Chaos, metastability and ergodicity in Bose-Hubbard superfluid circuits

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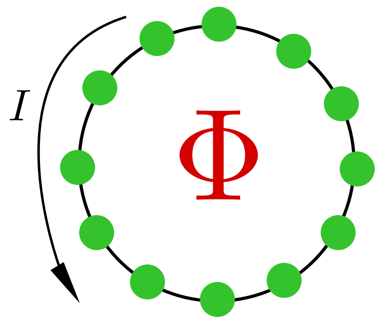
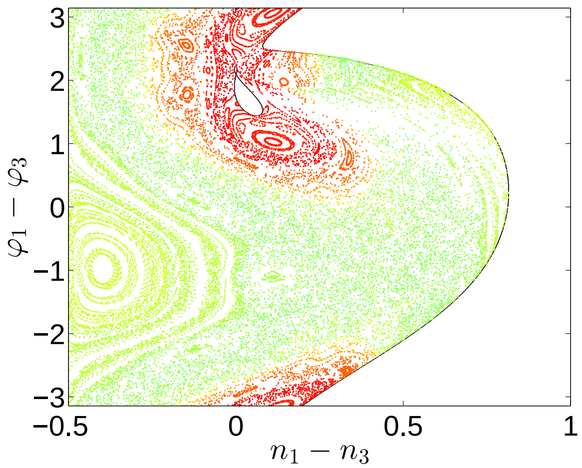
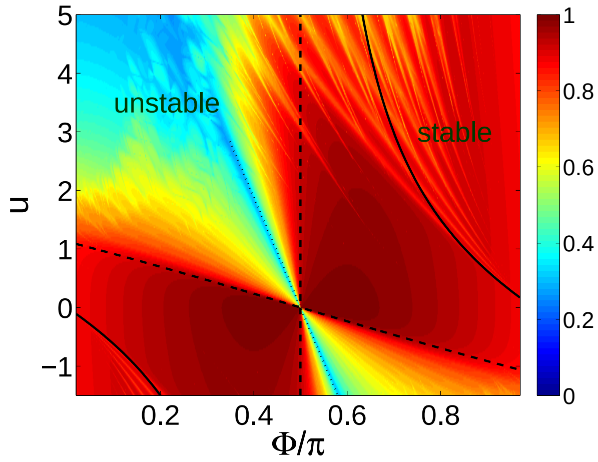
[**1**] Geva Arwas, Doron Cohen [**Physical Review B 2017**]
[**2**] Geva Arwas, Doron Cohen [**New Journal of Physics 2016**]
[**3**] Geva Arwas, Amichay Vardi, Doron Cohen [**Scientific Reports 2015**]
[**4**] Related work with Amichay Vardi and Christine Khripkov (see next page)



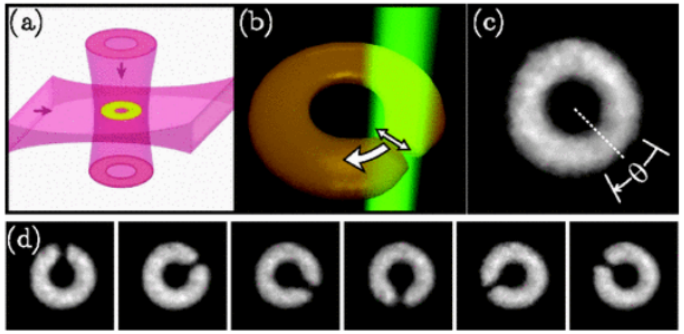




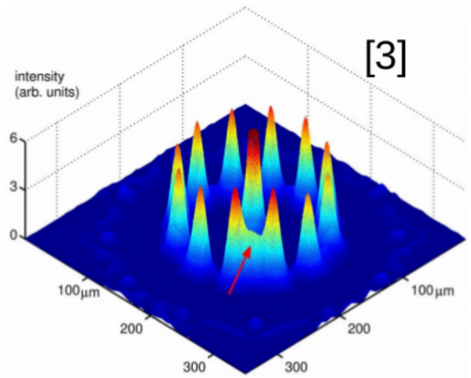
Metastability versus **Ergodicity** – beyond the "Landau criterion"
Oscillations – beyond the "pendulum physics" of the Joesephosn SQUID Hamiltonian
Thermalization – percolation in phase-space; semiclassical vs dynamical localization
Quantum Chaos!

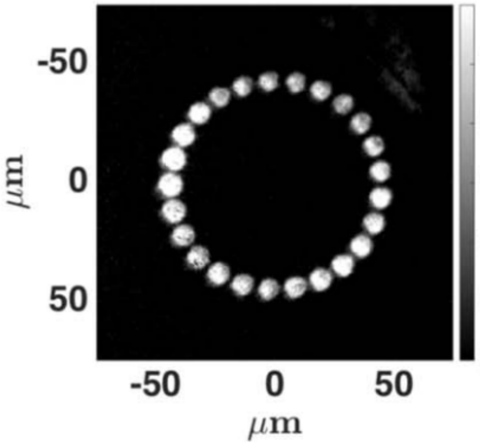
Motivating the interest in Atomtronic circuits

Toroidal ring [1,2]



Discrete ring (optical lattice) [4]



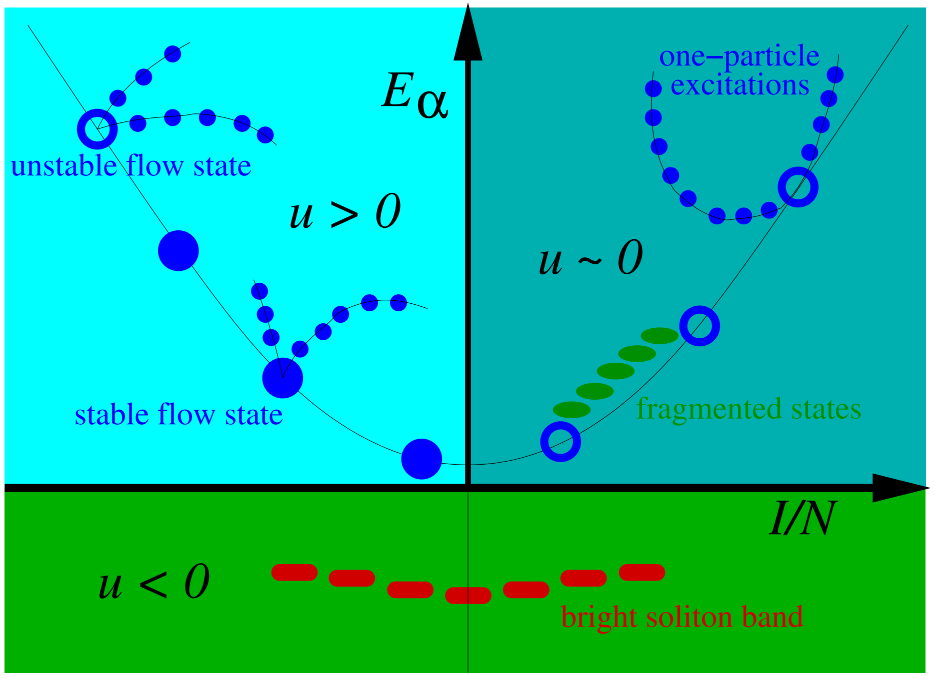


- Recent experiments [1,2,3] have opened a new arena: superfluidity in low dimensional circuits.
- The hallmark of superfluidity is a metastable persistent current: flow-state.
- A stability regime diagram of the flow states in the toroidal ring has been explained by following the reasoning of the Landau superfluidity criterion.
- We claim that a theory for the stability of the flow states in a discrete ring that are described by the Bose-Hubbard Hamiltonian requires a quantum chaos perspective.
- We demonstrate how the stability is affected by non-linear resonances, in regimes where the dynamics is traditionally considered to be stable.

[1] Wright, Blakestad, Lobb, Phillips, Campbell (PRL 2013)
[2] Ekel, Lee, Jendrzeejewski, Murray, Clark, Lobb, Phillips, Edwards, Campbell (Nature 2014)
[3] Amico, Aghamalyan, Auksztol, Crepaz, Dumke, Kwek (Sci. Rep. 2014)
[4] Gauthier, Lenton, Parry, Baker, Davis, Rubinsztein-Dunlop, Neely (Optica 2016)

The traditional view of Superfluidity

- Superfluidity means that there is a feasibility to observe a metastable persistent current.
- This definition has nothing to do with the thermodynamic limit!
  It is not a phase transition.
- This leads to the Landau criterion.
  More generally one can carry out Bogoliubov stability analysis [1-4].



- See also: persistent currents for interacting Bosons on a ring with a gauge field [5]
- The quantum chaos perspective of "superfluidity" has not been considered so far
- Related theme: non-linear resonances in Bose-Hubbard model [6]

[1] Smerzi, Trombettoni, Kevrekidis, Bishop (PRL 2002)

[2] Wu, Niu (NJP 2003)

[3] Polkovnikov, Altman, Demler, Halperin, Lukin (PRA 2005)

[4] De Sarlo, Fallani, Lye, Modugno, Saers, Fort, Inguscio (PRA 2005)

[5] Cominotti, Rossini, Rizzi, Hekking, Minguzzi (PRL 2014)

[6] Kolovsky (PRE 2007)

The Bose Hubbard Hamiltonian

The system consists of N bosons in M sites. Later we add a gauge-field Φ.

$$\mathcal{H}_{\mathrm{BHH}} \;=\; \frac{U}{2}\sum_{j=1}^{M} a_j^\dagger a_j^\dagger a_j a_j \;-\; \frac{K}{2}\sum_{j=1}^{M}\left(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}\right)$$

$$u \;\equiv\; \frac{NU}{K} \qquad \text{[classical, stability, supefluidity, self-trapping]}$$

$$\gamma \;\equiv\; \frac{Mu}{N^2} \qquad \text{[quantum, Mott-regime]}$$

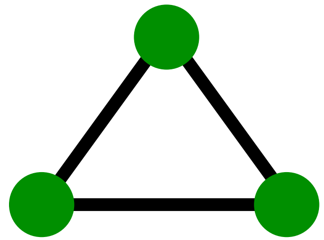
Dimer (M=2): Minimal BHH; Bosonic Josephson junction; Pendulum physics [1,5].
Driven dimer: Landau-Zener dynamics [2], Kapitza effect [3], Zeno effect [4], Standard-map physics [5].
Trimer (M=3): Minimal model for low-dimensional chaos; Coupled pendula physics.
Triangular trimer (M=3): Minimal model with topology, Superfluidity [6], Stirring [7].
Larger rings (M>3) High-dimensional chaos; web of non-linear resonances [7].
Coupled subsystems (M>3): Minimal model for Thermalization [8,9].

[1] Chuchem, Smith-Mannschott, Hiller, Kottos, Vardi, Cohen (PRA 2010).
[2] Smith-Mannschott, Chuchem, Hiller, Kottos, Cohen (PRL 2009).
[3] Boukobza, Moore, Cohen, Vardi (PRL 2010).
[4] Khripkov, Vardi, Cohen (PRA 2012)
[5] Khripkov, Cohen, Vardi (JPA 2013, PRE 2013).
[6] Geva Arwas, Vardi, Cohen (PRA 2014, SREP 2015, NJP 2016, PRB 2017).
[7] Hiller, Kottos, Cohen (EPL 2008, PRA 2008).
[8] Tikhonenkov, Vardi, Anglin, Cohen, (PRL 2013).
[9] Christine Khripkov, Vardi, Cohen (NJP 2015).

The “Quantum Chaos” perspective

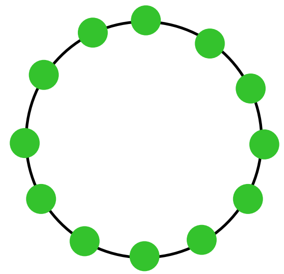
Stability of flow-states (I):

- Landau stability of flow-states (“Landau criterion”)
- Bogoliubov perspective of dynamical stability
- KAM perspective of dynamical stability



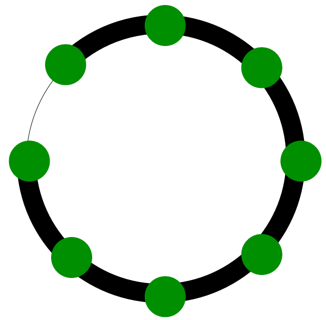
Stability of flow-states (II):

- Considering high dimensional chaos ($M > 3$).
- Web of non-linear resonances.
- Irrelevance of the the familiar Beliaev and Landau damping terms.
- Analysis of the quench scenario.



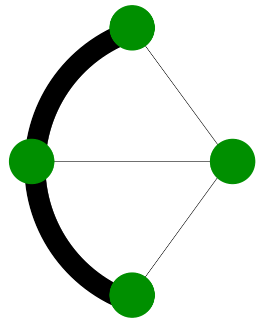
Coherent Rabi oscillations:

- The hallmark of coherence is Rabi oscillation between flow-states.
- Ohmic-bath perspective $\rightsquigarrow$ $\eta = (\pi/\gamma) > 1$
- Feasibility of Rabi oscillation for $M < 6$ devices.
- Feasibility of of chaos-assisted Rabi oscillation.



Thermalization:

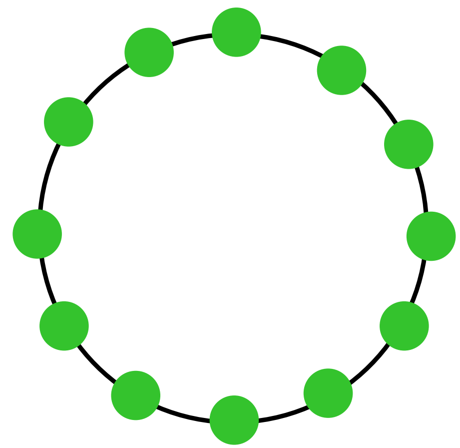
- Spreading in phase space is similar to Percolation.
- Resistor-Network calculation of the diffusion coefficient.
- Observing regions with Semiclassical Localization.
- Observing regions with Dynamical Localization.



The Model (non-rotating ring)

A Bose-Hubbard system with M sites and N bosons:

$$\mathcal{H} \;=\; \sum_{j=1}^{M} \left[\frac{U}{2} a_j^{\dagger} a_j^{\dagger} a_j a_j - \frac{K}{2} \left(a_{j+1}^{\dagger} a_j + a_j^{\dagger} a_{j+1} \right) \right]$$



In a semi-classical framework:

$$a_j \;=\; \sqrt{\boldsymbol{n}_j}\, \mathrm{e}^{i\boldsymbol{\varphi}_j} \quad , \quad [\boldsymbol{\varphi}_j, \boldsymbol{n}_i] \;=\; i\delta_{ij}$$

$$z \;=\; (\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_M,\;\; \boldsymbol{n}_1, \cdots, \boldsymbol{n}_M)$$

This is like M coupled oscillators with $\mathcal{H} = H(z)$

$$H(z) \;=\; \sum_{j=1}^{M} \left[\frac{U}{2} \boldsymbol{n}_j^2 - K \sqrt{\boldsymbol{n}_{j+1} \boldsymbol{n}_j} \cos\left(\boldsymbol{\varphi}_{j+1} - \boldsymbol{\varphi}_j \right) \right]$$

The dynamics is generated by the Hamilton equation:

$$\dot{z} \;=\; \mathbb{J} \boldsymbol{\partial} H \quad , \quad \mathbb{J} \;=\; \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

(DNLS)

Classically there is a single dimensionless parameter:

$$u \;=\; \frac{NU}{K}$$

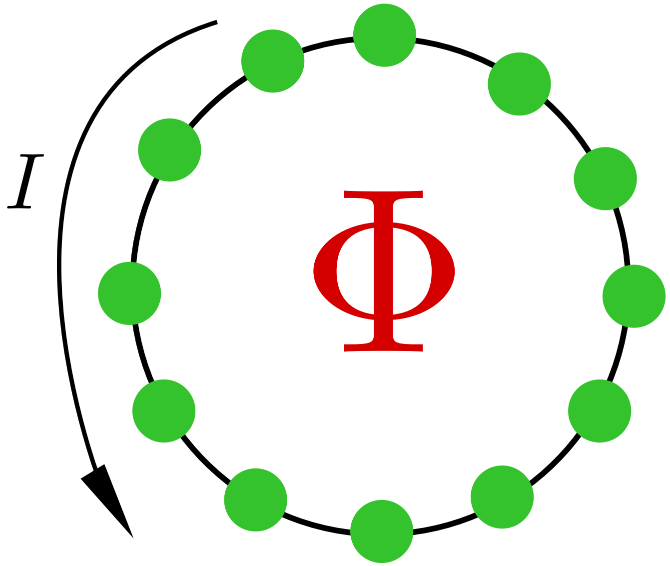
Rescaling coordinates:

$$\tilde{\boldsymbol{n}} \;=\; \boldsymbol{n}/N$$

$$[\boldsymbol{\varphi}_j, \tilde{\boldsymbol{n}}_i] \;=\; i \frac{1}{N} \delta_{ij}$$

$$\gamma \;\equiv\; \frac{\mathsf{m}^* g}{\rho} \;=\; \frac{Mu}{N^2}$$

The model (rotating ring)



In the rotating reference frame we have a Coriolis force, which is like magntic field $\mathcal{B} = 2\mathsf{m}\Omega$. Hence is is like having flux

$$\Phi \;=\; 2\pi R^2 \mathsf{m}\,\Omega$$

Note: there are optional experimental realizations.

$$\mathcal{H} \;=\; \sum_{j=1}^{M} \left[\frac{U}{2} a_j^\dagger a_j^\dagger a_j a_j - \frac{K}{2} \left(\mathrm{e}^{i(\Phi/M)} a_{j+1}^\dagger a_j + \mathrm{e}^{-i(\Phi/M)} a_j^\dagger a_{j+1} \right) \right]$$

Summary of model parameters:

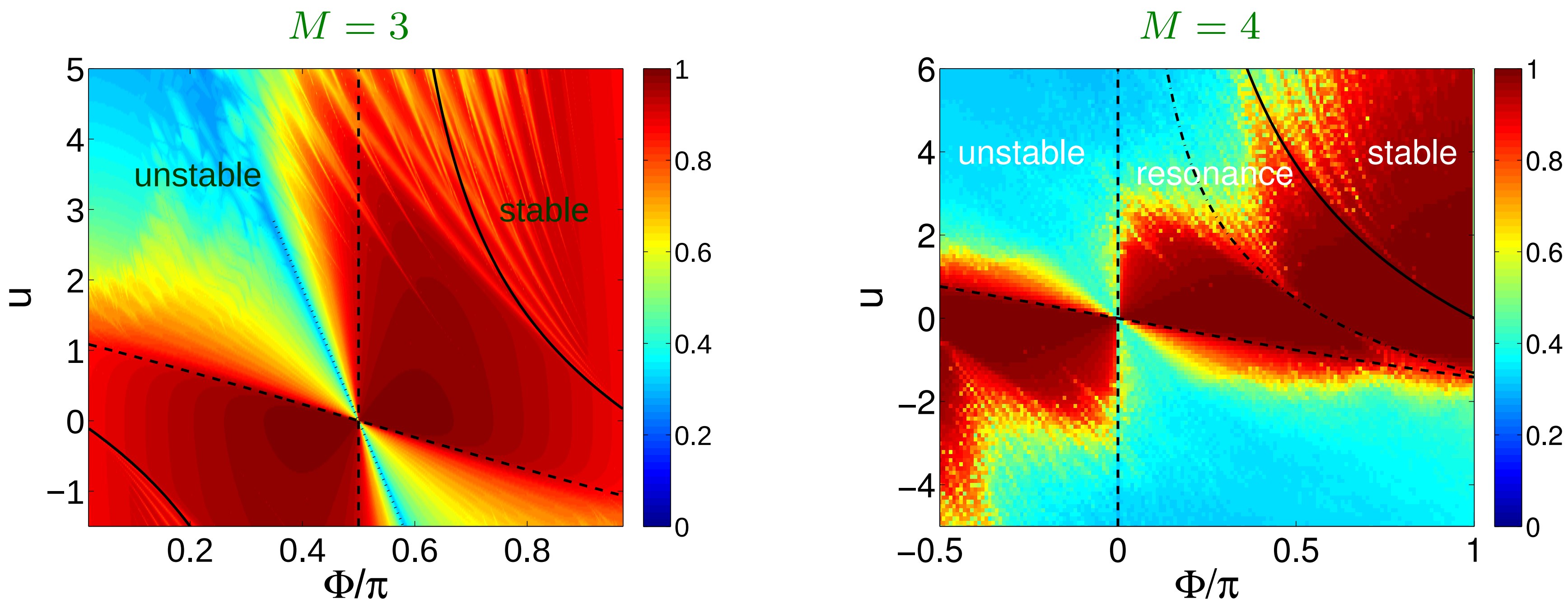
The "classical" dimensionless parameters of the DNLS are u and Φ.
The mumber of particles N is the "quantum" parameter (optionally γ).
The system has effectively $d = M-1$ degrees of freedom.

$M = 2$	Bosonic Josephson junction (Integrable)
$M = 3$	Minimal circuit (mixed chaotic phase-space)
$M > 3$	High dimensional chaos (Arnold diffusion)
$M \to \infty$	Continuous ring (Integrable)

Flow-state stability regime diagram

The I of the maximum current state is imaged as a function of (Φ, u)

- solid lines = energetic stability borders (Landau)
- dashed lines = dynamical stability borders (Bogoliubov)



The traditional paradigm associates flow-states with stationary fixed-points in phase space. Consequently the Landau criterion, and more generally the Bogoliubov linear-stability-analysis, are used to determine the viability of superfluidity.

Non-linear resonances

Regime diagram for flow-state metastability:

- Via quantum eigenstates
- Via quantum quench simulation
- Via semiclassical simulation

Observation:

The linear-stability analysis of Bogoliubov is not a sufficient condition for strict dynamical stability. A non-linear resonance between the frequencies can destroy the dynamical stability.

The "1:2" resonance

for the $m = 1$ flow-state of M=4 ring:

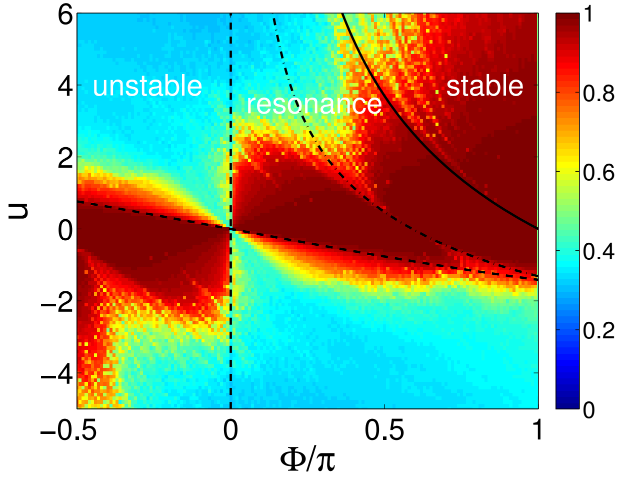
$$u \;=\; 4\cot\left(\frac{\Phi}{4}\right)\left[3\cos\left(\frac{\Phi}{4}\right) - \sqrt{6+2\cos\left(\frac{\Phi}{2}\right)}\right]$$

Addressing all flow-states in one diagram:

$$\phi \;=\; \Phi - 2\pi m \;=\; \text{unfolded phase} \;\; \in [-M\pi, M\pi]$$

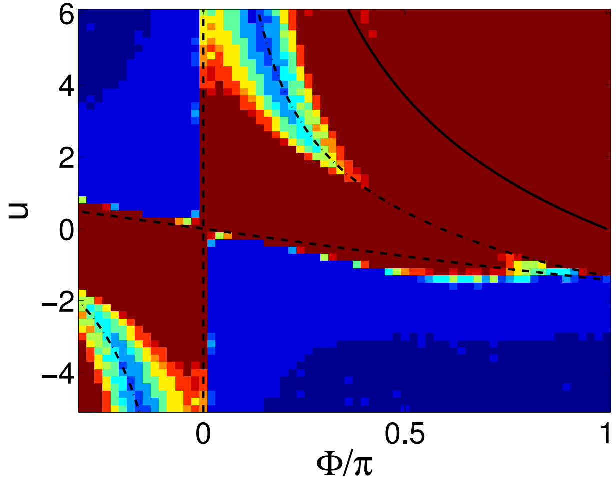
Quantum regimes $M = 4$

$N = 32$ particles



Classical reconstruction

Small cloud



The non-linear terms

$$\mathcal{H} \quad = \quad \sum_k \epsilon_k \boldsymbol{b}_k^\dagger \boldsymbol{b}_k \; + \; \frac{U}{2M} \sum_{\langle k_1..k_4 \rangle} \boldsymbol{b}_{k_4}^\dagger \boldsymbol{b}_{k_3}^\dagger \boldsymbol{b}_{k_2} \boldsymbol{b}_{k_1}$$

Assuming condensation at the k=0 orbital
the Hamiltonian can be expressed in terms
of Bogoliubov quasi-particles creation operators:

$$q = \frac{2\pi}{M} m$$

$$m = \text{integer} \neq 0$$

$$\boldsymbol{b}_q^\dagger \quad = \quad u_q \boldsymbol{c}_q^\dagger + v_q \boldsymbol{c}_{-q}$$

$$\frac{M}{2} < m \leq \frac{M}{2}$$

Approximated Hamiltonian at the vicinity of the condensate:

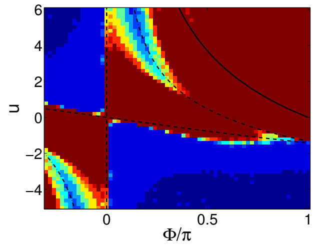
$$\mathcal{H} \quad = \quad \sum_q \omega_q \boldsymbol{c}_q^\dagger \boldsymbol{c}_q \; + \; \frac{\sqrt{N}U}{M} \sum_{\langle q_1, q_2 \rangle} \left[A_{q_1,q_2} \left(\boldsymbol{c}_{-q_1-q_2} \boldsymbol{c}_{q_2} \boldsymbol{c}_{q_1} + \text{h.c.} \right) + B_{q_1,q_2} \left(\boldsymbol{c}_{q_1+q_2}^\dagger \boldsymbol{c}_{q_2} \boldsymbol{c}_{q_1} + \text{h.c.} \right) \right]$$

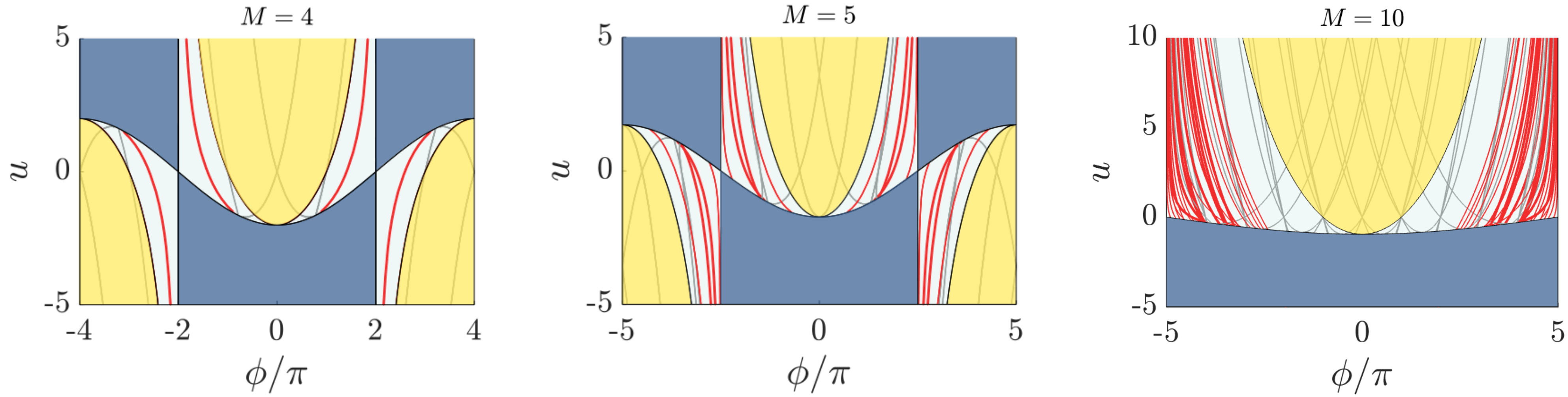
- The "B" terms are the Beliaev and Landau damping terms. [gray lines]
- The "A" terms are usually ignored. [red lines]

Mapping the non-linear resonances

$$\omega_{q_1} + \omega_{q_2} - \omega_{q_1+q_2} = 0 \qquad \text{[gray lines]}$$

$$\omega_{q_1} + \omega_{q_2} + \omega_{-q_1-q_2} = 0 \qquad \text{[red lines]}$$

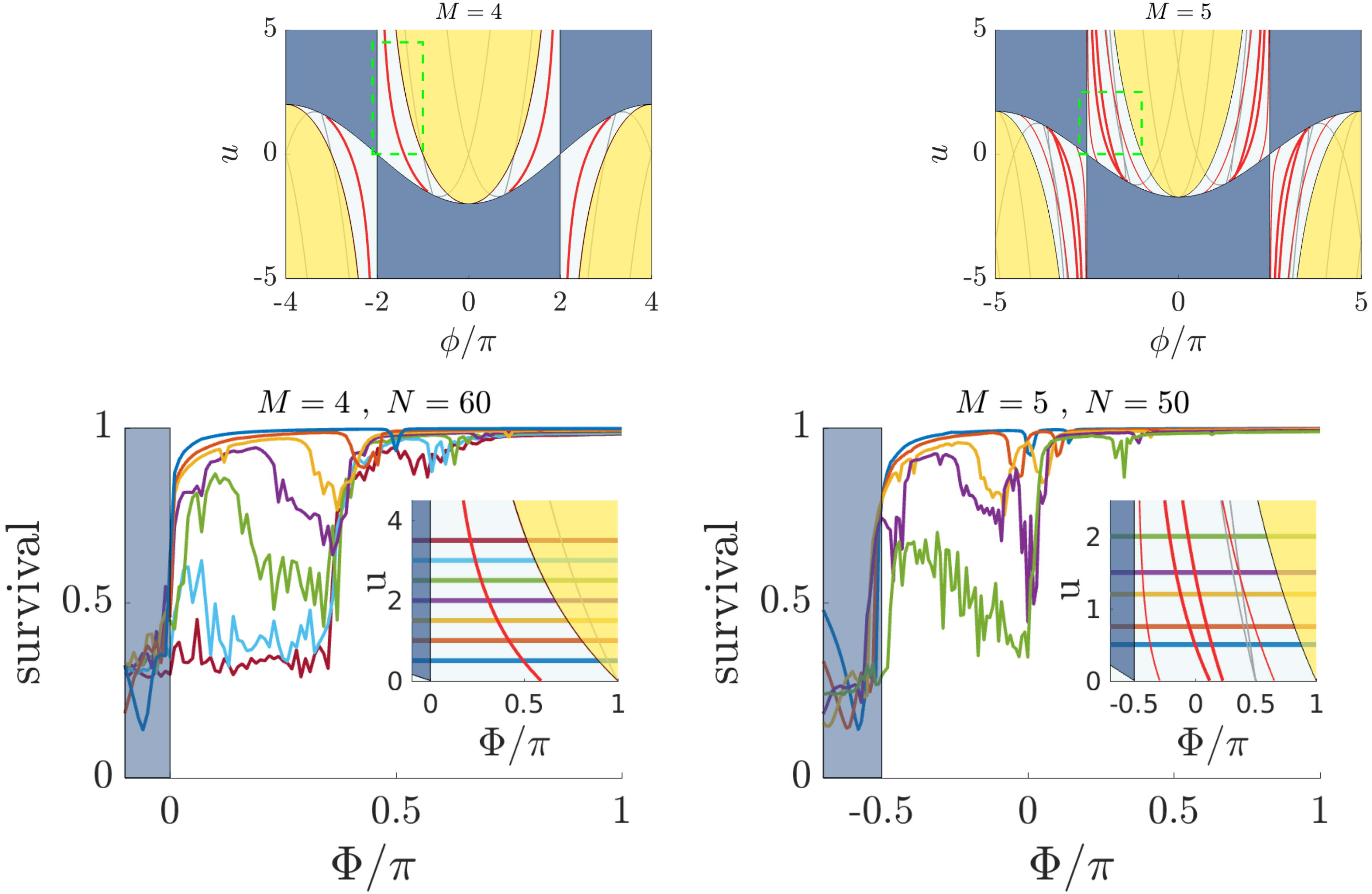




Considering the "1:2" resonance for the $m = 1$ flow-state of the $M = 4$ ring, setting $q_1 = q_2 = q = 2\pi/4$, we get from $2\omega_q + \omega_{-2q} = 0$ the resonance condition

$$u \;=\; 4\cot\left(\frac{\Phi}{4}\right)\left[3\cos\left(\frac{\Phi}{4}\right) - \sqrt{6 + 2\cos\left(\frac{\Phi}{2}\right)}\right]$$

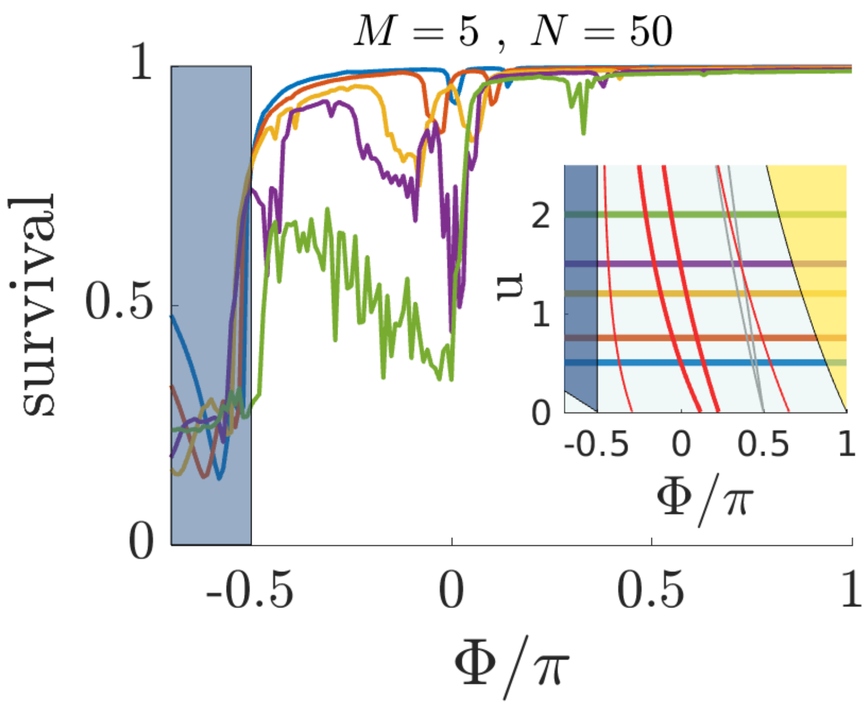
Survival of the flow-state

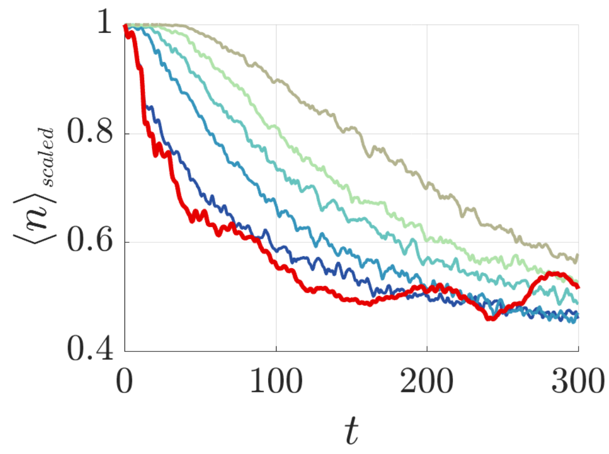


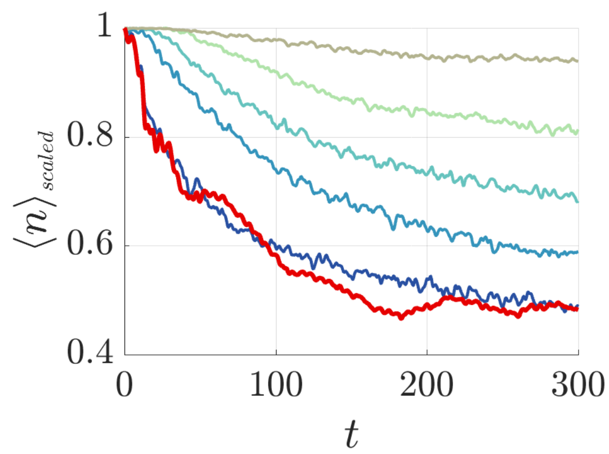
The decay of the flow-state

Observations:

- One can resolve dips in the dependence of the stability on Φ, provided if u is small.
- Even in the center of a dip the stability is better compared with the linear unstable regime.
- These dips broaden and merge as u becomes larger.
- Off-resonance there is strong sensitivity to N.





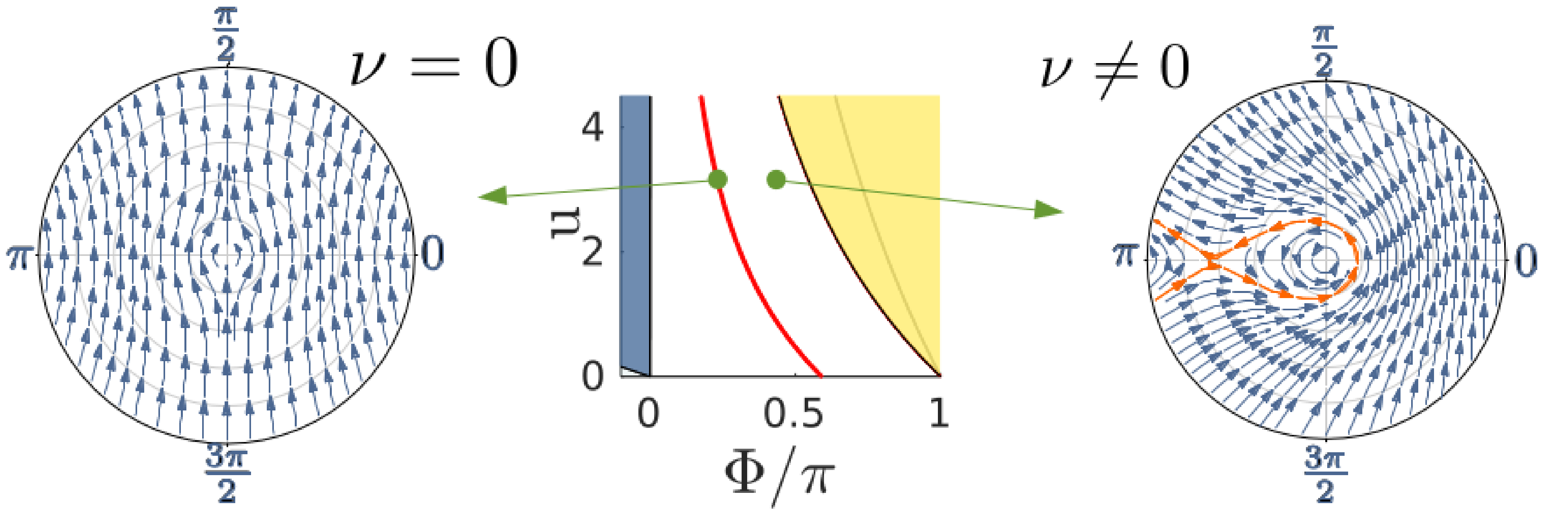


$N = 120, 500, 1000, 2000, 4000$

The N dependence

The flow state is represented in phase-space by a Gaussian-like cloud of uncertainty width $1/N$.

The size of the stability-island depends on the detuning: $\nu \equiv 2\omega_q + \omega_{-2q}$



ALL trajectories $\to \infty$

A stability island exists

The radial coordinate represents the quasiparticle occupation $\tilde{n}_q$.

The secular approximation

Considering the "1:2" resonance, we keep the two modes that are coupled by the resonance.

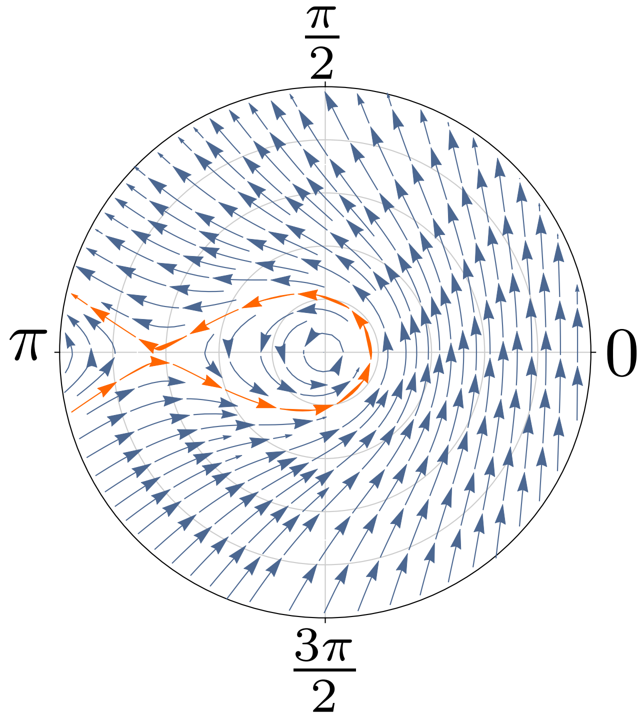
We get the "Cherry Hamiltonian" of celestial mechanics

$$H_q \;=\; \omega J + \nu I + \mu I \sqrt{(J/2) + I}\; \cos(\varphi)$$

$$\nu \;\equiv\; 2\omega_q + \omega_{-2q} \;=\; \text{detuning}$$

$$I \;=\; \tilde{n}_q/(2N), \qquad \text{conjugate } \varphi$$

$$J \;=\; (2\tilde{n}_{-2q} - \tilde{n}_q)/N \;=\; \text{const of motion}$$



Width of the resonance region:

$$\left|\nu\right| \;<\; A\left(\frac{1}{N}\right)^{1/2} \frac{u}{M} K$$

Note: in contrast, the Beliaev and Landau terms do not generate an escape route.

Hyperbolic escape

Exponential escape followed by hyperbolic escape

$$\tilde{n}_q \quad \propto \quad \frac{1}{(t_e - t)^2} \quad \text{for } t < t_e$$

After that transition to chaos.

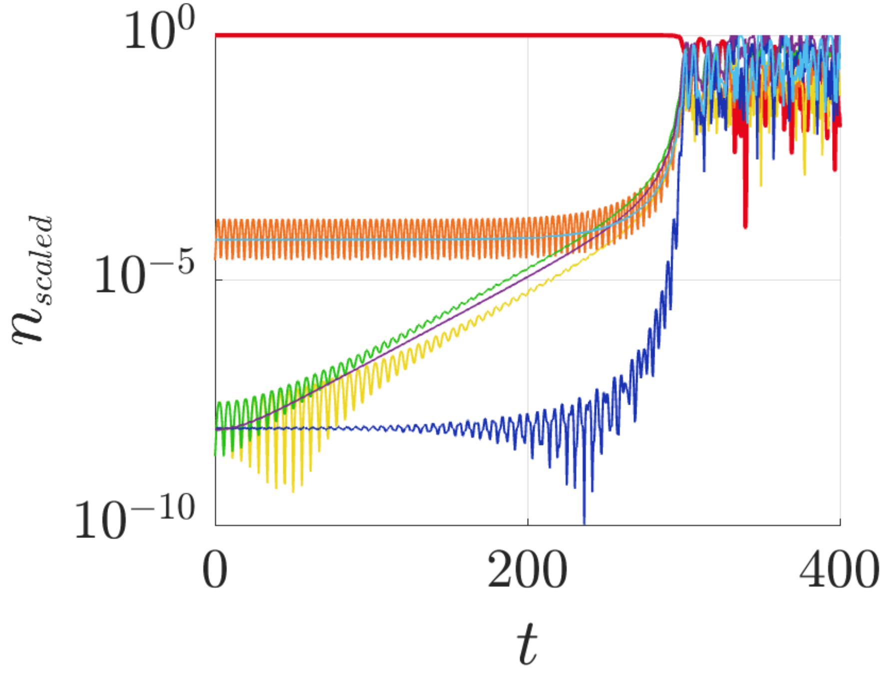
Complete decay as in the linear unstable regime.

Here the detuning is zero and u is large.

For small u the decay process is suppressed.

Re-injection scenario.

Dynamical localization.

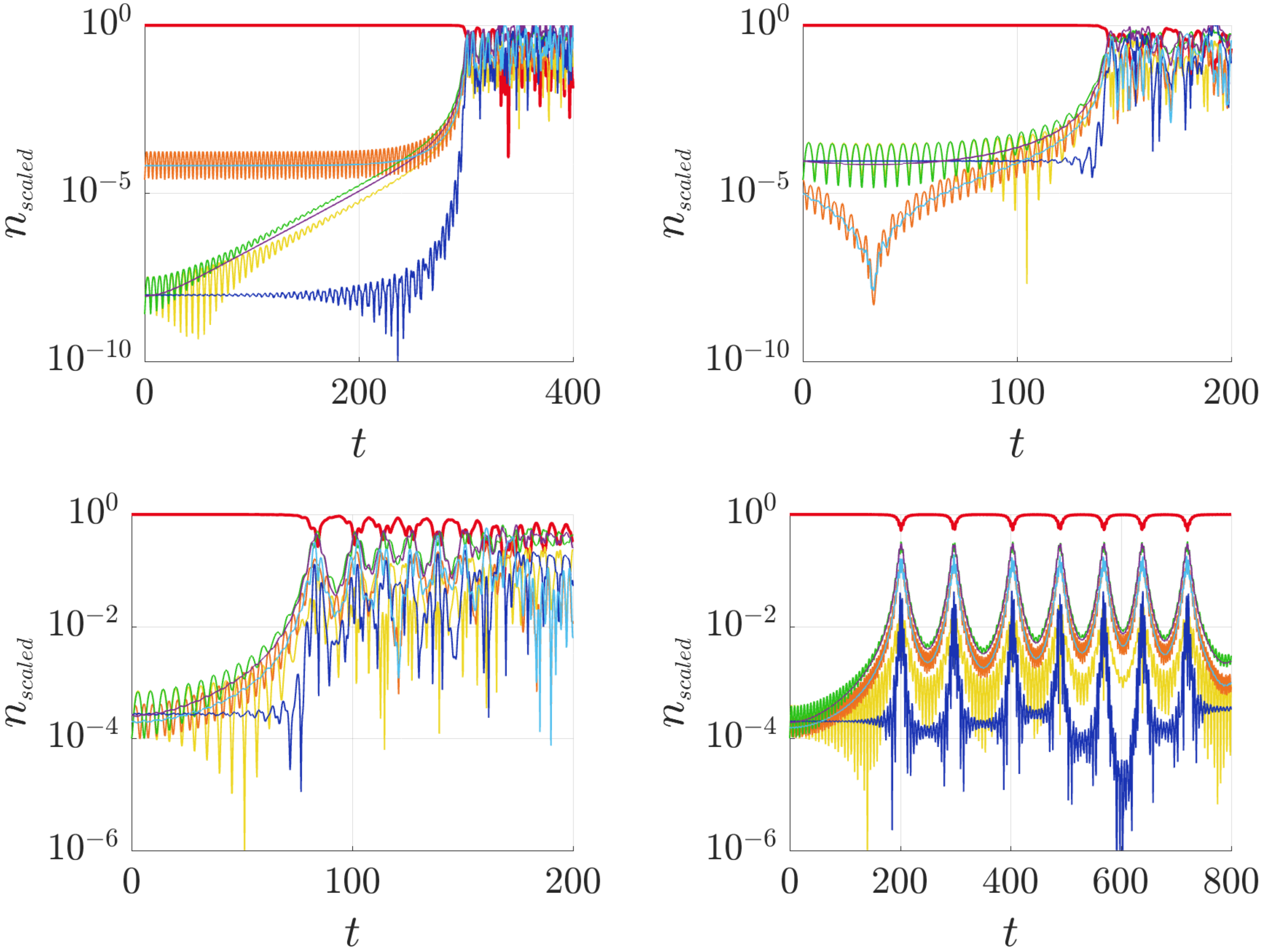


n_0 (red) occupation of the flow-state orbital

n_k occupation of the other momentum orbital

$\tilde{n}_q$ quasi-particle occupations

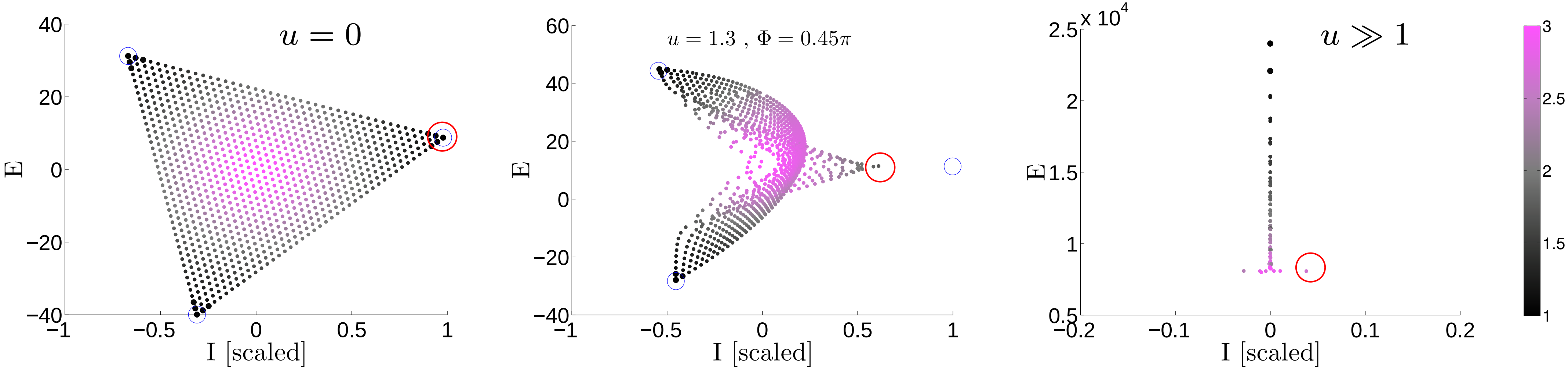
Hyperbolic escape - the possible scenarios



The many-body spectrum for $M = 3$ ring

We characterize each eigenstate $|\alpha\rangle$ of the BHH by $(\mathcal{I}_\alpha, E_\alpha)$ and colorcode by $\mathcal{M}_\alpha$

The expected location of a flow-state, and the maximum current state, are encircled by ◯ and ◯



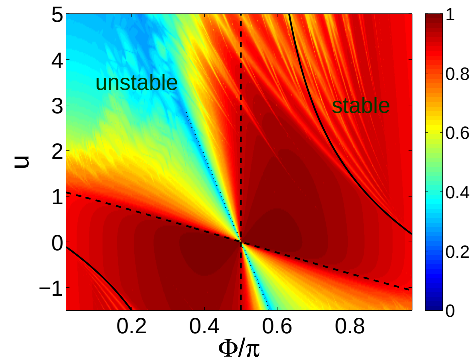
$$|m\rangle \quad = \quad \left(\tilde{a}_m^\dagger\right)^N |0\rangle \qquad\qquad m = 1...M$$

$$\mathcal{I}_m \quad = \quad N \times \left(\frac{K}{M}\right) \sin\left(\frac{1}{M}(2\pi m - \Phi)\right)$$

$$\mathcal{I}_\alpha \quad \equiv \quad -\left\langle \frac{\partial \mathcal{H}}{\partial \Phi} \right\rangle_\alpha$$

Constructing the regime diagram:

For every (Φ, u) value we plot $\max\{I_\alpha\}$



Energetic vs Dynamical stability

Poincare section $n_2 = n_3$ at the flow-state energy.

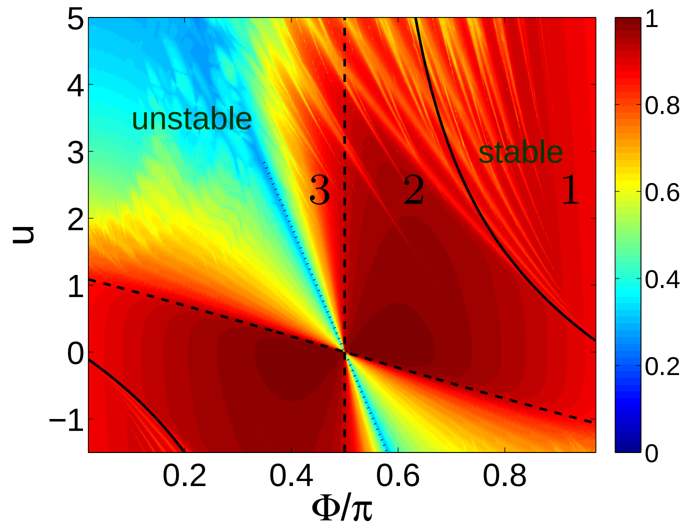
(1) Energetic stability; **(2)** Dynamical stability.

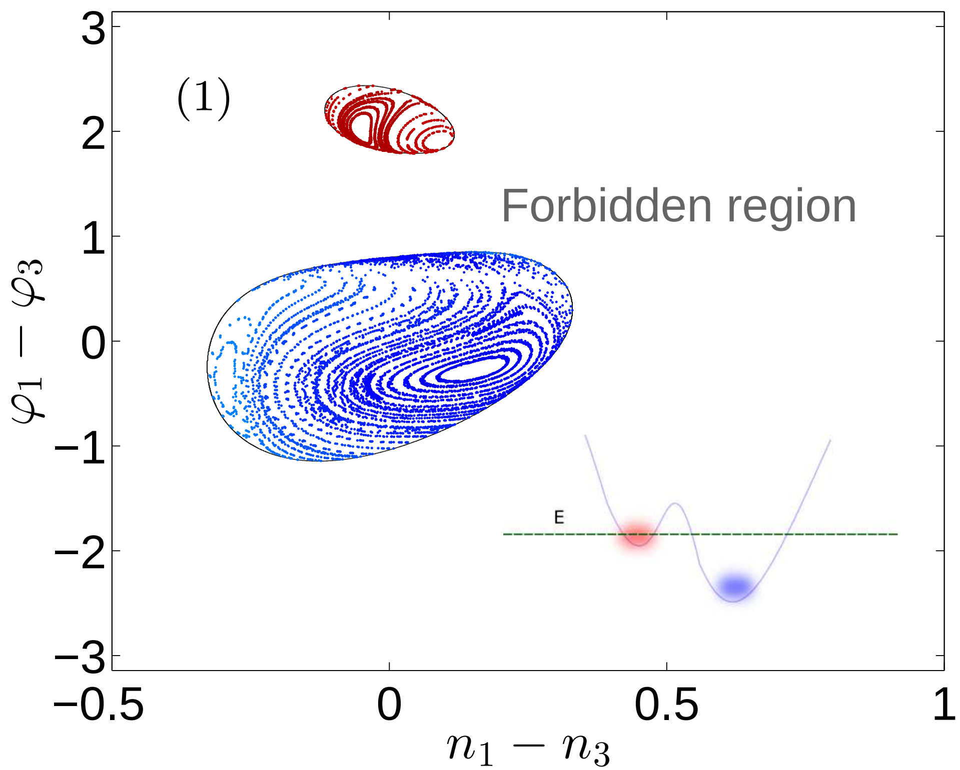
red trajectories = large positive current

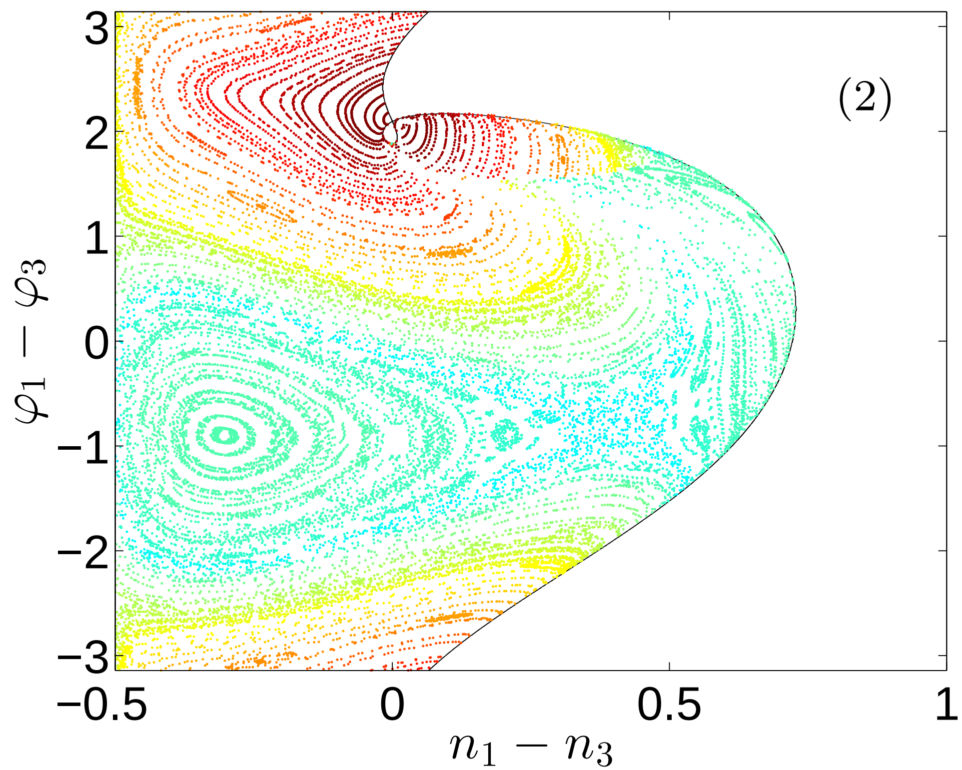
blue trajectories = large negative current

The flow-state fixed-points are located along the symmetry axis:

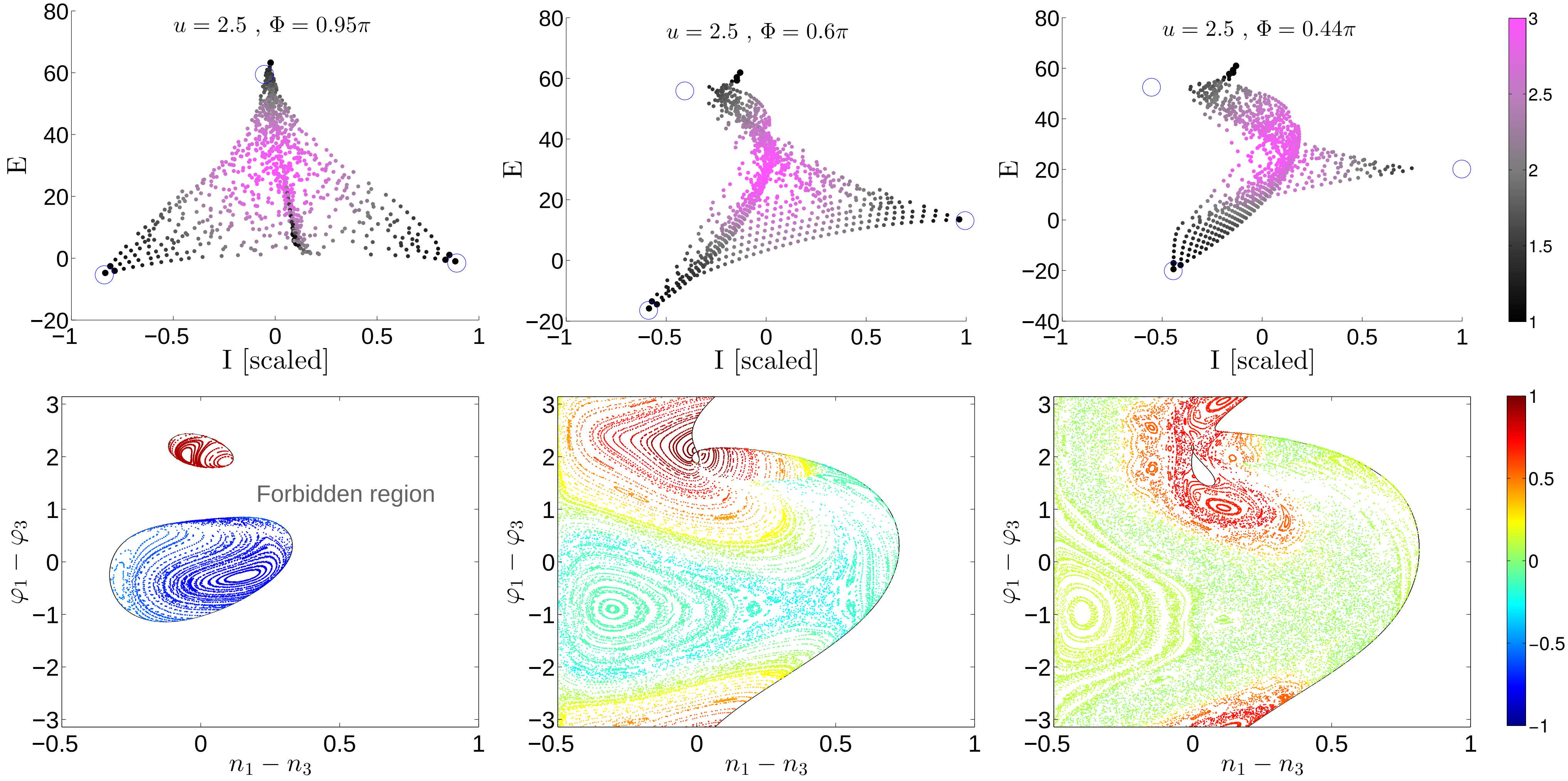
$$n_1 = n_2 = \cdots = N/M, \qquad \varphi_i - \varphi_{i-1} = \left(\frac{2\pi}{M}\right) m$$





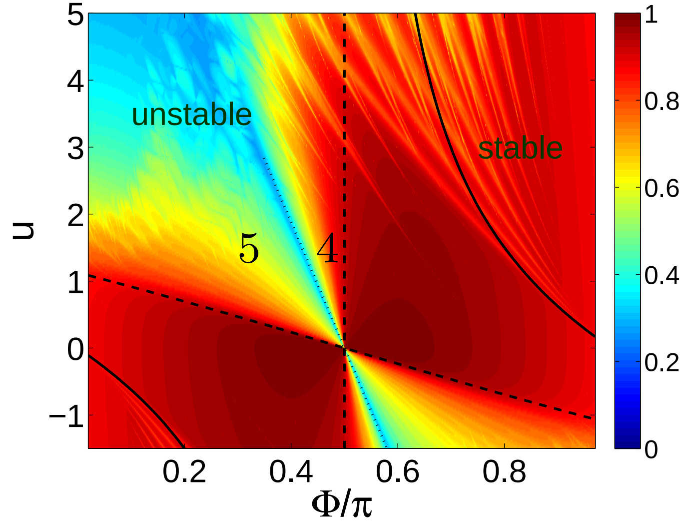


KAM stability - elliptic islands and chaotic ponds



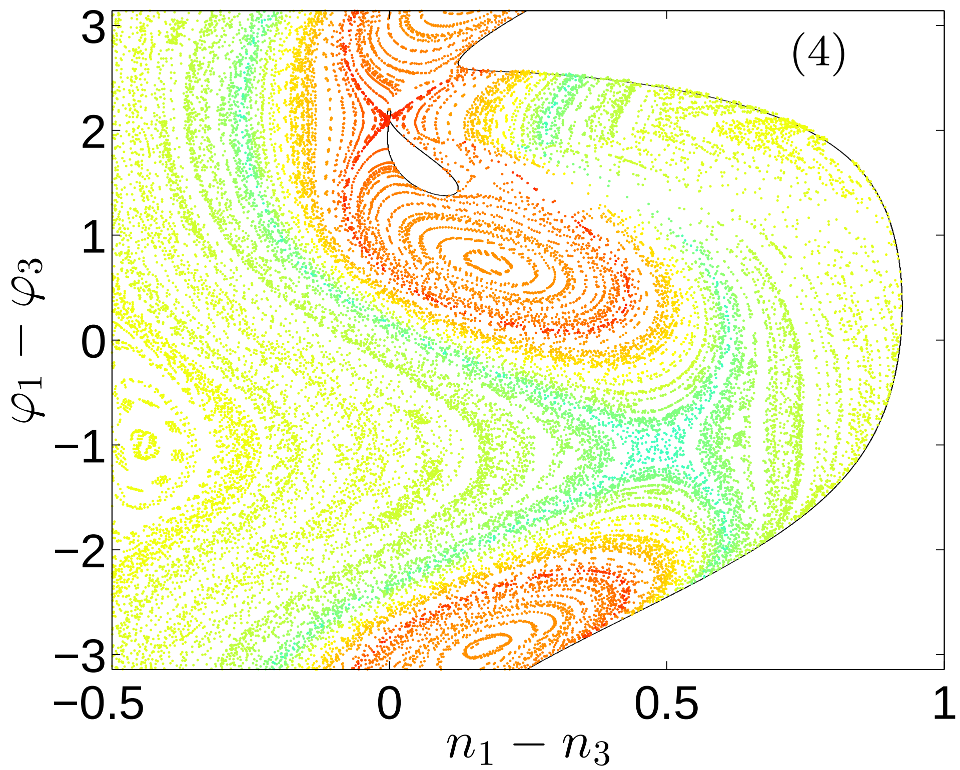
Swap transition

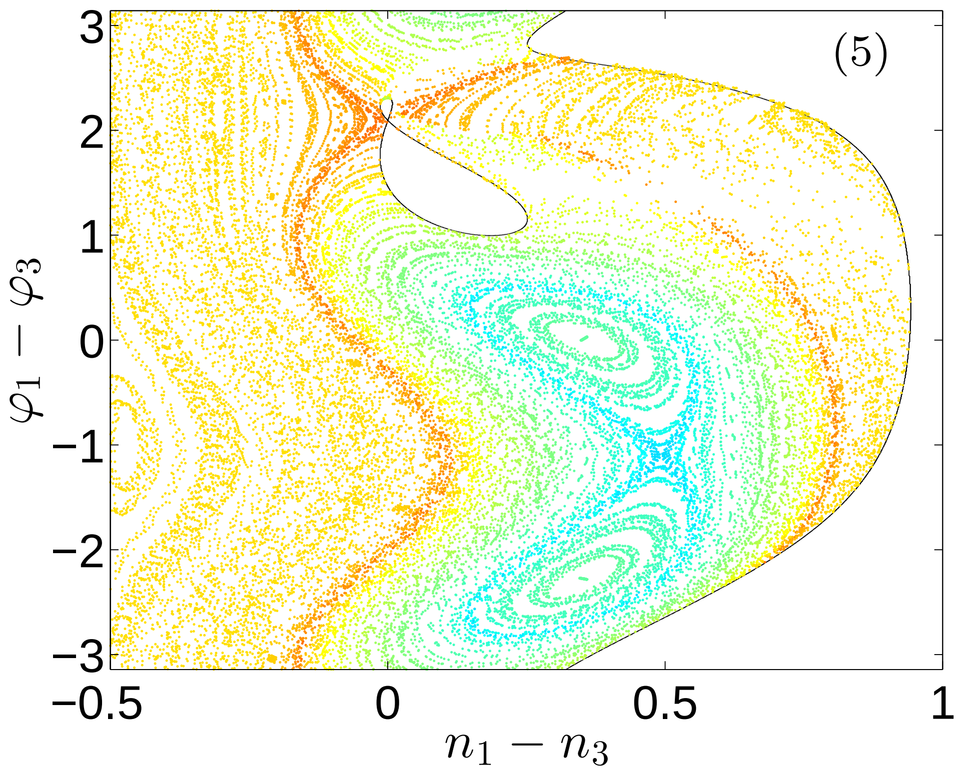
In **(4)** and **(5)** dynamical stability is lost $\rightsquigarrow$ chaotic motion.
But the chaotic trajectory is confined within a chaotic pond;
uni-directional chaotic motion; superfluidity persists!
At the separatrix swap-transition superfluidity diminishes.



Swap transition (dotted line):

$$u = 18 \sin\left(\frac{\pi}{6} - \frac{\Phi}{3}\right)$$





Manifestation of phase space topology for $M > 3$ circuits

Number of freedoms: $d = (M-1)$

$d = 2$ Mixed phase space: islands, ponds, and chaotic sea

$d > 2$ High dimensional chaos: Arnold web and chaotic sea

- The energy surface is $2d-1$ dimensional
- KAM tori are d dimensional
- The KAM tori are not effective in blocking the transport on the energy shell if $d > 2$.
- Resonances form an "Arnold Web" $\rightsquigarrow$ "Arnold diffusion"
- As u becomes larger this non-linear leakage effect is enhanced, stability of the motion is deteriorated, and the current is diminished.

For $M = 3$ the 3 dimensional energy surface is divided into territories by the 2 dimensional KAM tori.

For $M = 4$ the 5 dimensional energy surface cannot be divided into territories by the 3 dimensional KAM tori.

Work in progress

Metastability - the big picture

- (traditional) Energetic metastability, aka Landau criterion.
- (traditional) Dynamical metastability via linear stability analysis, aka BdG.
- Strict dynamical metastability (KAM, applies if $d = 2$)
- Quasi dynamical metastability (might be the case for $d > 2$)

In the absence of constants of motion, a generic system with $d > 2$ degrees-of-freedom is always ergodic. But the equilibration might be an extremely slow process.

Quasi stability might become Quantum stability due to dynamical localization.
The breaktime is determined from the breakdown of the QCC requirement:

$$t \quad \ll \quad t_H[\Omega(t)] \qquad \rightsquigarrow \qquad t^*$$

Chirikov, Izrailev, Shepelyansky [SovSciRevC 1981]; Shepelyansky [PhysicaD 1987];

Heller, *Quantum localization and the rate of exploration of phase space* [PRA 1987];

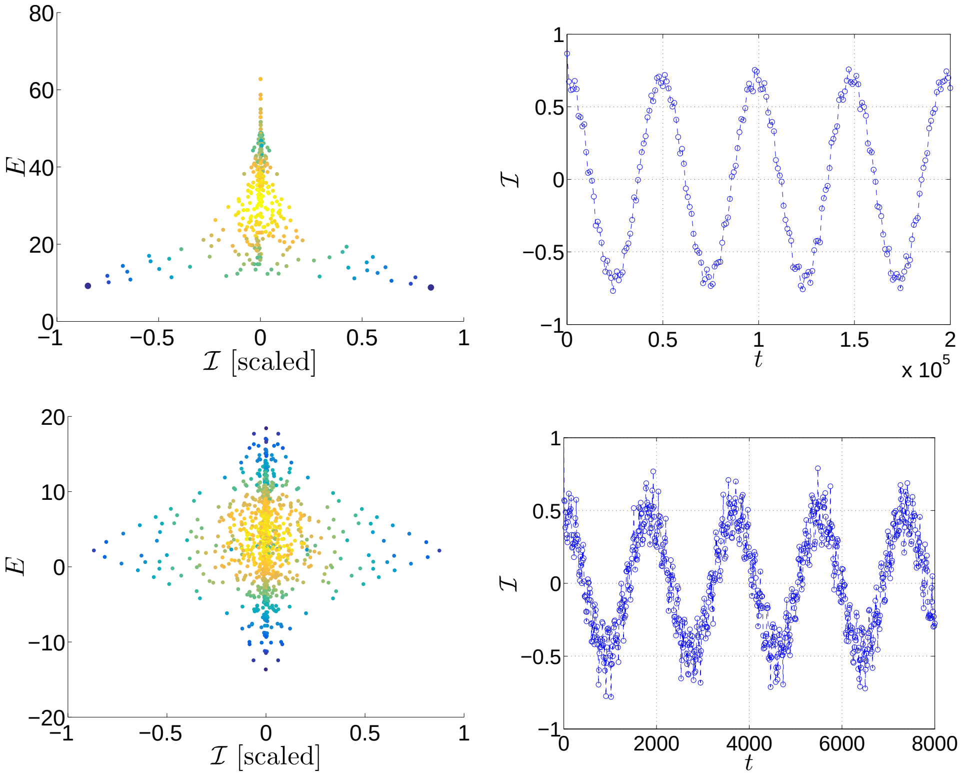
Dittrich, *Spectral statistics for 1D disordered systems* [Phys Rep 1996];

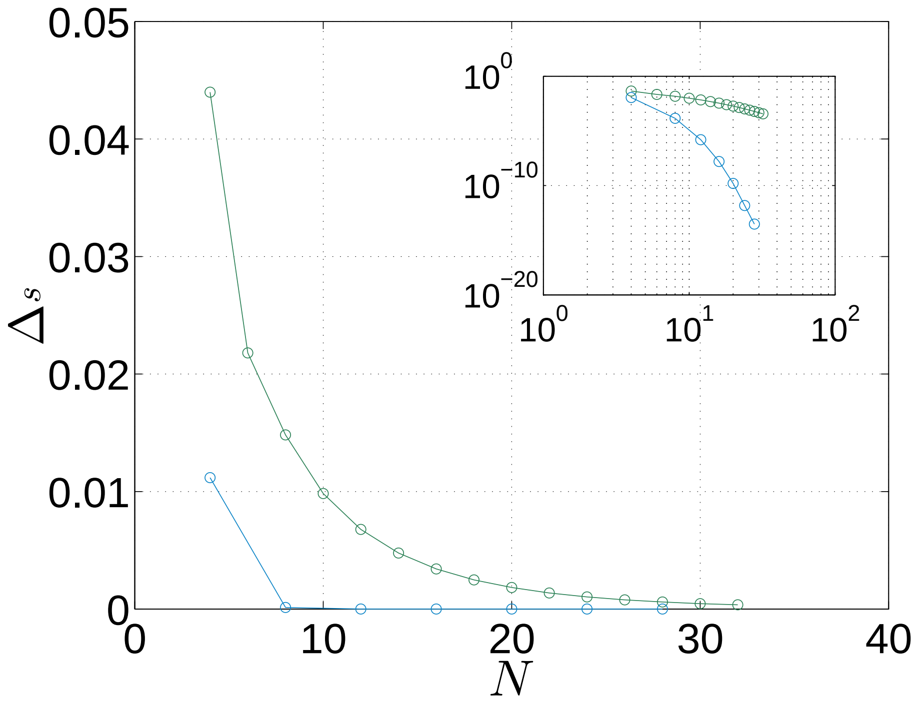
Cohen, *Periodic Orbits Breaktime and Localization* [JPA 1998];

Cohen, Yukalov, Ziegler, *Hilbert-space localization in closed quantum systems* [PRA 2016].

Implication: violation of the Eigenstate Thermalization Hypothesis.

Coherent Rabi oscillations between flow-states





Contrasting $\Phi\!=\!\pi$ with $\Phi\!=\!0$ for the $M\!=\!4$ circuit.

Upper panel: $N = 24$ bosons in $M = 3$ ring with $u = 5$, and $\Phi = \pi$. Under the barrier tunneling.

Lower panel: $N = 16$ bosons in $M = 4$ ring with $u = 1$, and $\Phi = 0$. Chaos-assisted tunneling.

The introduction of a weak link

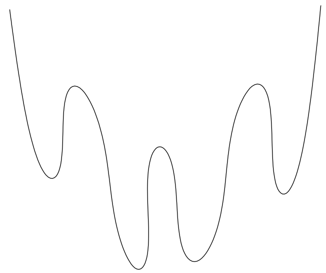
The "system plus bath" perspective is expected to be valid if $M \gg 1$.

$$\mathcal{H}_{\mathrm{JCH}} \;=\; E_C\, \boldsymbol{n}^2 + \frac{1}{2} E_L \boldsymbol{\varphi}^2 - E_J\, \cos(\boldsymbol{\varphi} - \Phi) + \mathcal{H}_{\mathrm{bath}}$$

with $E_C = U$, and $E_L = [(N/M)/(M-1)]K$, and $E_J = (N/M)K'$.

$$\alpha \equiv \frac{E_J}{E_L}$$

$\alpha > 1$

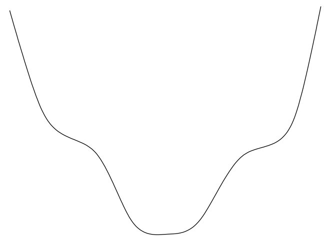


The bath Hamiltonian has the standard Caldeira-Leggett form

$$\mathcal{H}_{\mathrm{bath}} \;=\; \sum_m \left(\frac{1}{2\mathsf{m}_m} \tilde{\boldsymbol{n}}_m^2 + \frac{1}{2} \mathsf{m}_m \omega_m^2 \left(\tilde{\boldsymbol{\varphi}}_m - \frac{c_m}{\mathsf{m}_m \omega_m^2} \boldsymbol{\varphi} \right)^2 \right)$$

$$J(\omega) \;\equiv\; \frac{\pi}{2} \sum_m \frac{c_m^2}{\mathsf{m}_m \omega_m} \delta(\omega - \omega_m) \;=\; \eta\omega\ (\omega < \omega_c),$$

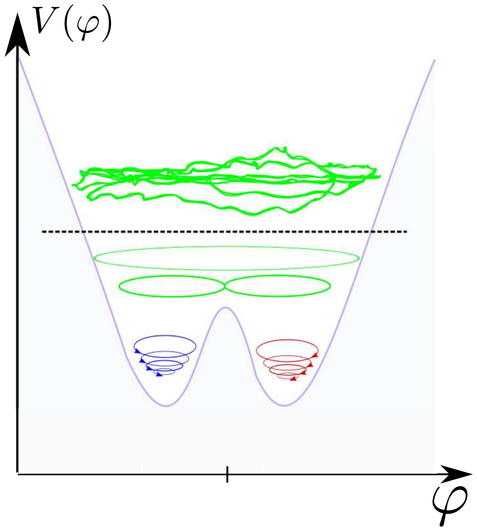
$\alpha < 1$

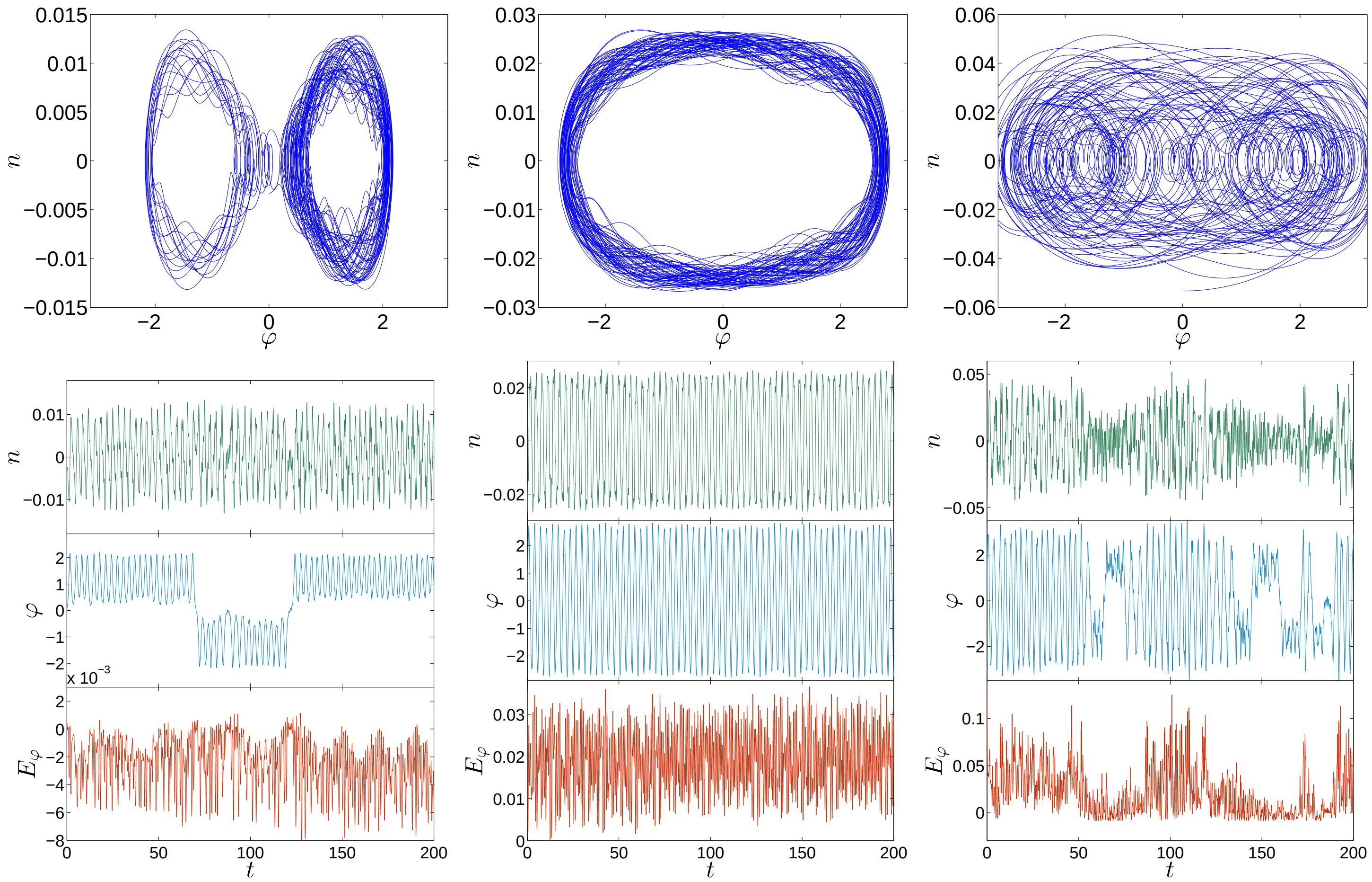


$$\eta \;=\; \frac{\pi}{\sqrt{\gamma}}, \qquad \gamma \;\equiv\; \frac{\mathsf{m}^* g}{\rho} \;=\; \frac{U}{\bar{n} K} \;=\; \frac{Mu}{N^2}$$

Coherent oscillations are feasible only in the Mott regime

The chaos threshold, rings with $M \geq 6$





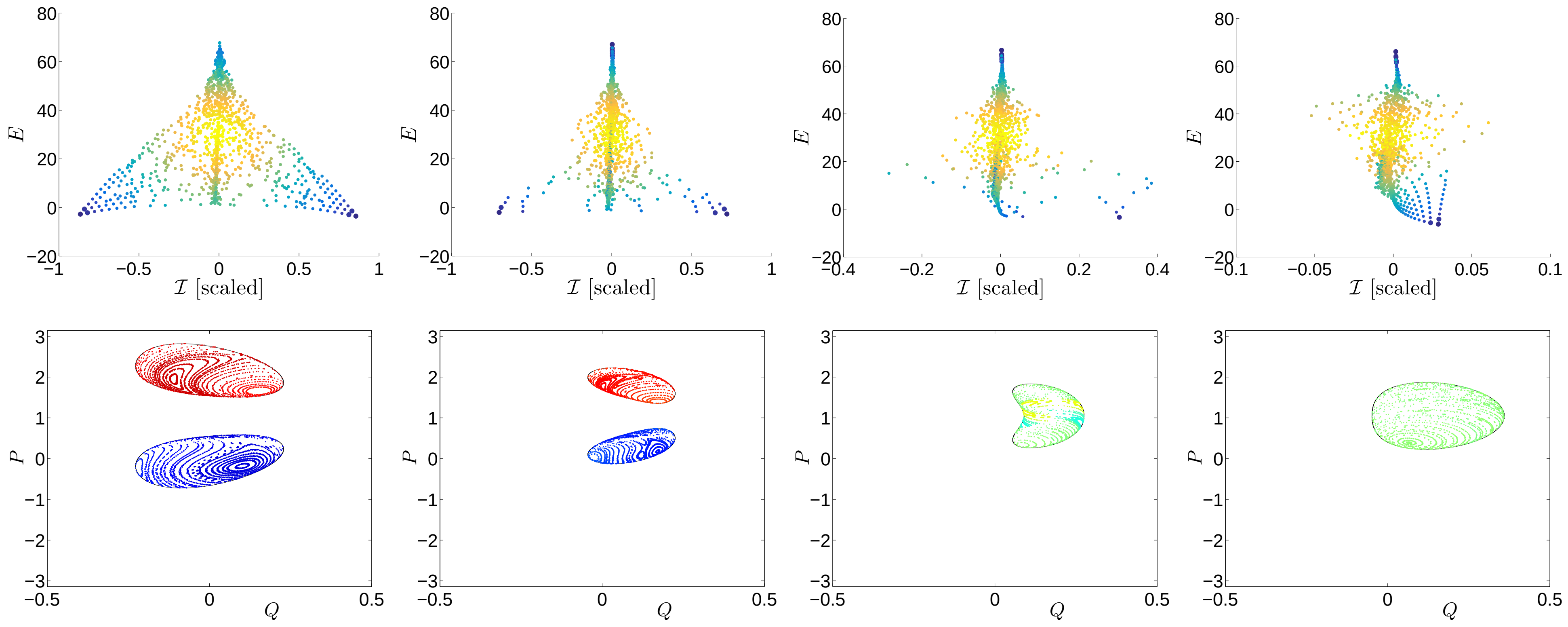
This picture is valid provided $M \geq 6$.

$$\alpha \;\equiv\; \frac{E_J}{E_L} \;=\; (M{-}1)\;\frac{K'}{K}$$

Double well for $\alpha > 1$

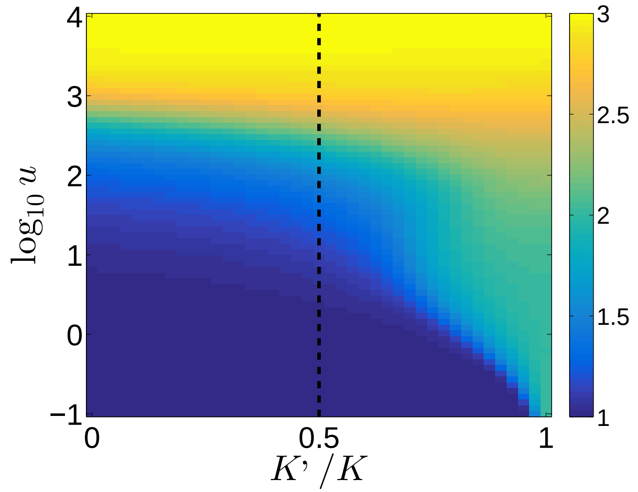
Here: $M = 6$, and $u = 200$, and $\Phi = \pi$, and $K'/K = 0.3$.

The phase-space for small rings $M < 6$



Here $M = 3$ and the interaction is $u = 2.5$.
The weak-link coupling ratio:
$K'/K = \ 1.0, \ 0.8, \ 0.65, \ 0.4.$

The global picture: see image on the right.
The vertical border assumes $N, u \to \infty$.
For finite N there is Mott transition at $u \sim N^2/M$.

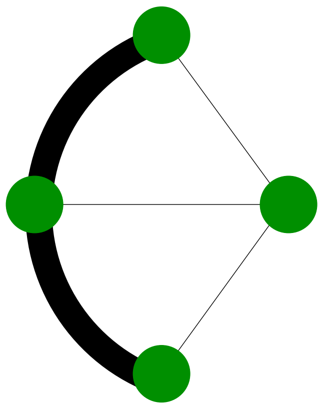


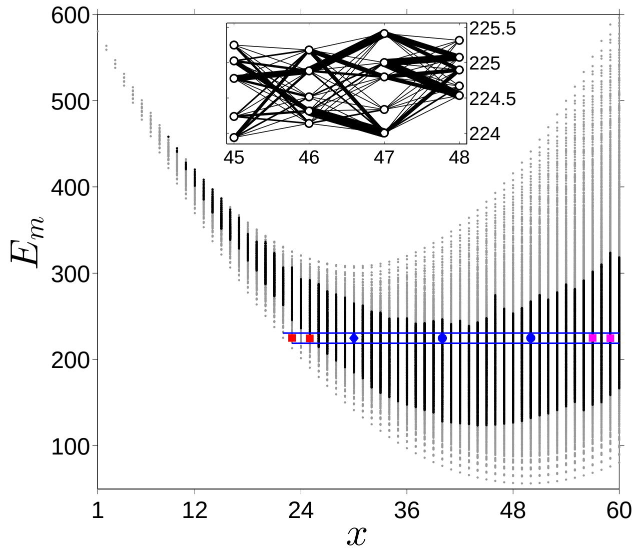
The minimal model for thermalization [CK,AV,DC (NJP 2015)]

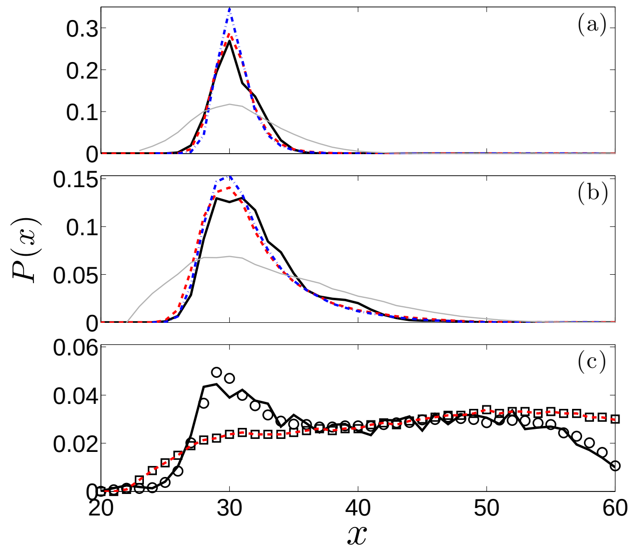
The FPE description makes sense if the sub-systems are chaotic.

Minimal model for a chaotic sub-system: BHH trimer.

Minimal model for thermalization: BHH trimer + monomer







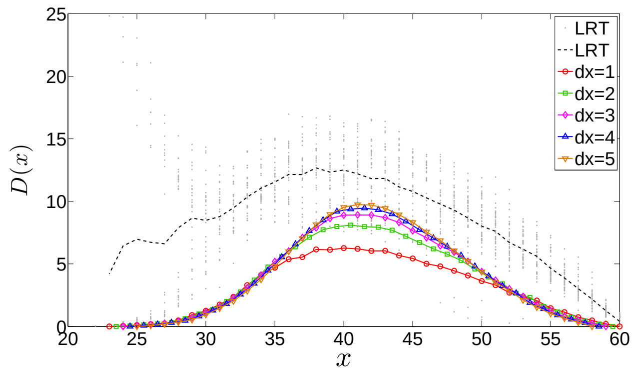
$N = 60 =$ number of particles

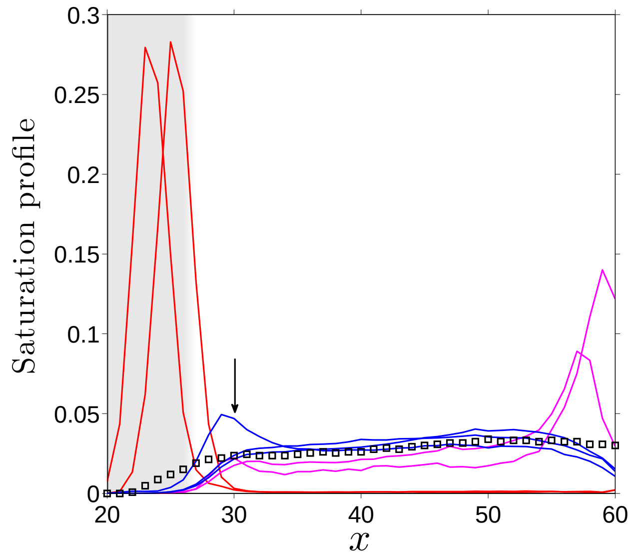
$x =$ occupation of the trimer

$N-x =$ occupation of the monomer

$\rho(x) =$ probability distribution

$$\frac{\partial \rho(x)}{\partial t} = \frac{\partial}{\partial x}\left[g(x) D(x) \frac{\partial}{\partial x}\left(\frac{\rho(x)}{g(x)}\right)\right]$$



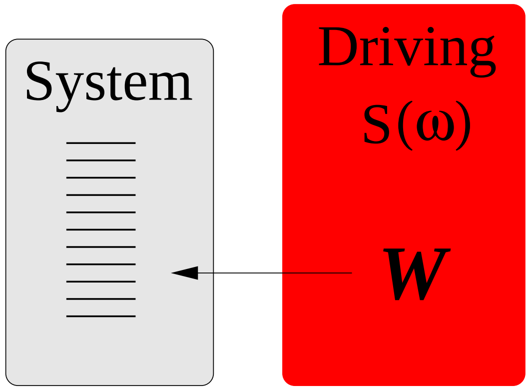


The fluctuation-diffusion-dissipation relation

Rate of energy absorption (work):

$$A(\varepsilon) \;=\; \partial_\varepsilon D_\varepsilon + \beta(\varepsilon) D_\varepsilon, \qquad\qquad \dot{W} = \langle A \rangle$$

$$D_\varepsilon \;=\; \int_0^\infty \frac{d\omega}{2\pi}\, \omega^2\, \tilde{C}_\varepsilon(\omega)\, \tilde{S}(\omega)$$



Derivation:

$$\frac{\partial \rho}{\partial t} \;=\; \frac{\partial}{\partial \varepsilon}\left(g(\varepsilon) D(\varepsilon) \frac{\partial}{\partial \varepsilon}\left(\frac{1}{g(\varepsilon)}\rho\right)\right) \;=\; -\frac{\partial}{\partial \varepsilon}\left(A(\varepsilon)\rho - \frac{\partial}{\partial \varepsilon}\left[D(\varepsilon)\rho\right]\right)$$

M. Wilkinson (1988), based on the diffusion picture of **Ott (1979)**

C. Jarzynski (1995) - adding FPE perspective.

D. Cohen (1999) - adding FDT perspective + addressing the quantum case.

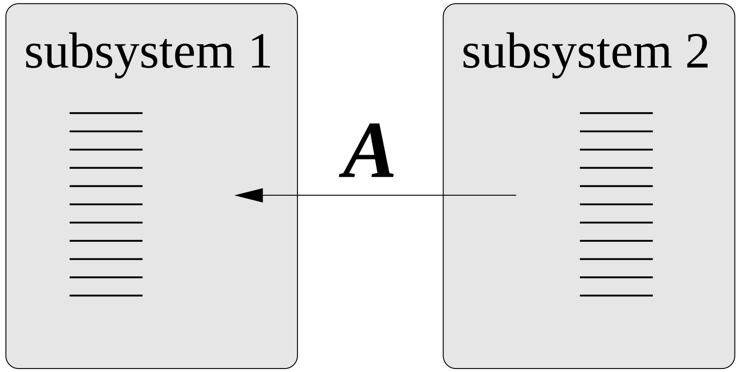
G. Bunin, L. D'Alessio, Y. Kafri, A. Polkovnikov (2011) - adding NFT based derivation.

Thermalization of two subsystems

Rate of energy transfer [FPE version]:

$$A(\varepsilon) \;=\; \partial_\varepsilon D_\varepsilon + (\beta_1 - \beta_2) D_\varepsilon$$

$$D_\varepsilon \;=\; \int_0^\infty \frac{d\omega}{2\pi}\, \omega^2\, \tilde{S}^{(1)}(\omega)\, \tilde{S}^{(2)}(\omega)$$



Derivation: **[Tikhonenkov, Vardi, Anglin, Cohen (PRL 2013)]**

The diffusion is along constant energy lines: $\varepsilon_1 + \varepsilon_2 = \mathcal{E}$

The proper Liouville measure is: $g(\varepsilon) = g_1(\varepsilon) g_2(\mathcal{E} - \varepsilon)$

Note: After canonincal preparation of the two subsystems:

$$\langle A(\varepsilon)\rangle \;=\; \left(\frac{1}{T_1} - \frac{1}{T_2}\right) \langle D_\varepsilon \rangle$$

MEQ version: **Hurowitz, Cohen (EPL 2011)**

NFT version: **Bunin, Kafri (JPA 2013)**