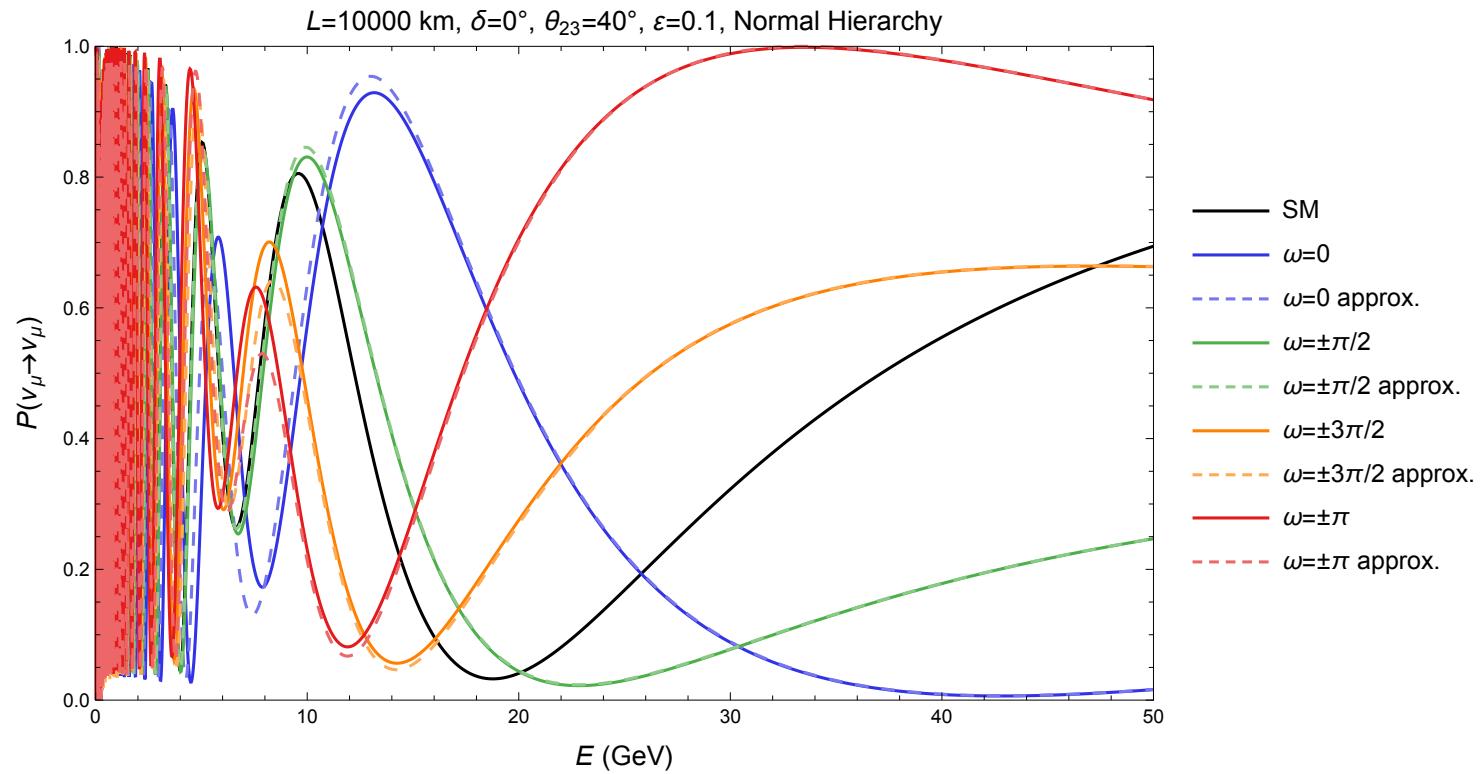
Case 2 : Running of Effective $\tilde{\Delta}_{21}$:



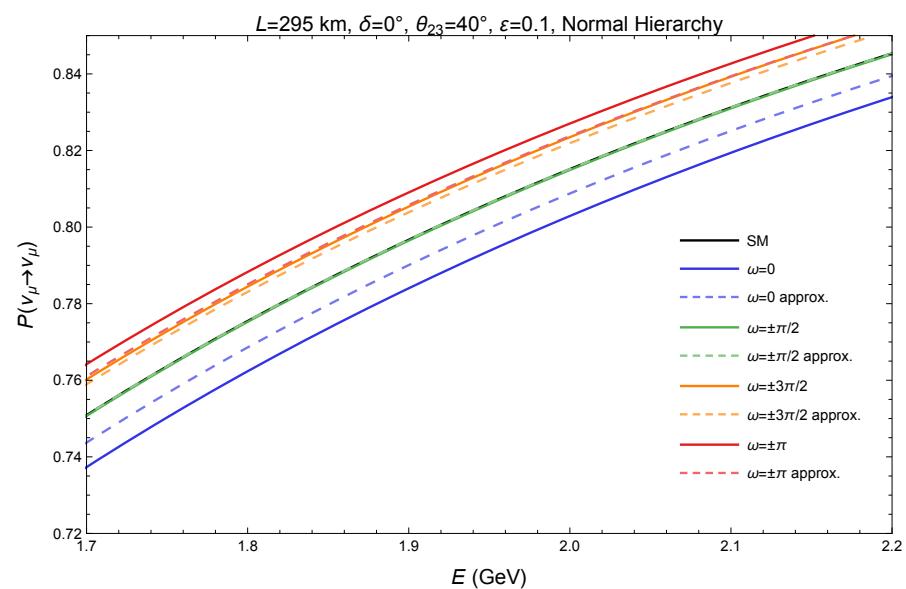
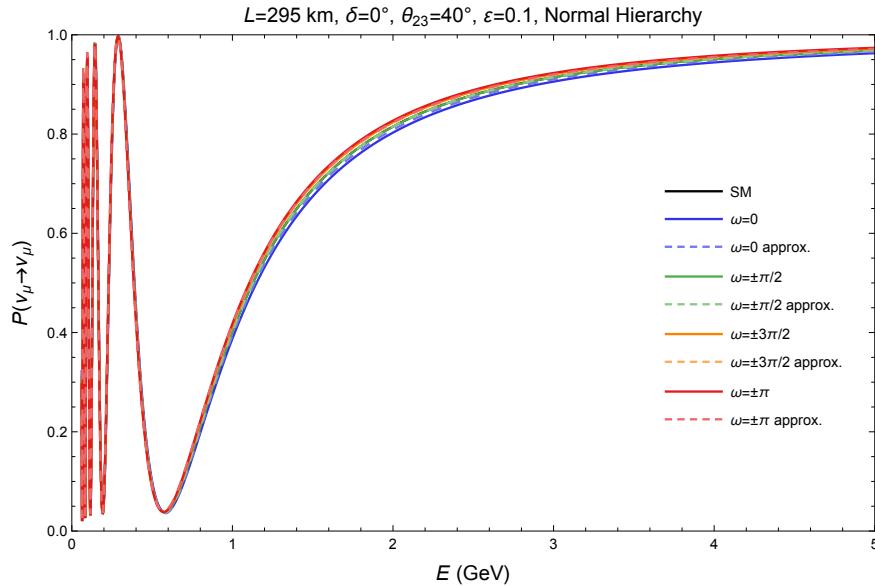
$$\tilde{P}(\nu_\mu \rightarrow \nu_\mu) \approx 1 - \sin^2(2\theta'_{23}) \sin^2 \frac{\tilde{\Delta}_{21}}{2}$$

$$\tilde{P}(\nu_\mu \rightarrow \nu_\tau) \approx \sin^2(2\theta'_{23}) \sin^2 \frac{\tilde{\Delta}_{21}}{2}$$

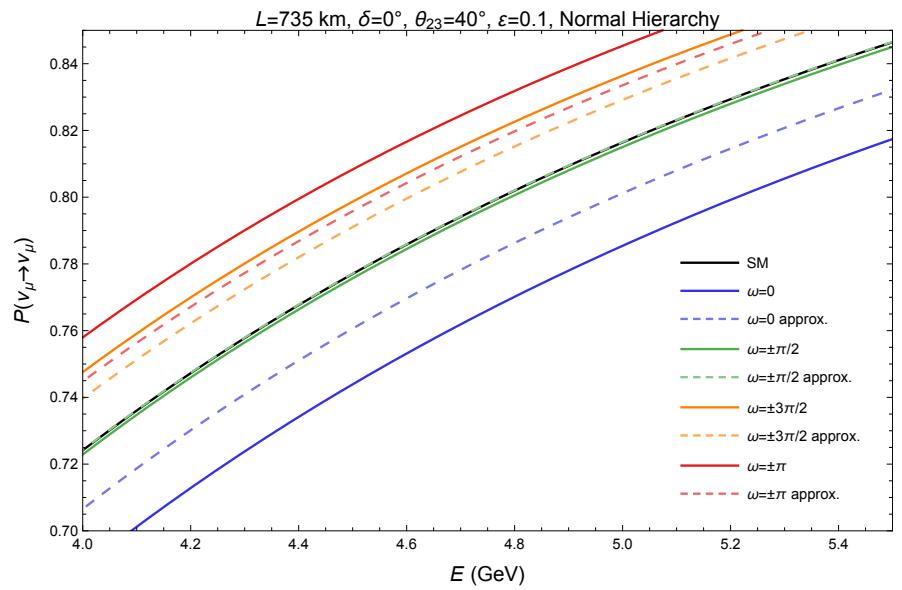
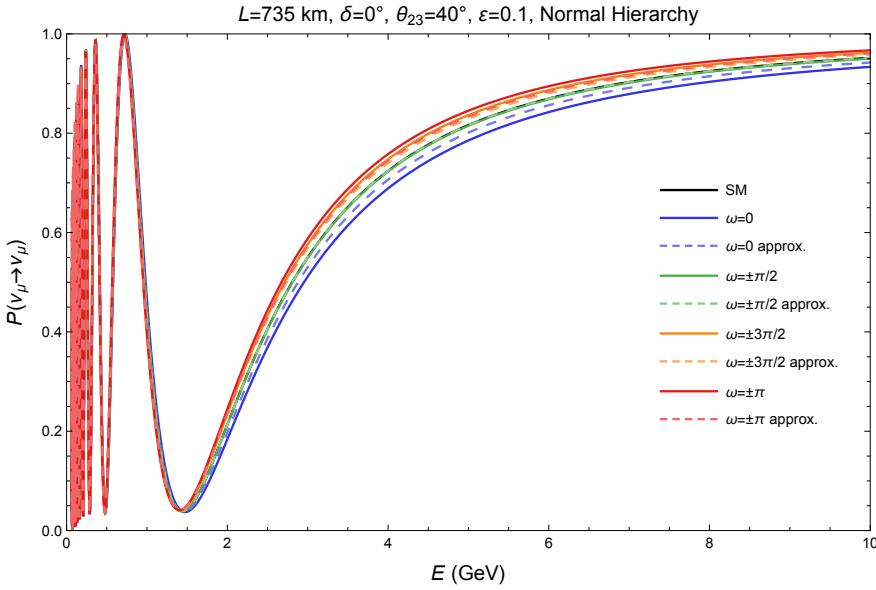
Case 2 : Oscillation Probabilities:



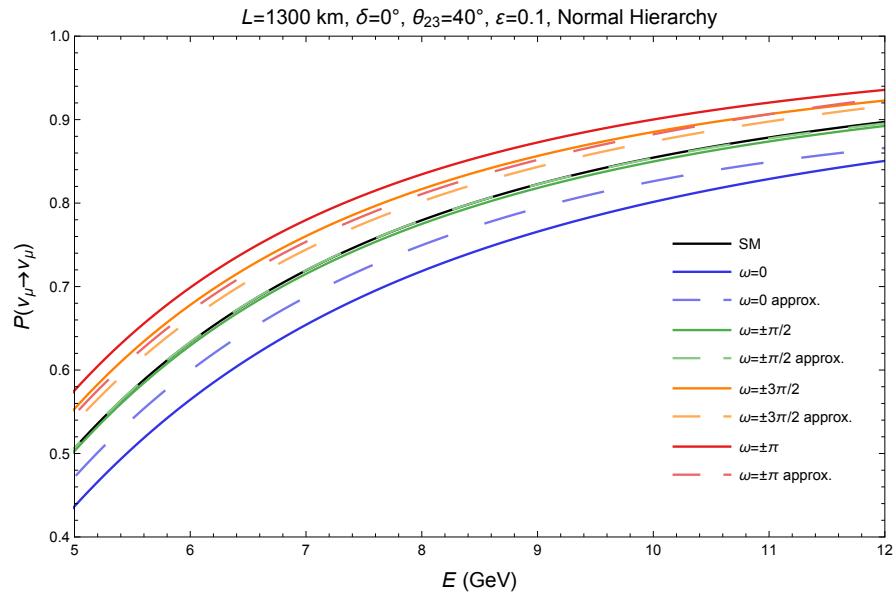
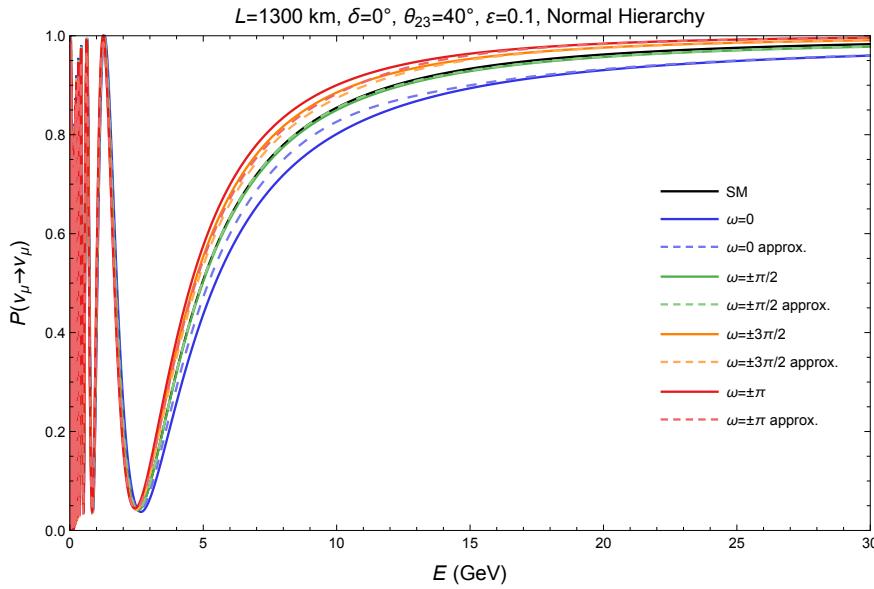
Case 2 : T2K



Case 2 : MINOS



Case 2 : DUNE



Conclusion:

- Matter effect is best understood as due to the “running” of the effective mixing angles and mass-squared differences.
- For the **SM** W-exchange interaction, only θ_{12} and θ_{13} run with $a = 2\sqrt{2}G_F N_e E$
- For **NSI**’s in the $\mu\tau$ sector, the running due to the NSI’s is confined to θ_{23} and δ
- The bound on the **absolute value** of $\varepsilon_{\mu\tau}$ depends crucially on its **phase**