Monodromy and entanglement in Dicke superradiance models

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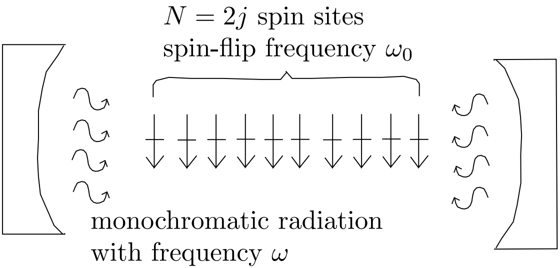
March 9, 2017

Model of atom-field interaction in a cavity



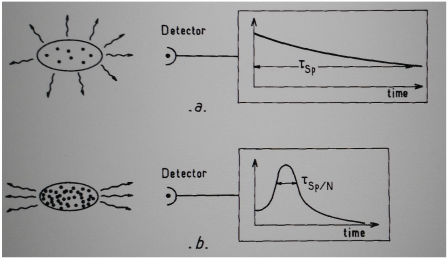


Figure: From Gross, Haroche, Physics reports 93.5 (1982): 301-396.

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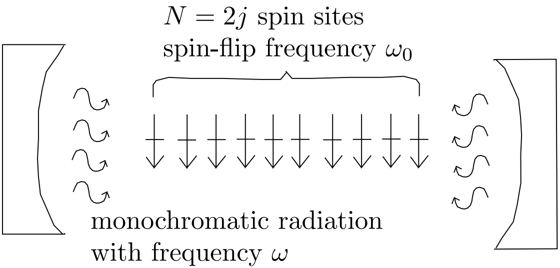

Dicke Hamiltonian

$$H_{\mathrm{D}} = \omega_0 J_0 + \omega b^\dagger b + \frac{\lambda}{\sqrt{2j}} \left(b + b^\dagger\right)\left(J_- + J_+\right)$$

R. H. Dicke, Phys. Rev. 93 (1954) 99

Tavis-Cummings Hamiltonian

$$H_{\mathrm{TC}} = \omega_0 J_0 + \omega b^\dagger b + \frac{\lambda}{\sqrt{2j}} \left(b J_+ + b^\dagger J_-\right)$$

Tavis, Cummings, Phys. Rev. 170 (1968) 379

'*collective*' operators:

$$J_+ = \sum_{i=1}^{2j} a^+_{\uparrow i} a_{\downarrow i},$$

$$J_- = \sum_{i=1}^{2j} a^+_{\downarrow i} a_{\uparrow i}$$

$$J_0 = \frac{1}{2} \sum_{i=1}^{2j} \left(a^+_{\uparrow i} a_{\uparrow i} - a^+_{\downarrow i} a_{\downarrow i}\right),$$

Superradiant phase transition

Quantum phase transition (QPT) - an abrupt change in properties of the **ground state** which becomes non-analytic in the thermodynamic limit $N = 2j \to \infty$

K. Hepp, E.H. Lieb, Phys. Rev. A 8, 2517 (1973)

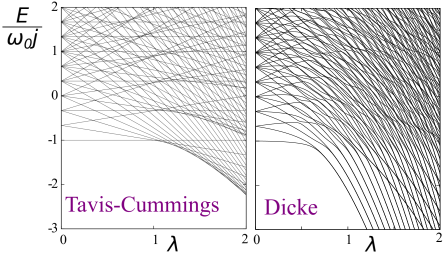

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = \omega = 1$

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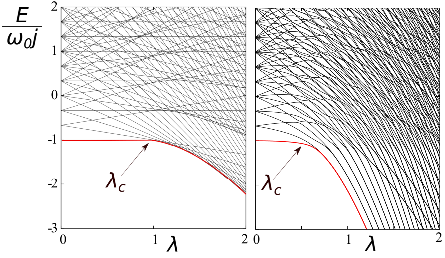

For $\lambda > \lambda_c$ (critical coupling) both matter and field aquire macroscopic excitation.

$\langle N_\gamma \rangle > 0$ & $\langle J_z \rangle + j > 0$

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = \omega = 1$

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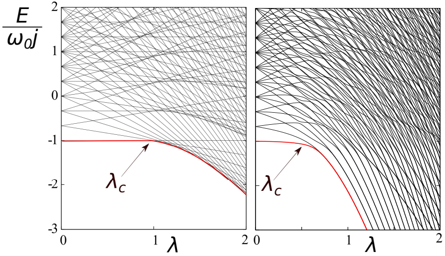

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = \omega = 1$

Experimental realization of QPT of the Dicke Hamiltonian using superfluid gases in optical cavity

K. Baumann et al, Phys. Rev. Lett. 107, 140402 (2011)

Superradiant phase transition

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K. Hepp, E.H. Lieb, Phys. Rev. A 8, 2517 (1973)

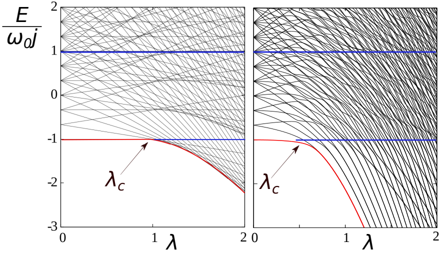

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = \omega = 1$

Singularities in level density – here in the first derivative of level density – as a generalization of QPTs → **Excited-state quantum phase transitions** (ESQPTs)

Tavis-Cummings regime

$H_{\mathrm{TC}} = \omega_0 J_0 + \omega b^\dagger b + \frac{\lambda}{\sqrt{2j}} \left(b J_+ + b^\dagger J_-\right)$

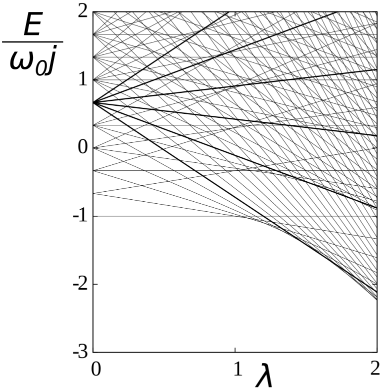

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = \omega = 1$

- Additional conserved quantity $M = \overbrace{b^\dagger b}^{n} + \overbrace{J_z + j}^{n_{ex}}$ counting the total number of field bosons and atomic excitations $\rightsquigarrow$ **integrable** system
- Level dynamics splits into invariant subspaces numbered by M which do not interact with each other.

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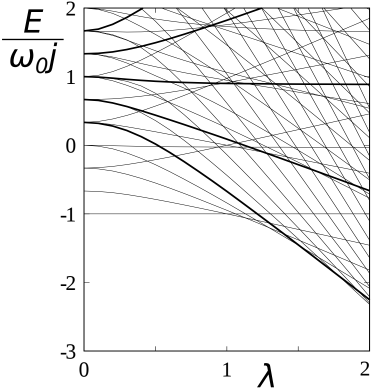

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = 1, \omega = 2$

- Additional conserved quantity
  $$M = \overbrace{b^\dagger b}^{n} + \overbrace{J_z + j}^{n_{ex}}$$
  counting the total number of field bosons and atomic excitations $\rightsquigarrow$ **integrable** system
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Quantum energy-momentum map

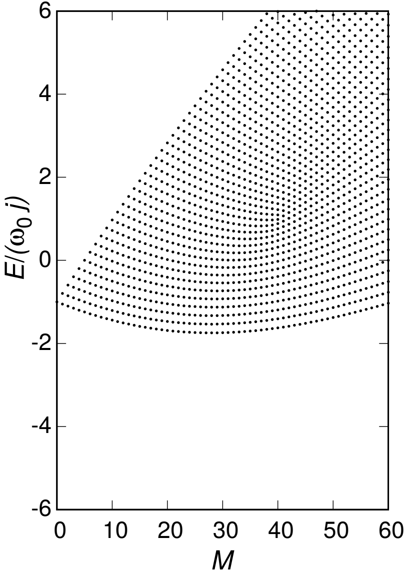
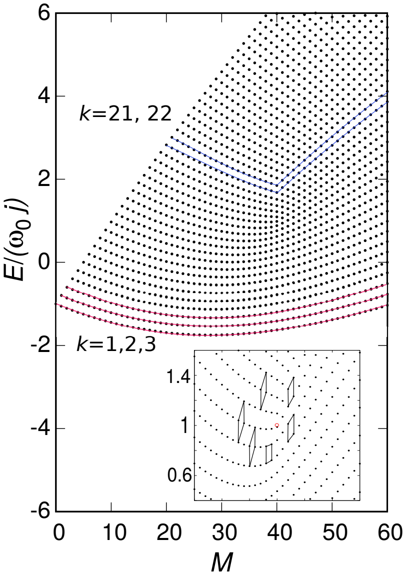


Figure: Quantum energy-momentum map for Tavis-Cummings model, parameters: $j = 20, \omega = 2,\ \omega_0 = 1, \lambda = 2.5 > \lambda_c$

defect in the lattice

Quantum energy-momentum map



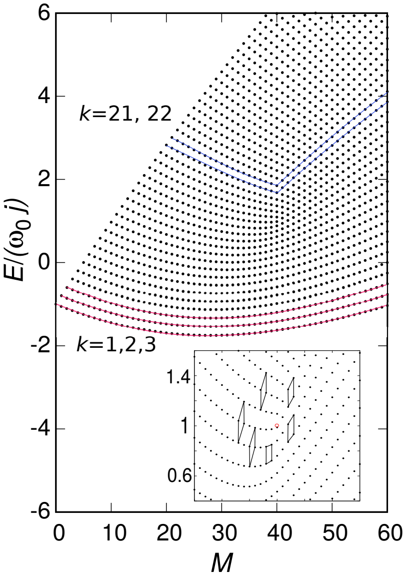
Quantum monodromy: absence of smooth global quantum numbers

$k \rightsquigarrow$ principle quantum number in each invariant subspace

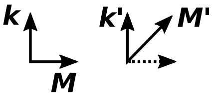
$$\oint_{E_k} p \, \mathrm{d}q = kh \, .$$

Ethymology:
monodromy $\leftrightarrow$ *once around*

Quantum energy-momentum map



Schematic cell transformation



$$\begin{pmatrix} \boldsymbol{M}' \\ \boldsymbol{k}' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{\mu}} \begin{pmatrix} \boldsymbol{M} \\ \boldsymbol{k} \end{pmatrix}$$

monodromy matrix μ of a so-called *focus-focus* singularity

'Critical point' $\rightsquigarrow [M, E/(\omega_0 j)] = [40, 1] = [2j, 1]$

Peres lattices

Peres lattice: quantum expectation value $\langle\psi_i|\bullet|\psi_i\rangle$ in individual energy eigenstates *vs.* the energy E_i

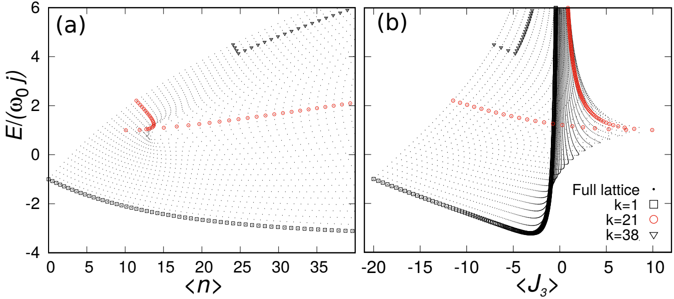


Figure: Peres lattices of observables $n = b^\dagger b$ and $J_3 = J_z$, parameters $j = 20,\ \omega = \omega_0 = 1$.

Reminder: Conserved quantity $M = n + J_3 + j$.

Monodromy and ESQPTs

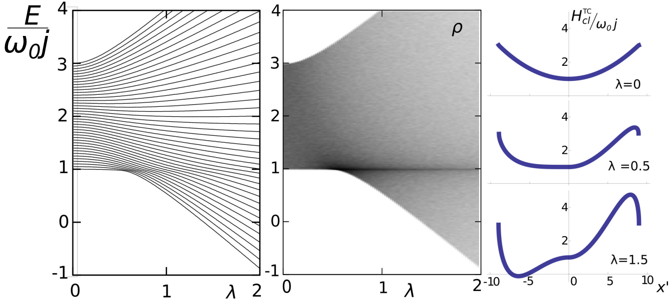
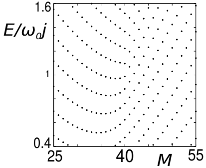


Figure: Energy spectrum and semiclassical level density for $M = 2j = 40$ invariant subspace. Other parameters $\omega = 2, \omega_0 = 1$.



Reminder: Monodromy plot...critical point $[M = 40, E/\omega_0 j = 1]$

Related literature

- L. M. Bates and R. H. Cushman, *Global Aspects of Classical Integrable Systems* (Basel: Birkhäuser), (1997)
- J. J. Duistermaat, Comm. Pure Appl. Math., 33, 687, (1980)
  On global action-angle coordinates
- H. Waalkens, A. Junge, H. R. Dullin, J. Phys. A, 36, L307 (2003)
  Quantum monodromy in the two-centre problem
- D. A. Sadovskii and B. I. Zhilinski, Mol. Phys., 104, 2595-2615 (2006)
  Quantum monodromy and its generalizations and molecular manifestations
- N. F. Zobov et al. , Chem. Phys. Lett., 414.1, 193-197, (2005)
  Monodromy in the water molecule
- P. Cejnar, M. Macek, S. Heinze, J. Jolie, J.Dobeš, J. Phys. A, 39, L515 (2006)
  Monodromy and excited-state quantum phase transitions in integrable systems...
- D. Larese and F. Iachello, J. Mol. Struct. 1006, 611 (2011)
  A study of quantum phase transitions and quantum monodromy in the bending motion of non-rigid molecules

Entanglement

- Atom-field entanglement

$$S(\psi) = -\frac{\mathrm{Tr}\left[\rho_{\mathrm{A}} \ln \rho_{\mathrm{A}}\right]}{\ln(2j+1)} = -\frac{\mathrm{Tr}\left[\rho_{\mathrm{F}} \ln \rho_{\mathrm{F}}\right]}{\ln(2j+1)}$$

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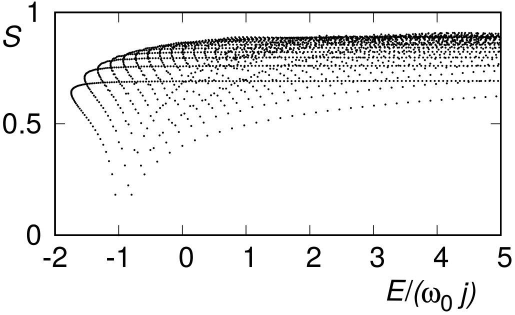

Figure: Entropic spectrum for Tavis-Cummings model, $\lambda = 2.5 > \lambda_c, j = 20, \omega = 2, \omega_0 = 2$.

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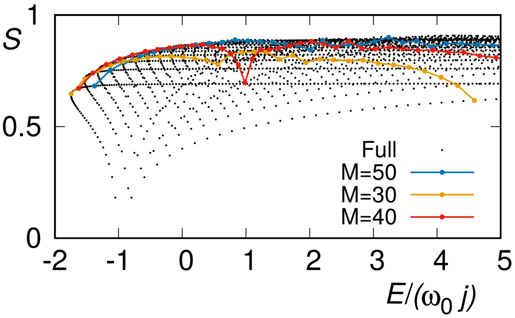

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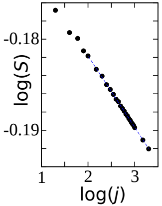

Figure: Scaling of entropy dip for $M = 2j,\ S_{\min} \propto 1/j^{0.008}$.

Non-integrable perturbation

The extended Hamiltonian

$$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$$

T. Brandes Phys. Rev. E 88, 032133 (2013)

$$\delta = \begin{cases} 0 & \rightarrow \quad \text{i.e. Tavis-Cummings model} \\ 1 & \rightarrow \quad \text{i.e. Dicke model} \\ \in (0,1) & \rightarrow \quad \text{extended Dicke model} \end{cases}$$

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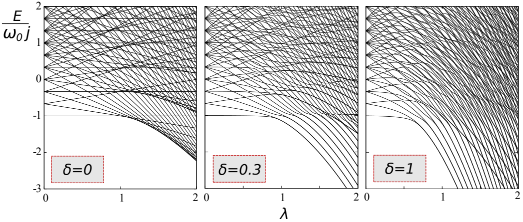

Figure: Spectra of quantum energies, $j = 3$, $\omega_0 = \omega = 1$

- for general δ we get $\lambda_c = \frac{\sqrt{\omega\omega_0}}{1+\delta}$

Decay of monodromy

$H(\lambda, \delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}} \left(b J_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_- \right)$

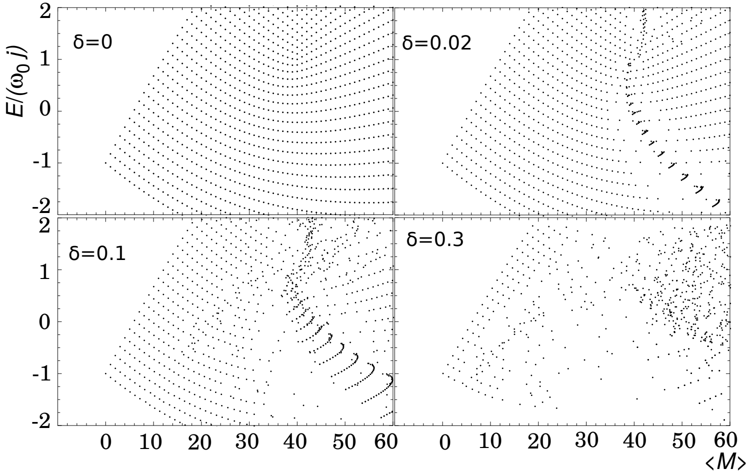

Figure: Mondromy decay under non-integrable perturbation. Parameters $\lambda = 2.5 > \lambda_c, j = 20, \omega = \omega_0 = 1$.

ESQPTs in extended Dicke model

$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$

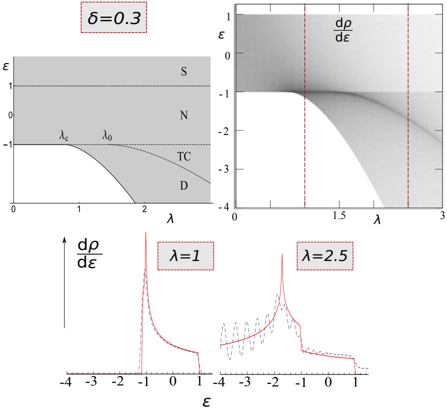

Figure: The derivative of semiclassical level density ρ with respect to $\varepsilon \equiv \frac{E}{\omega_0 j}$

Singularities in level density as a generalization of QPTs $\rightarrow$ **Excited-state quantum phase transitions** (ESQPTs)

$$\lambda_c = \frac{\sqrt{\omega\omega_0}}{1+\delta}\ ,\ \lambda_0 = \frac{\sqrt{\omega\omega_0}}{1-\delta}$$

'Phase diagram' - - →

D - Dicke phase

TC - Tavis-Cummings phase

N - Normal phase

S - Saturated phase

Specification of the phases

$$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$$

Peres lattice: quantum expectation value $\langle\psi_i|\bullet|\psi_i\rangle$ in individual energy eigenstates *vs.* the energy E_i

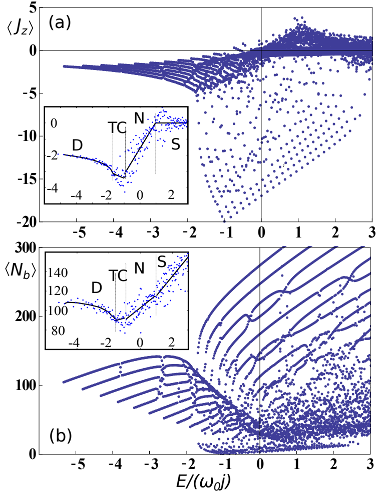

Figure: Peres lattices, $j = 20$, $\omega = \omega_0 = 1$, $\lambda = 2.5$, $\delta = 0.3$.

Inset: Averaged data over 20 neighbouring eigenstates.

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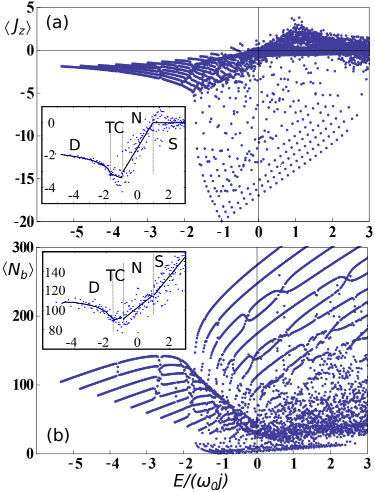

Figure: Peres lattices, $j = 20$, $\omega = \omega_0 = 1$, $\lambda = 2.5$, $\delta = 0.3$.

Inset: Averaged data over 20 neighbouring eigenstates.

Different energy dependence of the averaged data $\leftrightarrow$ *quantum phases*.

Specification of the phases

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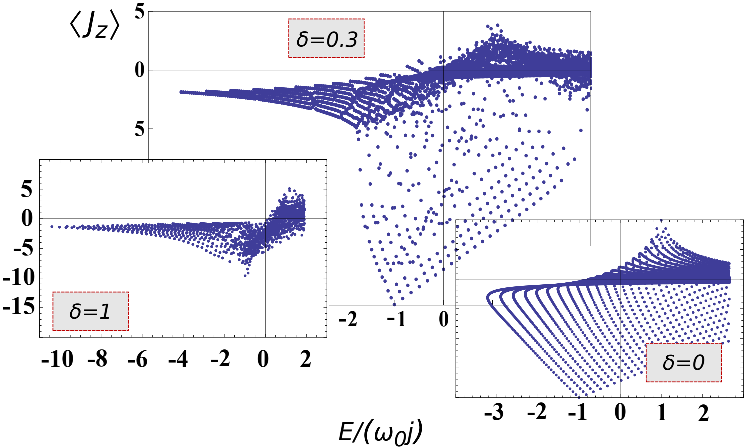
Comparing the phases with the limiting regimes, i. e. the Dicke ($\delta = 1$) and Tavis-Cummings ($\delta = 0$) models



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Maybe more illustrative approach:

Feynman-Hellmann theorem $\frac{dE_i}{d\lambda} = \langle \frac{dH}{d\lambda} \rangle_i = \frac{\langle H_{\text{int}} \rangle_i}{\sqrt{2j}}$

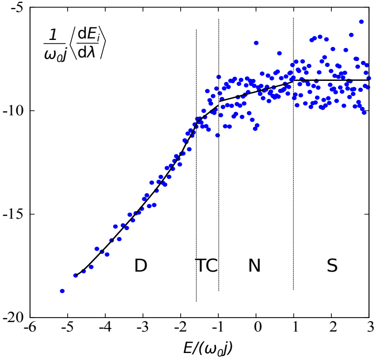

Figure: The slopes averaged over 20 neighbouring levels, $j = 20$, $\omega = \omega_0 = 1$, $\lambda = 2.5$, $\delta = 0.3$.

Atom-field entanglement in excited states

$$H(\lambda,\delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}}\left(bJ_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_-\right)$$

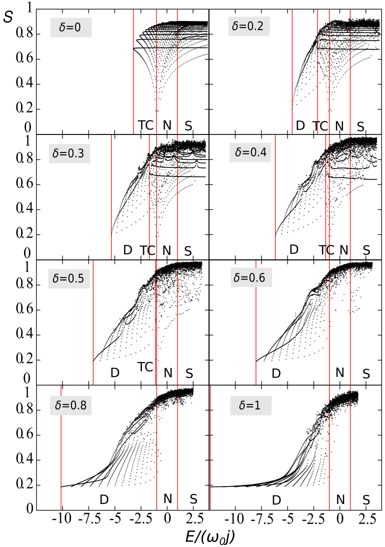

Figure: Evolution of the $j = 20$ entropic spectra of the atom-field entanglement with increasing parameter $\delta \in [0, 1]$ at $\lambda = 2.5 > \lambda_c$, $\omega = \omega_0 = 1$. The ESQPT energies are indicated by vertical lines.

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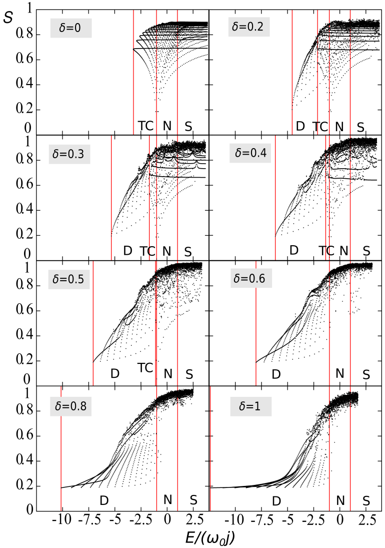

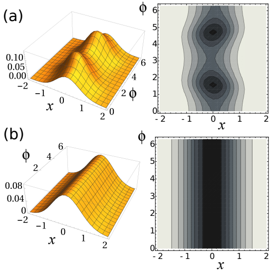

Figure: A detailed view of wave functions for (a) the eigenstate closest to the transition between the TC and N phases at $\lambda = 2.5$, $\delta = 0.3$ and (b) the factorized $E = -\omega_0 j$ eigenstate for $\delta = 0$. The other parameters are the same as in previous figures.

Summary

1. We reported on the occurence of quantum monodromy in the integrable version of Dicke model
2. We classified the type of monodromy and linked it with a certain ESQPT
3. We used a 'lattice plot' to visualize atom-field entanglement in excited states (*entropic spectra*)
4. We showed the scaling of atom-field entanglement of the 'critical' state in the center of monodromy with j as $S_{\min} \propto 1/j^{0.008}$
5. For smoothly increasing non-integrable perturbation we observed a decaying monodromy
6. For smoothly increasing non-integrable perturbation we observed an evolution of entropic spectrum with a focus on ESQPTs

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Thank you for your attention

Back-up slides

Properties of the model

- In thermodynamic limit $N \to \infty$ the number of dof's remains constant due to the collectivity $\leftrightsquigarrow$ *finite* model
- for general δ the number of degrees of freedom $f = 2$
- Algebraically speaking the model is $SU(2) \times HW(1)$
- such a class of *finite* models can be treated semiclassically in thermodynamic limit

Semiclassical analysis

QPTs rooted in classical dynamics

The Hamiltonian

$$H(\lambda, \delta) = \omega b^\dagger b + \omega_0 J_0 + \frac{\lambda}{\sqrt{2j}} \left(b J_+ + b^\dagger J_- + \delta b^\dagger J_+ + \delta b J_- \right)$$

M. A. Bastarrachea-Magnani et al Phys. Rev. A 89, 032101 (2014)

$$\begin{aligned} (J_x, J_y, J_z) &\mapsto j(\sin\theta\cos\phi, \sin\theta\sin\phi, -\cos\theta), \\ (b, b^\dagger) &\mapsto \frac{1}{\sqrt{2}}\left(x + ip, x - ip\right). \end{aligned}$$

$$\{x, p\} = 1, \quad \{\phi, j_z\} = 1, \text{ where } j_z = -j\cos\theta$$

x, p and ϕ, j_z are canonically conjugate variables

$$H_{\rm cl} = \omega_0 j_z + \frac{\omega}{2}\left(p^2 + x^2\right) + \lambda\sqrt{j}\sqrt{1 - \frac{j_z^2}{j^2}}\left[(1+\delta) q\cos\phi - (1-\delta) p\sin\phi\right],$$

Semiclassical analysis

Hamiltonian equations

$$
\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H_{\mathrm{cl}}}{\partial p} &= \omega p - (1-\delta)\,\lambda\sqrt{j}\sqrt{1-\frac{j_z^2}{j^2}}\sin\phi, \\
\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial H_{\mathrm{cl}}}{\partial q} &= -\omega q - (1+\delta)\,\lambda\sqrt{j}\sqrt{1-\frac{j_z^2}{j^2}}\cos\phi, \\
\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H_{\mathrm{cl}}}{\partial j_z} &= \omega_0 - \frac{\lambda j_z}{j^{\frac{3}{2}}\sqrt{1-\frac{j_z^2}{j^2}}}\left[(1+\delta)\,q\cos\phi - (1-\delta)\,p\sin\phi\right], \\
\frac{\mathrm{d}j_z}{\mathrm{d}t} = -\frac{\partial H_{\mathrm{cl}}}{\partial \phi} &= 2\lambda\sqrt{j}\sqrt{1-\frac{j_z^2}{j^2}}\left[(1+\delta)\,q\sin\phi + (1-\delta)\,p\cos\phi\right].
\end{aligned}
$$

Semiclassical analysis

Hamiltonian equations

$$0 = \omega p - (1-\delta)\,\lambda\sqrt{j}\sqrt{1-\frac{j_z^2}{j^2}}\,\sin\phi, \quad (1)$$

$$0 = -\omega q - (1+\delta)\,\lambda\sqrt{j}\sqrt{1-\frac{j_z^2}{j^2}}\,\cos\phi, \quad (2)$$

$$0 = \omega_0 - \frac{\lambda j_z}{j^{\frac{3}{2}}\sqrt{1-\frac{j_z^2}{j^2}}}\left[(1+\delta)\,q\cos\phi - (1-\delta)\,p\sin\phi\right],$$

$$0 = 2\lambda\sqrt{j}\sqrt{1-\frac{j_z^2}{j^2}}\left[(1+\delta)\,q\sin\phi + (1-\delta)\,p\cos\phi\right].$$

searching for the **fixed** points (stationary points of the Hamiltonian)

Semiclassical analysis

Fixed points can be visualized (and searched for) as a stationary points of the energy functional (using $j_z = -j\cos\theta$) $h_{\rm cl}(\phi, j_z) = H_{\rm cl}(\phi, j_z, x = x_{\rm fixed}, p = p_{\rm fixed})$ on the Bloch sphere.

$$h_{\rm cl}(\phi, j_z) = \omega_0 j_z - \frac{\omega_0 j}{2}\frac{\lambda^2}{\lambda_c^2}\left(1 - \frac{j_z^2}{j^2}\right)\left[1 - \frac{4\delta}{(1+\delta)^2}\sin^2\phi\right]$$

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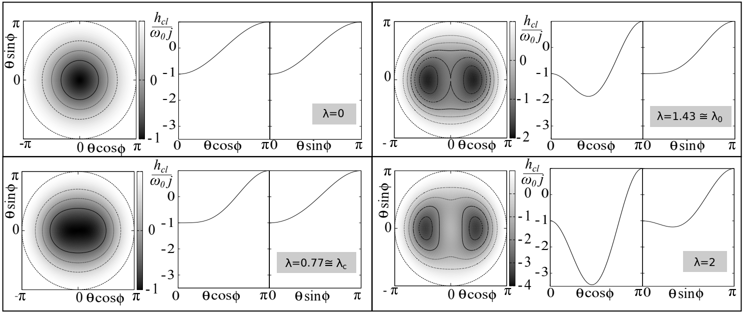


Figure: The function $h_{\rm cl}$ in the phase space of the atomic subsystem defined by spherical angles θ and ϕ for $\omega = \omega_0 = 1$ and $\delta = 0.3$

Semiclassical analysis

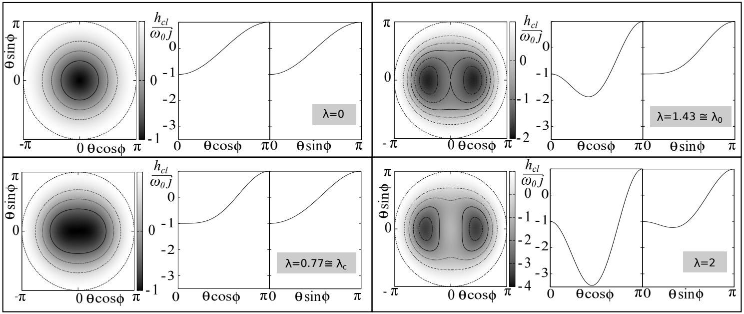


Figure: The function $h_{\rm cl}$ in the phase space of the atomic subsystem defined by spherical angles θ and ϕ for $\omega = \omega_0 = 1$ and $\delta = 0.3$

$$\lambda_c = \frac{\sqrt{\omega\omega_0}}{1+\delta} \;, \quad \lambda_0 = \frac{\sqrt{\omega\omega_0}}{1-\delta}$$

$$E_{\rm g.s.} = \begin{cases} -\omega_0 j & \lambda \in [0, \lambda_c] \,, \\ -\frac{1}{2}\omega_0 j \left(\frac{\lambda_c^2}{\lambda^2} + \frac{\lambda^2}{\lambda_c^2} \right) & \lambda \in (\lambda_c, \infty) \,. \end{cases}$$

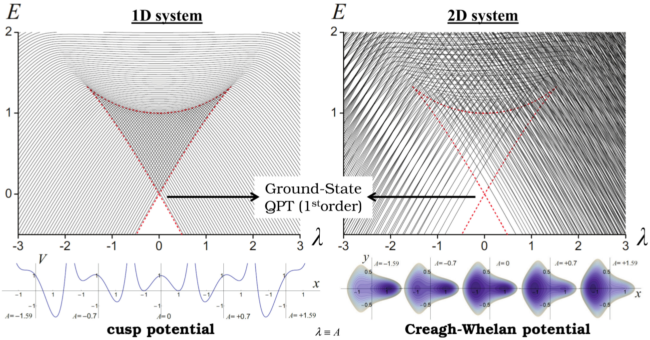
$$E_{\rm saddle} = \begin{cases} -\omega_0 j & \lambda \in [0, \lambda_0] \,, \\ -\frac{1}{2}\omega_0 j \left(\frac{\lambda_0^2}{\lambda^2} + \frac{\lambda^2}{\lambda_0^2} \right) & \lambda \in (\lambda_0, \infty) \,. \end{cases}$$

Definition

ESQPTs are

‘Singularities’ in

1. energy eigenstate density as a function of excitation energy
2. flow of energy spectrum as a function of the control parameter



ESQPTs and related literature

- P. Cejnar, M. Macek, S. Heinze, J. Jolie, J.Dobeš, J. Phys. A, 39, L515 (2006)
  Monodromy and excited-state quantum phase transitions in integrable systems...
- M. A. Caprio, P. Cejnar, F. Iachello, Ann. Phys. 323, 1106-1135 (2008)
  Excited state quantum phase transitions in many-body systems
- P. Cejnar, P. Stránský, Phys. Rev. E 78, 031130 (2008)
  Impact of quantum phase transitions on excited-level dynamics

What properties can be affected by (ES)QPTs?

Entanglement measures

- Atom-atom entanglement

$$S(\psi) = -\frac{\mathrm{Tr}\left[\rho_{\mathrm{A}} \ln \rho_{\mathrm{A}}\right]}{\ln(2j+1)} = -\frac{\mathrm{Tr}\left[\rho_{\mathrm{F}} \ln \rho_{\mathrm{F}}\right]}{\ln(2j+1)}$$

- Atom-atom entanglement

$$C(\psi) = (N-1)\max\left\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\,, 0\right\}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are eigenvalues (real and non-negative) of a non-Hermitian matrix $\varrho = \rho_{\mathrm{A}}^{kl}\,\left(\sigma_y^k \otimes \sigma_y^l\right)\rho_{\mathrm{A}}^{kl*}\left(\sigma_y^k \otimes \sigma_y^l\right)$.

$\rho_{\mathrm{A}}^{kl} \leftrightarrow$ density matrix of a pair of qubits. Due to the symmetry can be expressed through the expectation values of the collective spin operators $\langle J_z\rangle$, $\langle J_z^2\rangle$ and $\langle J_+\rangle$.

X. Wang and K. Mølmer, Eur. Phys. J. D 18 385 (2002)

Entanglement in QPT

N. Lambert, C. Emary, T. Brandes, Phys. Rev. Let 92(7), 073602 (2008)

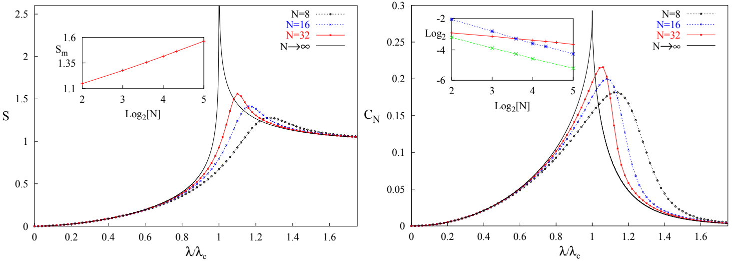


Figure: Anomalous behaviour of entanglement in QPT in Dicke model.

Entanglement in QPT

Extended Dicke model

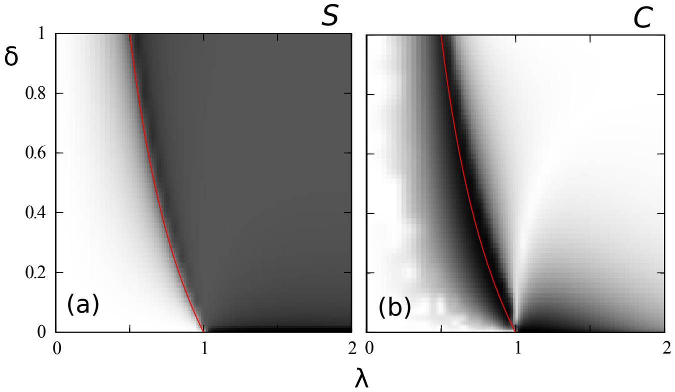


Figure: Entanglement properties of the ground state of the $j = N/2$ model in the plane of control parameters λ and δ (for $\omega = \omega_0 = 1$, $N = 40$). The red curve indicates the QPT critical coupling λ_c

Entropic spectrum

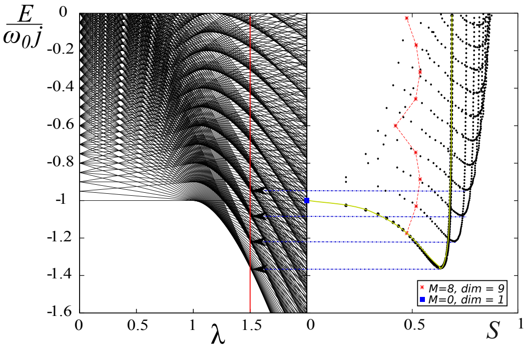
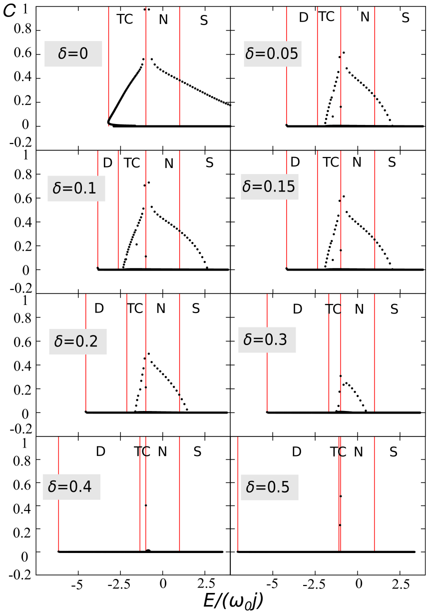


Figure: The full energy spectrum of the $\delta = 0$ model with $\omega = \omega_0 = 1$ for $j = N/2$ and $N = 40$ (left panel), and the atom–field entanglement entropies S in individual eigenstates corresponding to $\lambda = 1.5$ cut of the spectrum (right panel).

Atom-atom entanglement in excited states



Entanglement in ESQPT

within Tavis-Cummings $\delta = 0$ invariant $M = 2j$

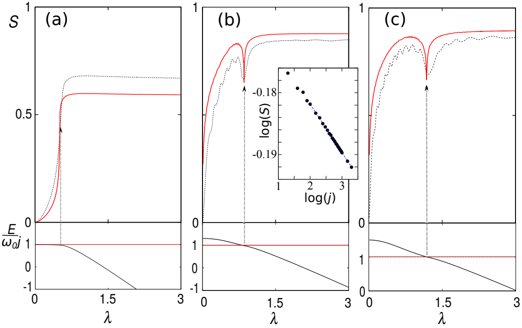


Figure: The atom–field entanglement entropy for three states from the $M = 2j$ subspace of the $\delta = 0$ detuned $(\omega = 2, \omega_0 = 1)$ model with $j = N/2$: The log-log plot in panel (b) inset shows the minimum value of S in the dip as a function of j for the given level (for this level, the dependence is roughly $S_{\min} \propto 1/j^{0.008}$).