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Random Matrix Theory for Transition Strengths in Finite Quantum Many-particle Systems: Applications and Open Questions

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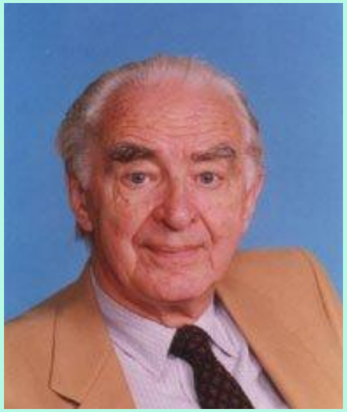
1. Introduction

E.P. Wigner



J.B. French

Finite (small) isolated many particle systems:

Complex nuclei, atoms, molecules (also biological molecules), small devices of condensed matter and quantum optics on Nano- and micro-scale (ex: quantum dots, small metallic grains), cold atoms in optical lattices, ion traps, implementations of quantum computers involving many interacting *q*-bits , ------

$$\boldsymbol{H} = \underbrace{\boldsymbol{h}(1)}_{one-body} + \underbrace{\boldsymbol{V}(2)}_{two-body}$$

$$H = h(1) + \lambda V(2)$$

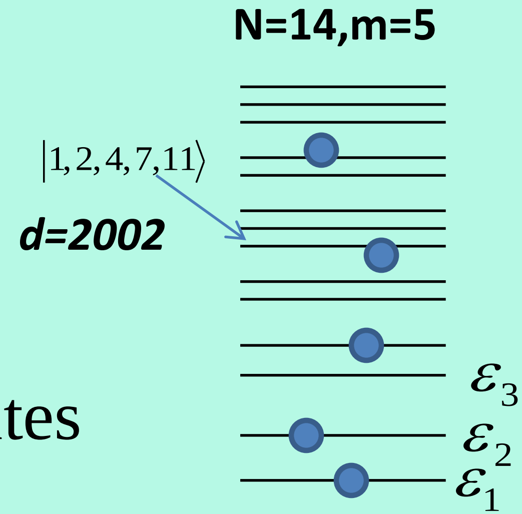
$$h(1) = \sum_i \varepsilon_i n_i \;\; ; \; n_i = a_i^\dagger a_i$$

$$V(2) = \sum_{i>j,k>l} \langle kl | V | ij \rangle a_k^\dagger a_l^\dagger a_j a_i$$

with m fermion (or bosons) in say N sp states

$$d = \binom{N}{m} \text{ or } \binom{N+m-1}{m}$$

Now, H will be $d \times d$ matrix in m particle spaces



Single particle spectrum

Large scale diagonalization using nuclear shell model, atomic structure calc, one dimensional interacting spins systems ------- all with fixed *h* and *V* showed clearly:

statistical regularities in spectral averages, due to quantum chaos coming into play, with λ increasing

Although used with increasing frequency in many branches of Physics, (classical) random matrix ensembles sometimes are <u>too unspecific</u> to account for important features of the physical system at hand. One important refinement which retains the basic stochastic approach but allows for such features (to describe statistical properties) consists in the use of <u>embedded ensembles</u>

interaction strength in units of Δ

$$\hat{H} = \hat{h}(1) + \lambda\left\{\hat{V}(2)\right\}$$

$$\hat{h}(1) = \sum_i \varepsilon_i \hat{n}_i \;;\; \varepsilon_i$$

fixed (TBRIM)

random (TBRIM)

drawn from GOE (RIMM)

$\hat{V}(2)$ a random interaction $\Leftrightarrow$

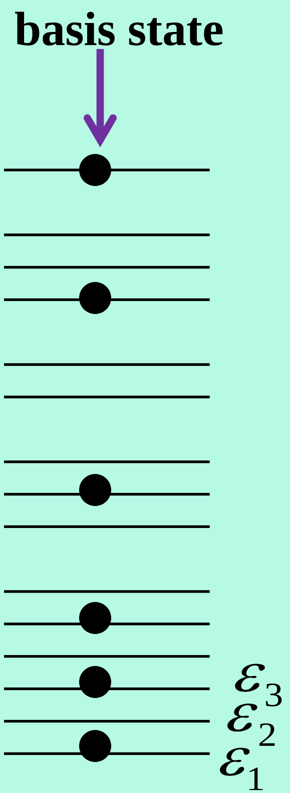
$\left\{\hat{V}(2)\right\}$ is GOE(1) in 2-particle space

Given *m* fermions in *N* sp states, we have in *m*-particle spaces embedded GOE of one plus two-body interactions: EGOE(1+2)

$$d_f(12,2) = 66, \quad d_f(16,2) = 120$$

$$d_f(12,6) = 924, \quad d_f(16,8) = 12870$$

EGOE(1+2) $\Leftrightarrow$ (m,N,λ/Δ)



Single particle spectrum

Δ is average spacing

for interacting boson systems we have BEGOE(1+2)

EE's generate Gaussian eigenvalue densities independent of λ; convergence is asymptotic; for small λ Poisson fluctuations

In EE, just as in many realistic systems, the behavior of various observables continues to evolve ,even after NNSD is stabilized, with the strength (λ) of the perturbation. Therefore, more generally, quantum chaos is defined in terms of the (chaotic) structure of eigenstates, rather than in terms of level statistics. [BISZ]

As λ increases strength functions change form from BW to Gaussian and with further increase there will be thermalization with maximal wavefunction delocalization (within a energy shell!). Here with $\lambda \sim \lambda_t$, the spreading produced by *h(1)* and *V(2)* will be equal and thus generate maximum mixing with strength functions Gaussian and fluctuations GOE.

Many-body chaos $\Leftrightarrow$ thermalization $\Leftrightarrow$ RMT-EE $\leftarrow$complex nuclei

K.K. Mon and J.B. French, Ann. Phys. (N.Y.) 95 (1975) 90

VKBK, Phys. Rep. 347 (2001) 223.

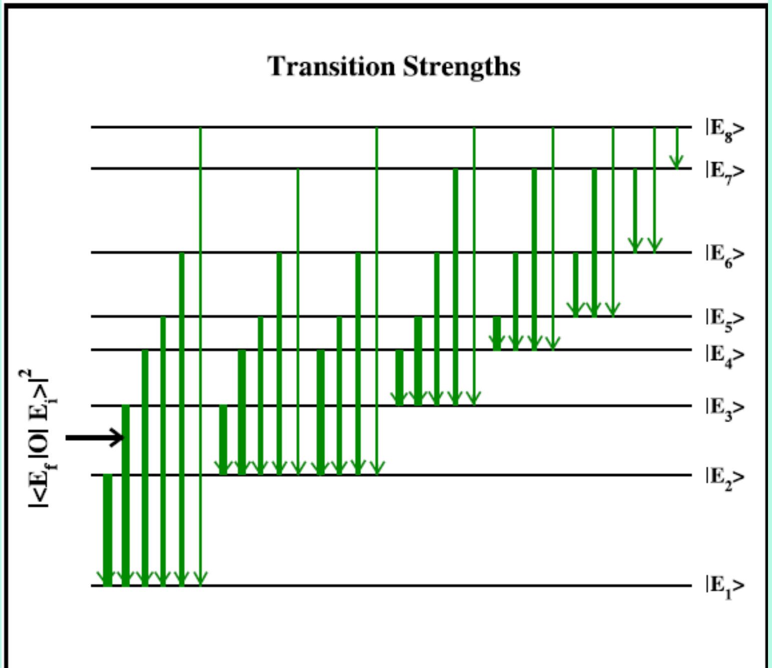
VKBK, Embedded Random Matrix Ensembles in Quantum Physics, Lecture Notes in Physics, Volume 884 (Springer, Heidelberg, 2014).

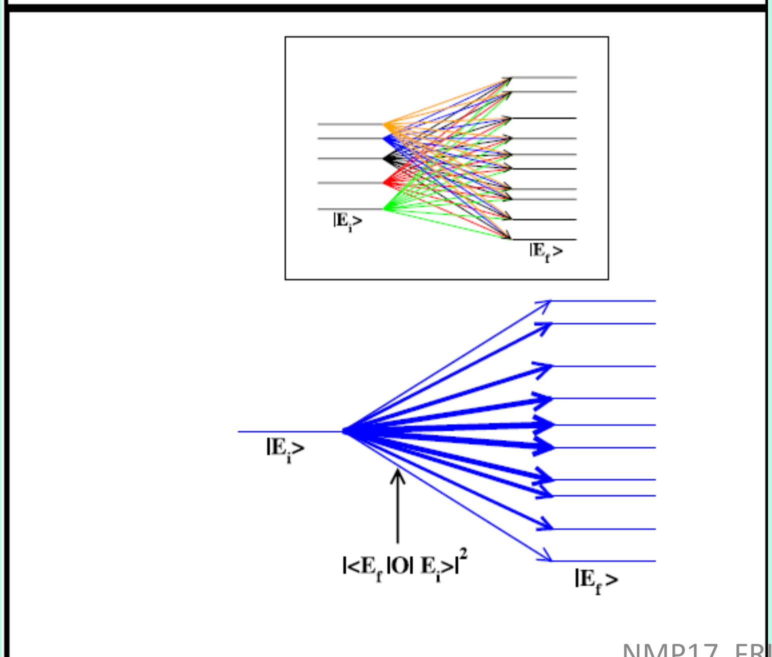
F. Borgonovi, F.M. Izrailev, L.F. Santos, and V.G. Zelevinsky, Phys. Rep. 626 (2016) 1.

Gaussian eigenvalue densities and Gaussian strength functions applied inconfiguration- *J* spaces led to the interacting particle theory for nuclear level densities: Sen'kov and Zelevinsky, Phys. Rev. C 93, (2016) 064304 ; French et al, Can. J. Phys. 84 (2006) 677.

In this talk we will describe the current status of EE theory for transition strengths and one new application

2. RMT-EE for Transition Strengths





Transition strength density

$$I_O(E_i,E_f) = I(E_f)\ |\langle E_f|O|E_i\rangle|^2\ I(E_i)$$
$$= \langle\langle O^{\dagger}\delta(H-E_f)O\delta(H-E_i)\rangle\rangle$$

It is a bivariate density (with other quantum numbers - multivariate)

What is the form of $\overline{I_O(E_i,E_f)}$?

(i) *H* is a EGOE/EGUE/EGSE
**   *O* is fixed : DFW/HBZ** } problems

(ii) *H* is a EGOE/EGUE/EGSE
**   *O* is another independent EGOE/EGUE/EGSE**
-- used first by FKPT (Fl-KS-KM)

#(ii) gives results consistent with numerical Embedded Ensemble/Nuclear Shell Model

What is the form of $\overline{\rho_{\boldsymbol{O}}^{\boldsymbol{H}}\left(\boldsymbol{E}_i,\boldsymbol{E}_f\right)} = \overline{\left\langle \boldsymbol{O}^{\dagger}\delta(\boldsymbol{H}-\boldsymbol{E}_f)\boldsymbol{O}\delta(\boldsymbol{H}-\boldsymbol{E}_i)\right\rangle}$?

Here we employ EGOE/EGUE/EGSE for both $\boldsymbol{H}$ and $\boldsymbol{O}$ operators

$$\boxed{\overline{\rho_O^H(E_i,E_f)} \Leftrightarrow \text{ moments } M_{PQ} = \overline{\left\langle O^{\dagger}H^{Q}OH^{P}\right\rangle}}$$

$$\hat{M}_{PQ} = M_{PQ} / M_{00}; \quad \mu_{PQ} = \left[\hat{M}_{20}\right]^{-P/2}\left[\hat{M}_{02}\right]^{-Q/2}\hat{M}_{PQ}$$

$\mu_{11} = \xi$ is the bivariate correlation coefficient

fourth order cumulants (shape parameters) are:

$$k_{40} = \mu_{40} - 3,\ k_{04} = \mu_{04} - 3,\ k_{31} = \mu_{31} - 3\xi,$$

$$k_{13} = \mu_{13} - 3\xi,\ k_{22} = \mu_{22} - 2\xi^2 - 1$$

Second and fourth order moments from RMT-EE:
H represented by EGUE(*k*)
O represented by an independent EGUE

VKBK and Manan Vyas, Ann. Phys. (N.Y.) 359 (2015) 252-289

Results are derived for:

O

I — ***t*-body operator ; identical spinless fermions [*H* a *k*-body operator – U(*N*) algebra]**

II — **k_0 number of particles addition or removal operator; identical spinless fermions [*H* a *k*-body operator – U(*N*) algebra]**

III — **β- decay or 0ν-$\beta\beta$ decay type operators; involves two types of spinless fermions [$H(k)=H_{11}+H_{22}+H_{12}$ and U(N_1)+U(N_2) algebra]**

Extensions to boson systems: some results are available

Independence of the EGUE's representing the *H* and *O* operators implies that we are removing the *H*-*O* correlated part from the transition operator *O*. It is well known* that

$$\langle O \rangle^m, \langle OH \rangle^m \text{ and } \langle OH^2 \rangle^m$$

determine the expectation values of *O* operator.

***Draayer, French and Wong, Ann. Phys. (N.Y.) 106 (1977) 472**
VKBK, Ann. Phys. (N.Y.) 306 (2003) 58-77

Example: **beta decay and neutrinoless double beta decay type transition operators**

$$\hat{H} = \sum_{i+j=k} \sum_{\alpha,\beta,a,b} V_{\alpha a;\beta b}(i,j) A^{\dagger}(f_i v_{\alpha}) A(f_i v_{\beta}) A^{\dagger}(f_j v_a) A(f_j v_b);$$

$$V_{\alpha a;\beta b}(i,j) = \left\langle f_i v_{\alpha} f_j v_a \left| \hat{H} \right| f_i v_{\beta} f_j v_b \right\rangle \ ; i+j=2 \text{ for nuclei}$$

$$\hat{O} = \sum O_{\lambda d} A^{\dagger}(f_{k_0} v_{\lambda}) A(f_{k_0} v_d)$$

$$O_{\lambda d} = \left\langle \#1 : k_0 \lambda \left| O \right| \#2 : k_0 d \right\rangle; \ \ k_0 = 1(\beta \text{ decay}),\ 2(\text{NDBD})$$

Note that we have $U(N_1)+U(N_2)$ symmetry for H. Given m_1 particles in #1 and m_2 in #2, the irreps are (m_1,m_2) and the action of O changes (m_1,m_2) to $(m_1+k_0 , m_2 - k_0)$

GUE representation for both $V_{\alpha a:\beta b}(i,j)$ and $O_{\alpha,a}$ implies

$$\overline{O_{\alpha_1,a_1} O^{\dagger}_{\alpha_2,a_2}} = V_O^2 \; \delta_{\alpha_1,\alpha_2} \delta_{a_1,a_2} \longrightarrow \textbf{\textit{O} matrix is rectangular}$$

$$\overline{V_{\alpha a:\beta b}(i,j) V_{\lambda c:\mu d}(i',j')} = V_H^2(i,j) \; \delta_{i,i'} \delta_{j,j'} \delta_{\alpha,\mu} \delta_{\beta,\lambda} \delta_{a,d} \delta_{b,c}$$

Unitary decomposition of *H* w.r.t. *U(N₁)+U(N₂)* gives

$$\hat{H} = \sum_{i+j=k} \sum_{\nu,\omega_\nu,\nu',\omega_{\nu'}} W_{ij}(\nu,\omega_\nu;\nu',\omega_{\nu'}) \underbrace{B_i(\nu,\omega_\nu)}_{\#1} \underbrace{C_j(\nu',\omega_{\nu'})}_{\#2}$$

W_{ij} are independent Gaussian variables with zero center and variance $V_H^2(i,j)$

$$\hat{O} \text{ is } T^{f_{k_0}} \text{ w.r.t. } U(N_1) \text{ and } T^{\overline{f_{k_0}}} \text{ w.r.t. } U(N_2)$$

$$\overline{\left\langle O^{\dagger}O\right\rangle}^{m_1m_2}=V_O^2\binom{N_1-m_1}{k_0}\binom{m_2}{k_0},\quad \overline{\left\langle OO^{\dagger}\right\rangle}^{m_1m_2}=V_O^2\binom{N_2-m_2}{k_0}\binom{m_1}{k_0}$$

$$\overline{\left\langle \hat{H}^2\right\rangle}^{m_1m_2}=\sum_{i+j=k}V_H^2(i,j)\,\Lambda^0(N_1,m_1,i)\Lambda^0(N_2,m_2,j)$$

gives M_{20} and M_{02}

$$\overline{\left\langle O^{\dagger}O\hat{H}^P\right\rangle}^{m_1m_2}=\overline{\left\langle O^{\dagger}O\right\rangle}^{m_1m_2}\,\overline{\left\langle \hat{H}^P\right\rangle}^{m_1m_2},\quad \overline{\left\langle O^{\dagger}\hat{H}^QO\right\rangle}^{m_1m_2}=\overline{\left\langle O^{\dagger}O\right\rangle}^{m_1m_2}\,\overline{\left\langle \hat{H}^Q\right\rangle}^{m_1+k_0,m_2-k_0}$$

gives M_{40} and M_{04}

$$\overline{\left\langle \hat{H}^4\right\rangle}^{m_1m_2}=2\left[\overline{\left\langle \hat{H}^2\right\rangle}^{m_1m_2}\right]^2+\sum_{i+j=k,r+l=k}V_H^2(i,j)\,V_H^2(r,l)\,(A)\,(B)$$

$$A=\binom{N_1}{m_1}^{-1}\sum_{\nu_1=0}^{\min(i,m_1-r)}\Lambda^{\nu_1}(N_1,m_1,m_1-i)\Lambda^{\nu_1}(N_1,m_1,r)d(\nu_1)$$

$B\leftrightarrow A$ *with* $1\rightarrow 2$ and $(i,r)\rightarrow(j,l)$

$$\Lambda^{\nu}(N,m,k)=\binom{m-\nu}{k}\binom{N-m+k-\nu}{k}$$

The first non-trivial bivariate moment is M_{11} and the formula for this is:

$$M_{11} = V_O^2 \left\{ \binom{N_1}{m_1} \binom{N_2}{m_2} \right\}^{-1} \sum_{i+j=k} V_H^2(i,j) \binom{N_1 - k_0}{m_1} \binom{N_2 - k_0}{m_2 - k_0}$$

$$\times \left[\sum_{\nu_1=0}^{i} X_{11}(N_1, m_1, k_0, i, \nu_1) \right] \left[\sum_{\nu_2=0}^{j} Y_{11}(N_2, m_2, k_0, j, \nu_2) \right] ;$$

$$X_{11}(N_1, m_1, k_0, i, \nu) = \left[\binom{N_1}{k_0} d(N_1 : \nu) \right]^{1/2}$$

$$\times \left[\Lambda^{\nu}(N_1, m_1, m_1 - i) \Lambda^{\nu}(N_1, m_1 + k_0, m_1 + k_0 - i) \right]^{1/2}$$

$$\times (-1)^{\phi(f_{m_1+k_0}, \overline{f_{m_1}}, f_{k_0}) + \phi(f_{m_1}, \overline{f_{m_1}}, \nu)} \; U(f_{m_1+k_0}, \overline{f_{m_1}}, f_{m_1+k_0}, f_{m_1}; f_{k_0}, \nu) \; ,$$

$$Y_{11}(N_2, m_2, k_0, j, \nu) = X_{11}(N_2, m_2 - k_0, k_0, j, \nu).$$

note: $f_r = \left\{1^r\right\}$, $\nu = \left\{2^{\nu}, 1^{N-2\nu}\right\}$. U-coefficient is w.r.t

$U(N_1)$ or $U(N_2)$ and formula for this was given by Hecht.

Formulas in terms of U-coefficients are also derived for $M_{04}, M_{40}, M_{13}, M_{31}, M_{22}$ and they in turn give formulas for all the fourth order cumulants.

asymptotic limit: $N_i \to \infty,\ m_i \to \infty,\ m_i / N_i \to 0$ and k and k_0 fixed

$$M_{11} \to V_O^2 \sum_{i+j=k} V_H^2(i,j) \binom{N_1 - m_1 - i}{k_0} \binom{m_2 - j}{k_0} \binom{N_1}{i} \binom{m_1}{i} \binom{N_2}{j} \binom{m_2}{j}$$

Similar asymptotic limit formulas are also derived for all the fourth order moments/cumulants.

bivariate correlation coefficient and fourth order cumulants for NDBD operator

Nuclei	N_1	m_1	N_2	m_2	k_{40}	k_{04}	k_{13}	k_{31}	k_{22}	ζ
${}^{100}_{42}Mo_{58}$	30	2	32	8	−0.45(−0.39)	−0.42(−0.38)	−0.24(−0.23)	−0.26(−0.25)	−0.20(−0.22)	0.57
${}^{150}_{60}Nd_{90}$	32	10	44	8	−0.27(−0.22)	−0.29(−0.23)	−0.22(−0.18)	−0.20(−0.17)	−0.19(−0.18)	0.72
${}^{154}_{62}Sm_{92}$	32	12	44	10	−0.24(−0.18)	−0.25(−0.18)	−0.19(−0.15)	−0.18(−0.15)	−0.17(−0.15)	0.76
${}^{180}_{74}W_{106}$	32	24	44	24	−0.19(−0.08)	−0.20(−0.08)	−0.17(−0.08)	−0.15(−0.08)	−0.15(−0.08)	0.77
${}^{238}_{92}U_{146}$	44	10	58	20	−0.18(−0.13)	−0.18(−0.13)	−0.15(−0.11)	−0.15(−0.11)	−0.13(−0.11)	0.83

K_0=2

EGOE but not GOE

Results for beta decay / EC (first four β^- , next four EC, remaing two β^+)

Nuclei	N_1	m_1	N_2	m_2	$\zeta_{biv}(m_1, m_2)$	k_{40}	k_{04}	k_{13}	k_{31}	k_{22}
${}^{62}_{27}Co_{35}$	20	7	30	15	0.72	−0.26(−0.18)	−0.27(−0.18)	−0.24(−0.16)	−0.23(−0.16)	−0.22(−0.16)
${}^{64}_{27}Co_{37}$	20	7	30	17	0.73	−0.27(−0.16)	−0.27(−0.16)	−0.24(−0.15)	−0.23(−0.15)	−0.21(−0.15)
${}^{62}_{26}Fe_{36}$	20	6	30	16	0.72	−0.28(−0.18)	−0.28(−0.18)	−0.24(−0.16)	−0.24(−0.16)	−0.22(−0.16)
${}^{68}_{28}Ni_{40}$	20	8	30	20	0.72	−0.27(−0.14)	−0.27(−0.14)	−0.24(−0.13)	−0.23(−0.13)	−0.21(−0.13)
${}^{65}_{32}Ge_{33}$	36	5	36	4	0.55	−0.45(−0.41)	−0.46(−0.42)	−0.35(−0.33)	−0.34(−0.32)	−0.34(−0.34)
${}^{69}_{34}Se_{35}$	36	7	36	6	0.66	−0.36(−0.29)	−0.34(−0.30)	−0.28(−0.25)	−0.28(−0.25)	−0.27(−0.25)
${}^{73}_{36}Kr_{37}$	36	9	36	8	0.72	−0.28(−0.23)	−0.28(−0.23)	−0.24(−0.20)	−0.24(−0.20)	−0.23(−0.20)
${}^{77}_{38}Sr_{39}$	36	11	36	10	0.76	−0.24(−0.19)	−0.24(−0.19)	−0.21(−0.17)	−0.21(−0.17)	−0.20(−0.17)
${}^{85}_{42}Mo_{43}$	36	15	36	14	0.79	−0.20(−0.14)	−0.21(−0.14)	−0.19(−0.13)	−0.18(−0.13)	−0.17(−0.13)
${}^{93}_{46}Pd_{47}$	36	19	36	18	0.80	−0.19(−0.11)	−0.19(−0.11)	−0.18(−0.10)	−0.17(−0.10)	−0.16(−0.10)

K_0=1

Bivariate transition strength density is a bivariate Gaussian

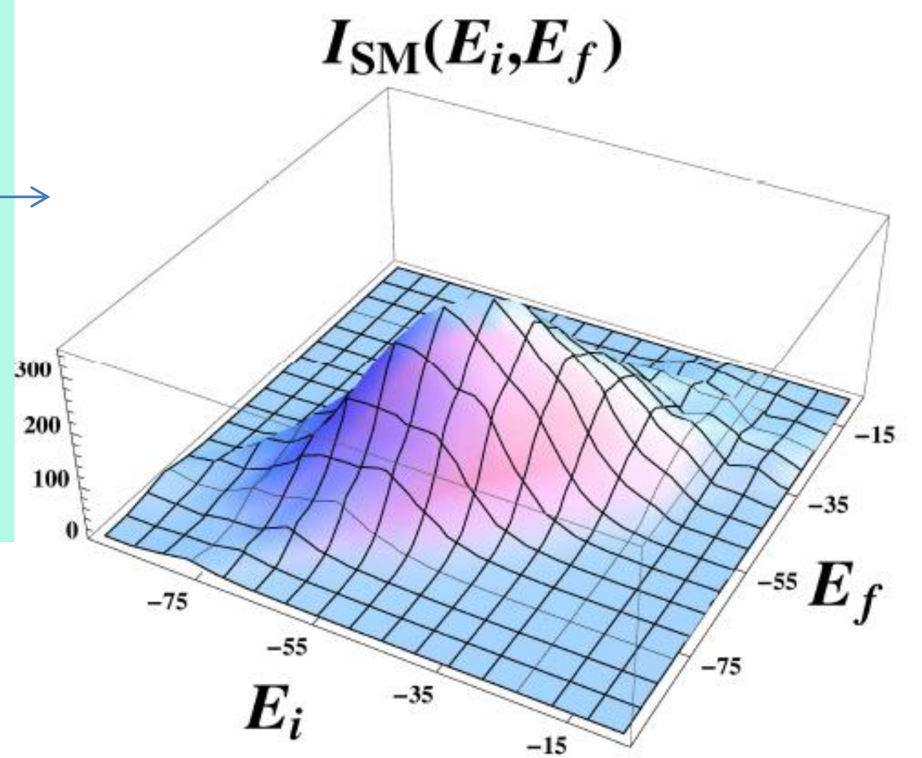
$$\rho_{biv-\mathcal{G}:\mathcal{O}}(E_i, E_f) = \rho_{biv-\mathcal{G}:\mathcal{O}}(E_i, E_f; \varepsilon_i, \varepsilon_f, \sigma_i, \sigma_f, \zeta_{biv}) = \frac{1}{2\pi\sigma_i\sigma_f\sqrt{1-\zeta_{biv}^2}}$$
$$\times \exp -\frac{1}{2(1-\zeta_{biv}^2)}\left\{\left(\frac{E_i-\varepsilon_i}{\sigma_i}\right)^2 - 2\zeta_{biv}\left(\frac{E_i-\varepsilon_i}{\sigma_i}\right)\left(\frac{E_f-\varepsilon_f}{\sigma_f}\right) + \left(\frac{E_f-\varepsilon_f}{\sigma_f}\right)^2\right\}$$

(1)ζ=0 implies GOE and for EGOE examples ζ ~0.6-0.9
∴strength will be around $\hat{E}_i \sim \hat{E}_f$
(2) Expanding the bivariate Gaussian in terms of the product of the marginal Gaussian densities will give

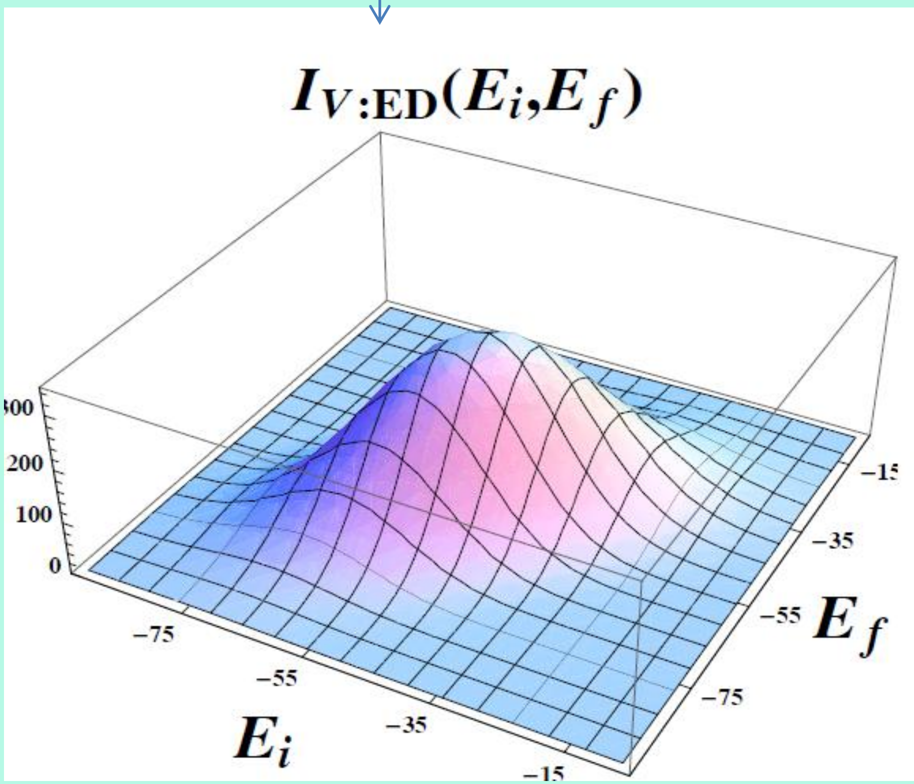
$$\rho_{biv-G}(x,y) = \rho_G(x)\rho_G(y)\sum_\mu \zeta^\mu P_\mu(x)P_\mu(y)$$

∴polynomial expansion of DFPW will not in general converge and this starts with GOE value
(3) EGUE results extend to EGOE
(4) In practice Edgeworth corrections to biv-Gaussian needed

(2s1d) Shell Model example : ^{24}Mg with $(J,T)=(0,0)$



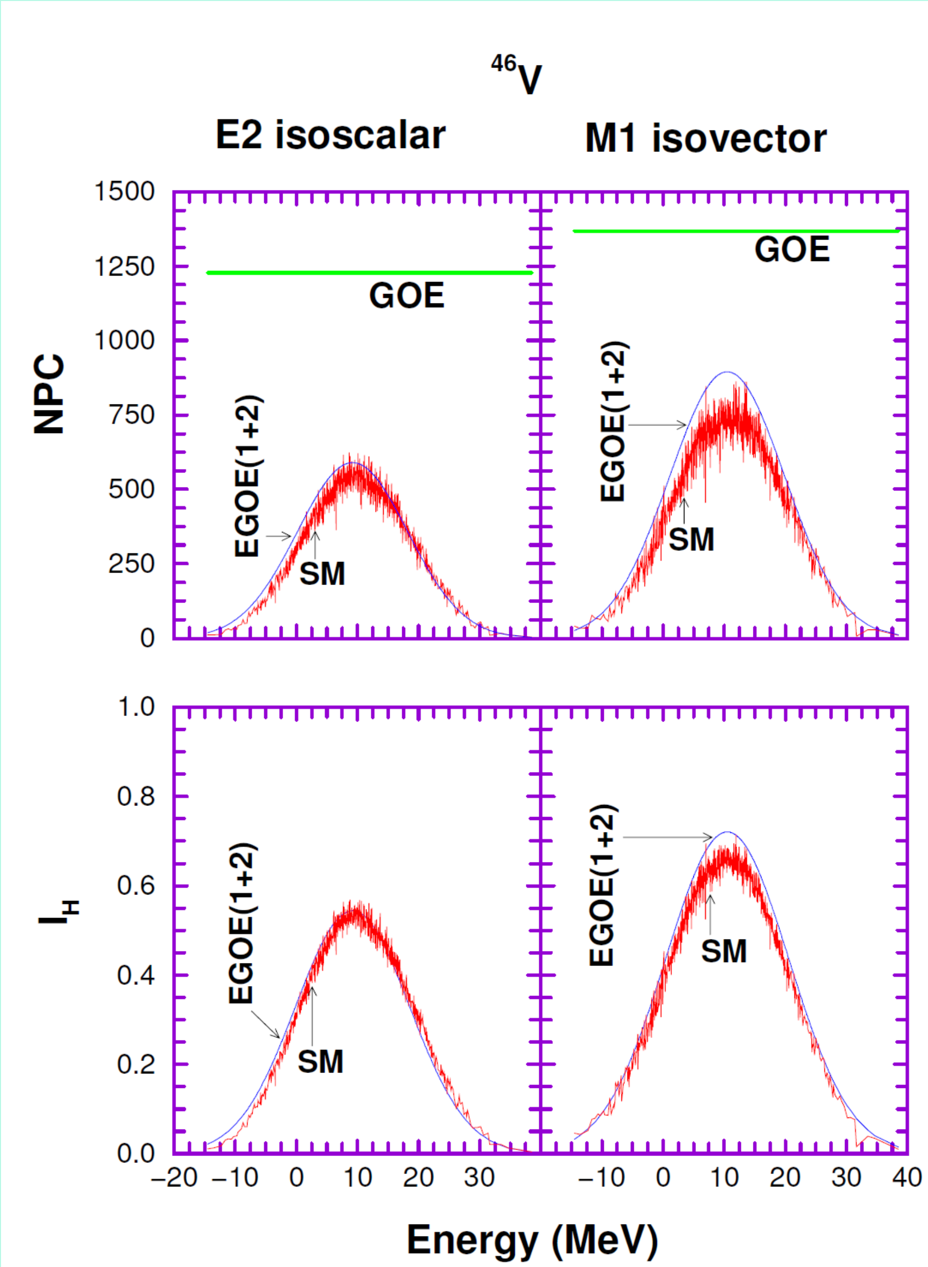
Edgeworth corrected Gaussian with $O=V^{\nu=2}$



Manan Vyas & VKBK

Eur. Phys. J. A 45, 111 (2010)

There are many other direct and indirect examples from nuclear shell model confirming bivariate Gaussian form for the transition strength densities



$$\mathcal{R}(E,E') = \{\langle E|\mathcal{O}^{\dagger}\mathcal{O}|E\rangle\}^{-1} |\langle E'|\mathcal{O}|E\rangle|^2,$$

$$(\mathrm{NPC})_E = \left\{ \sum_{E'} \{\mathcal{R}(E,E')\}^2 \right\}^{-1},$$

$$l_H(E) = \exp[(S^{info})_E]/(0.48\, d')$$

$$(S^{info})_E = -\sum_{E'} \mathcal{R}(E,E') \ln \mathcal{R}(E,E').$$

EGOE formulas are derived using
(i) bivariate Gaussian form for the transition strength densities
(ii) P-T for strength fluctuations

^{46}V: $J=0^+$, $T=0$: $d=814$

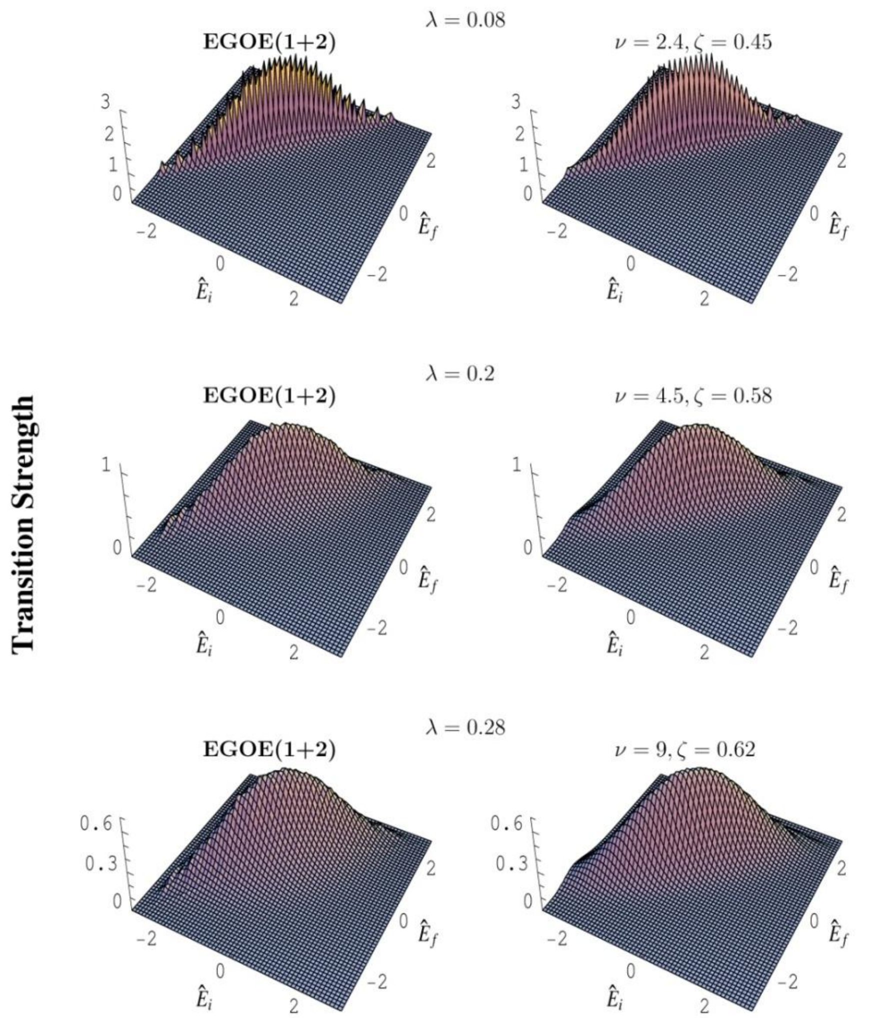
$J=2^+$, $T=0$: $d=3683$

$J=1^+$, $T=1$: $d=4105$

GKKMR, Phys. Rev. C 69 (2004) 057302;
KS, Phys. Lett. B 429 (1998) 1

VKBK,Sahu, Chavda: PRE 73 (2006) 047203

EGOE(1+2) results for *m=6, N=12* with *O* changing a particle from orbit *2* to orbit *9*



With H=h(1)+λ{V(2)} and λ increasing, just as the situation with the strength functions, transition Strength densities change from biv-BW to biv-Gauss

chaos-therm → biv-Gaussian

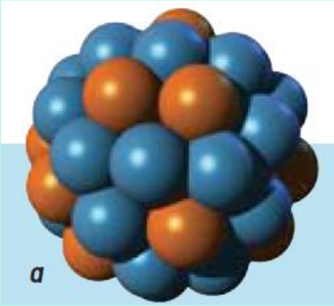
An useful interpolating function is the biv-*t* distribution:

$$\rho_{biv-t:\mathcal{O}}(E_i,E_f;\varepsilon_i,\varepsilon_f,\sigma_i,\sigma_f,\zeta_{biv};\nu)_{\nu\ge 1}=\frac{1}{2\pi\sigma_i\sigma_f\sqrt{1-\zeta_{biv}^2}}$$

$$\times\left[1+\frac{1}{\nu(1-\zeta_{biv}^2)}\left\{\left(\frac{E_i-\varepsilon_i}{\sigma_i}\right)^2-2\zeta_{biv}\left(\frac{E_i-\varepsilon_i}{\sigma_i}\right)\left(\frac{E_f-\varepsilon_f}{\sigma_f}\right)+\left(\frac{E_f-\varepsilon_f}{\sigma_f}\right)^2\right\}\right]^{-(\nu+2)/2}$$

(i) ν is the shape parameter
(ii) ν=1 gives bivariate BW
(iii) ν→∞ gives bivariate Gaussian
(iv) σ_i and σ_f are marginal widths only when ν→∞ and they are spreading widths when ν=1
(v) ζ remains the bivariate correlation coefficient

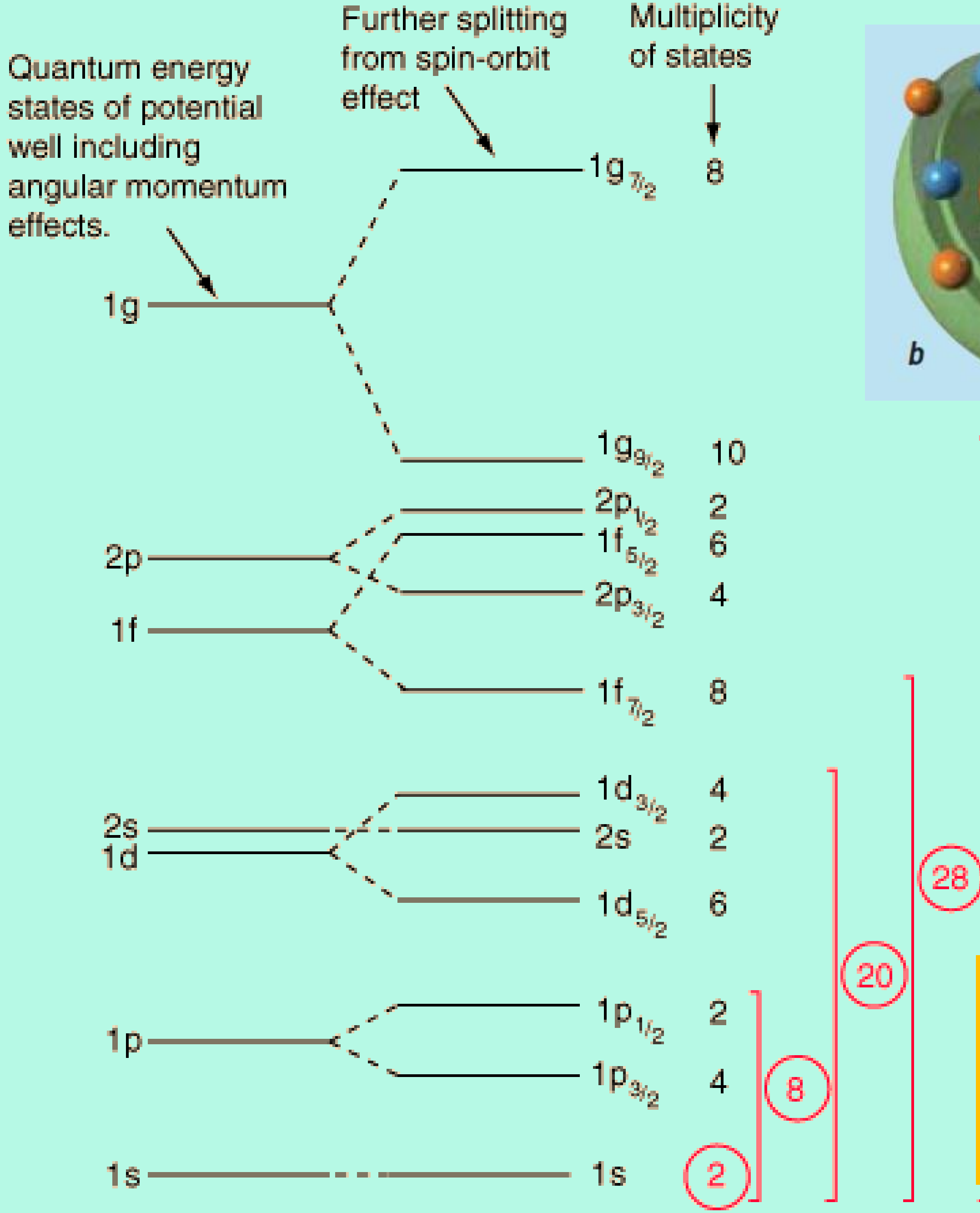
3. Extension to Transition Strengths With Partitioning

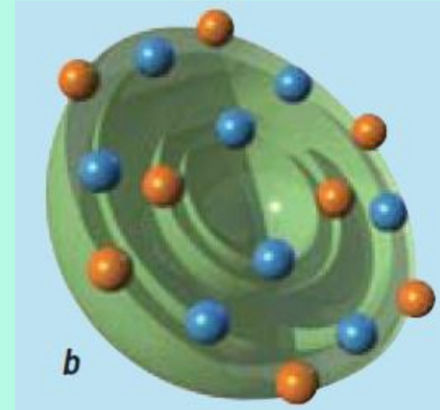


Nuclear Shell Model

Maria G Mayer

Rochester-Oak Ridge Code - 1966





NuShellX

ANTOINE

In larger spectroscopic spaces instead of using a single bivariate Gaussian (or *t*-) distribution, it is more appropriate to partition the space (physically motivated one) and then apply EGOE result appropriately:

$$p \rightarrow (j_1^p, j_2^p, \ldots, j_r^p), \quad n \rightarrow (j_1^n, j_2^n, \ldots, j_s^n)$$

$$\tilde{m}_p = [m_p^1, m_p^2, \ldots, m_p^r], \quad \tilde{m}_n = [m_n^1, m_n^2, \ldots, m_n^s]$$

$$m_p = \sum_{i=1}^{r} m_p^i \ , \ m_n = \sum_{j=1}^{s} m_n^j$$

$\widetilde{m} = \left(\widetilde{m_p}, \widetilde{m_n}\right)$ is a p - n configuration $\leftarrow$ mean-field $h(1)$ basis states

$$I^{(m_p,m_n)}(E) \xrightarrow[\text{Chaos-therm (nuclei)}]{RMT-EGOE} \sum_{(\tilde{m}_p,\tilde{m}_n)} I_{\mathcal{G}}^{(\tilde{m}_p,\tilde{m}_n)}(E)$$

$$\boxed{E_c(\tilde{m}_p,\tilde{m}_n) = \left\langle H \right\rangle^{(\tilde{m}_p,\tilde{m}_n)}, \ \sigma^2(\tilde{m}_p,\tilde{m}_n) = \left\langle H^2 \right\rangle^{(\tilde{m}_p,\tilde{m}_n)} - \left[E_c(\tilde{m}_p,\tilde{m}_n)\right]^2}$$

we need these with *J* – exact/approx – we will return to this later

To develop the ensemble theory for (smoothed) transition strength densities with $H = h(1) + V(2)$ and partitioning, we will start with h(1)=$\sum_r \varepsilon_r n_r$,

$$I_O^{h(1)}(x_i, x_f) = \sum_{\widetilde{m_i}, \widetilde{m_f}} \sum_{\gamma_i \in \widetilde{m_i}, \gamma_f \in \widetilde{m_f}} \left| \left\langle \widetilde{m_f}, \gamma_f \left| O \right| \widetilde{m_i}, \gamma_i \right\rangle \right|^2$$

$$\times\, \delta(x_i - \varepsilon(\widetilde{m_i}))\ \delta(x_f - \varepsilon(\widetilde{m_f})\ ;\ \varepsilon(\widetilde{m}) = \sum_i m_i \varepsilon_i$$

The role of interactions ($\boldsymbol{V}(2)$) is to generate local spreadings of the bivariate density due to $\boldsymbol{h}(1)$

$\Rightarrow \rho_O^H = \rho_O^h \otimes \rho_O^V$ and then,

RMT-EE/chaos-therm ?

$$I_O^{H=h+V}(E_i, E_f) \approx \int I_O^h(x, y)\ \rho_O^V(E_i - x, E_f - y) dxdy$$

corrections to the convolution form? ignored (*h,V,O*)- cross correlations

$$\left|\left\langle E_f \,|\, \mathcal{O} \,|\, E_i \right\rangle\right|^2 =$$

$$\sum_{(\tilde{m}_p,\tilde{m}_n)_i,(\tilde{m}_p,\tilde{m}_n)_f} \frac{\dfrac{d\left[(\tilde{m}_p,\tilde{m}_n)_i\right] d\left[(\tilde{m}_p,\tilde{m}_n)_f\right]}{I^{(m_p,m_n)}(E_i)\, I^{(m_p,m_n)}(E_f)}}{\times\ \left|\left\langle (\tilde{m}_p,\tilde{m}_n)_f \,|\, \mathcal{O} \,|\, (\tilde{m}_p,\tilde{m}_n)_i \right\rangle\right|^2}$$

$$\times\ \rho_{biv-t\,:\,\mathcal{O}\,:\,V}^{(\tilde{m}_p,\tilde{m}_n)_i,(\tilde{m}_p,\tilde{m}_n)_f}(E_i, E_f; E_c^i, E_c^f, \sigma_i, \sigma_f, \zeta)$$

$\Rightarrow E_c^i, E_c^f, \sigma_i, \sigma_f, \zeta$ using only some approximations

Example of a one-body transition operator : $O=\sum_{\alpha,\beta}\varepsilon_{\alpha\beta}a_{\alpha}^{\dagger}a_{\beta}$

$$d((\tilde{m}_p,\tilde{m}_n)_f)\ \overline{\left|\left\langle(\tilde{m}_p,\tilde{m}_n)_f\ \middle|\ \mathcal{O}\ \middle|(\tilde{m}_p,\tilde{m}_n)_i\right\rangle\right|^2}=\left\langle\frac{n_\beta(N_\alpha-n_\alpha)}{N_\beta(N_\alpha-\delta_{\alpha\beta})}\right\rangle^{(\tilde{m}_p,\tilde{m}_n)_i}\left|\varepsilon_{\alpha\beta}\right|^2;$$

$$(\tilde{m}_p,\tilde{m}_n)_f=(\tilde{m}_p,\tilde{m}_n)_i\times\left(1_\alpha^\dagger 1_\beta\right);\ \ E_c^f=E_c^i-\varepsilon_\beta+\varepsilon_\alpha$$

assume constancy of $\sigma_1,\sigma_2,\zeta,\nu\ \Rightarrow$

$$\left|\left\langle E_f\left|O\right|E_i\right\rangle\right|^2=\sum_{\alpha,\beta}\left|\varepsilon_{\alpha\beta}\right|^2\left\langle n_\beta(1-n_\alpha)\right\rangle^{E_i}\ \overline{D(E_f)}\ \Im\ ;$$

$$\Im=\int_{-\infty}^{+\infty}\rho_{biv-t:O}(E_i,E_f;E_c^i,E_c^f=E_c^i-\varepsilon_\beta+\varepsilon_\alpha,\sigma_1,\sigma_2,\zeta;\nu)\ dE_c^i$$

$$=\frac{\Gamma[(\nu+1)/2]}{\sqrt{\pi}\Gamma(\nu/2)}\frac{1}{\sqrt{\nu(\sigma_1^2+\sigma_2^2-2\zeta\sigma_1\sigma_2)}}\left[1+\frac{\Delta^2}{\nu(\sigma_1^2+\sigma_2^2-2\zeta\sigma_1\sigma_2)}\right]^{-\left(\frac{\nu+1}{2}\right)},\ \Delta=E_f-E_i+\varepsilon_\beta-\varepsilon_\alpha$$

with ν=1 and ζ=0 along with $\sigma_1^2+\sigma_2^2=[(\Gamma_1+\Gamma_2)/2]^2$**, the above will reduce to the theory given by V. V. Flambaum *et al.;* see for example Phys. Rev. A 58, 230 (1998).**

For completing the statistical theory for systems such nuclei or atoms, we need *J*-projection of all the quantities as the eigenstates carry *J* quantum number. This is indeed complicated. A simple method is used in the example to be discussed next.

4. Neutrinoless double beta decay NTME for ^{130}Te and ^{136}Xe

NDBD half life for gs-gs of an e-e nucleus to a final e-e nucleus

$$\left[T_{1/2}^{0\nu}(0_i^+ \to 0_f^+)\right]^{-1} = G^{0\nu}\left|M^{0\nu}(0^+)\right|^2\left(\frac{\langle m_\nu \rangle}{m_e}\right)^2$$

Phase space factor (atomic physics)

NTME (nuclear physics)

Neutrino mass (particle physics)

$$\left.\begin{array}{l}\textbf{Kamland-Zen: PRL 110, 062502 (2013)}\\ \textbf{EXO-200: PRL 109, 032505}\end{array}\right\} \quad {}^{136}\mathrm{Xe} > 3.4 \times 10^{25}\ \mathrm{yr}$$

GERDA-phase-I: PRL 111, 122503 (2013) ^{76}Ge $> 3 \times 10^{25}$ yr

more recent from K-Z : ^{136}Xe $> 1.1 \times 10^{26}$ yr

For nuclei, there is good evidence that $\rho_{biv-t\,:\,\mathcal{O}\,:\,V}^{(\tilde{m}_p,\tilde{m}_n)_i,(\tilde{m}_p,\tilde{m}_n)_f}$ is a bivariate Gaussian

need to use following approximations from RMT-EGOE

marginal centroids and variances:

$$E_c^i = E_c^{O:H}((\tilde{m}_p,\tilde{m}_n)_i) \approx \left\langle H \right\rangle^{(\tilde{m}_p,\tilde{m}_n)_i},$$

$$E_c^f = E_c^{O:H}((\tilde{m}_p,\tilde{m}_n)_f) \approx \left\langle H \right\rangle^{(\tilde{m}_p,\tilde{m}_n)_f},$$

$$\sigma_i^2 = \sigma_{O:H}^2((\tilde{m}_p,\tilde{m}_n)_i) \approx \left\langle V^2 \right\rangle^{(\tilde{m}_p,\tilde{m}_n)_i},$$

$$\sigma_f^2 = \sigma_{O:H}^2((\tilde{m}_p,\tilde{m}_n)_f) \approx \left\langle V^2 \right\rangle^{(\tilde{m}_p,\tilde{m}_n)_f}$$

Trace propagation formulas due to CFT will give configuration averages starting with the shell model inputs (i.e. spe and TBME)

complicated is the bivariate correlation coefficient:

$$\zeta(m_p,m_n) = \frac{\left\langle O^\dagger V O V \right\rangle^{(m_p,m_n)}}{\sqrt{\left\langle O^\dagger V^2 O \right\rangle^{(m_p,m_n)} \left\langle O^\dagger O V^2 \right\rangle^{(m_p,m_n)}}}$$

estimates/values 0.6 to 0.8 from RMT-EGOE

definition of ζ involving configurations not known yet

$$\left|\left\langle\left(\widetilde{m_p},\widetilde{m_n}\right)_f\left|\mathcal{O}(2:0\nu)\right|\left(\widetilde{m_p},\widetilde{m_n}\right)_i\right\rangle\right|^2 d\left[\left(\widetilde{m_p},\widetilde{m_n}\right)_f\right]$$

$$=\sum_{\alpha,\beta,\gamma,\delta}\frac{m_n^i(\alpha)\left[m_n^i(\beta)-\delta_{\alpha\beta}\right]\left[N_p(\gamma)-m_p^i(\gamma)\right]\left[N_p(\delta)-m_p^i(\delta)-\delta_{\gamma\delta}\right]}{N_n(\alpha)\left[N_n(\beta)-\delta_{\alpha\beta}\right]N_p(\gamma)\left[N_p(\delta)-\delta_{\gamma\delta}\right]}$$

$$\times\sum_{J_0}\left[\mathcal{O}^{J_0}_{\gamma^p\delta^p\alpha^n\beta^n}(0\nu)\right]^2(2J_0+1),$$

$$\left(\widetilde{m_p},\widetilde{m_n}\right)_f=\left(\widetilde{m_p},\widetilde{m_n}\right)_i\times\left(1^+_{\gamma^p}1^+_{\delta^p}1_{\alpha^n}1_{\beta^n}\right)$$

In reality we need $\left|\left\langle E_f, J_f=0\right| O \left|E_i, J_i=0\right\rangle\right|^2$

Therefore *J*-projection is needed

$$\left|\left\langle E_f, J_f = 0\right| O \left|E_i, J_i = 0\right\rangle\right|^2 = \left[I(E_i, J_i) I(E_f, J_f)\right]^{-1}$$

$$\times \left\langle\left\langle O^\dagger \; \delta(H - E_f)\delta(J^2 - J_f(J_f+1)) \; O \; \delta(H - E_i)\delta(J^2 - J_i(J_i+1))\right\rangle\right\rangle^{(m_p, m_n)_i}$$

$$= \frac{I_O(E_i, E_f, J_i, J_f)}{I(E_i, J_i) I(E_f, J_f)} = \frac{\rho_O(J_i, J_f : E_i, E_f) I_O(E_i, E_f)}{\rho(J_i : E_i) I(E_i) \rho(J_f : E_f) I(E_f)}$$

$$\approx \frac{I_O(E_i, E_f) \sqrt{C_{J_i}(E_i) C_{J_f}(E_f)}}{I(E_i) \; C_{J_i}(E_i) \; I(E_f) \; C_{Jf}(E_f)}$$

Note: $J_i = j_f = 0$

$$\Rightarrow \left|\left\langle E_f, J_f = 0\right| O \left|E_i, J_i = 0\right\rangle\right|^2 = \frac{1}{\sqrt{C_{J_i=0}(E_i) C_{J_f=0}(E_f)}} \left|\left\langle E_f \right| O \left| E_i \right\rangle\right|^2$$

$$C_{J_r}(E) \sim \frac{(2J_r+1)}{\sqrt{8\pi} \; \sigma_J^3(E)} \; \exp \; -\frac{(2J_r+1)^2}{8\sigma_J^2(E)} \xrightarrow{J_r=0} \frac{1}{\sqrt{8\pi} \; \sigma_J^3(E)}$$

$\sigma_J(E)$ is spin cut-off factor : in the gs region it is ~ *3-6.*

we have applied RMT-EE theory with partitioning by treating ζ and *$\sigma_J(E)$* as free parameters

SDM for ^{130}Te$\rightarrow$ ^{130}Xe NDBD NTME: first results

(i) sp space, sp energies and effective interaction:

$^{0}g_{7/2}, {}^{1}d_{5/2}, {}^{1}d_{3/2}, {}^{2}s_{1/2}, {}^{0}h_{11/2}$ **space and JJ55 as in SM**

(ii) parameters in the transition operator:

Jastrow parameters $(\gamma_1 , \gamma_2 , \gamma_3) = (1.1, 0.68, 1)$;

$R = 1.2\, A^{1/3}$ **fm,** $g_A/g_V=1$ **(quenched);** $b = 1.003\, A^{1/6}$ **fm,** $\bar{E} = 1.12\, A^{1/2}$ **MeV**

(iii) ζ **and** $\sigma_J = \sigma_J(E_i(gs)) = \sigma_J(E_j(gs))$ **free parameters**

(iv) no. of TBME = 327; no. of SPE = 5

(v) for ^{130}Te, $(E_R , J_R , N_R)= (1.633$ **MeV,** $4^+ , 20)$,

for ^{130}Xe, $(E_R , J_R , N_R)= (1.205$ **MeV,** $4^+ , 20)$ $\rightarrow$ *for gs*

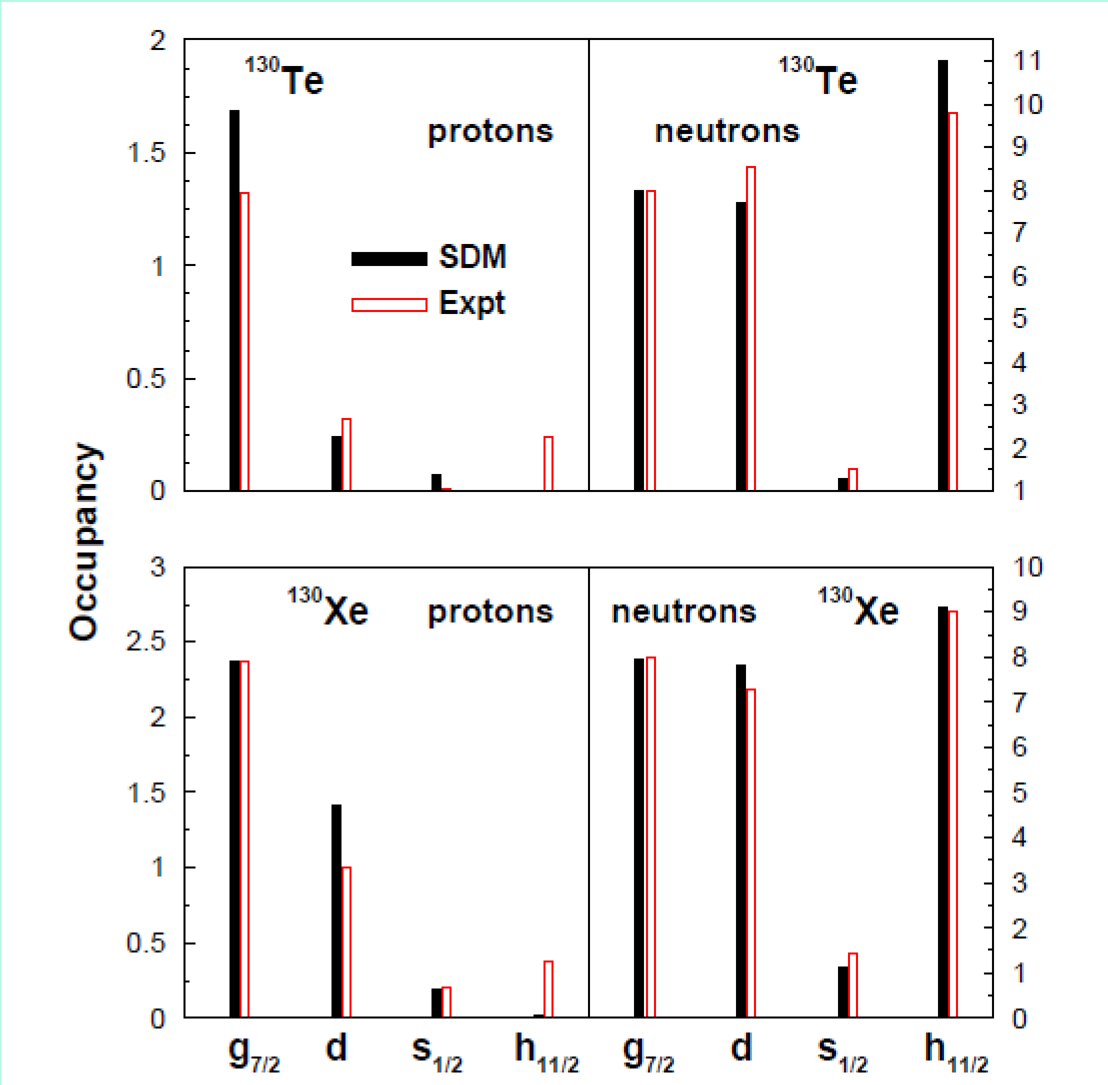
(vi) +ve parity configurations for ^{130}Te and ^{130}Xe are 554 and 5848 respectively

(vii) average width ~ 1.64 MeV with 9% fluctuation for ^{130}Te and 2.65 MeV with 6%

. fluctuation for ^{130}Xe

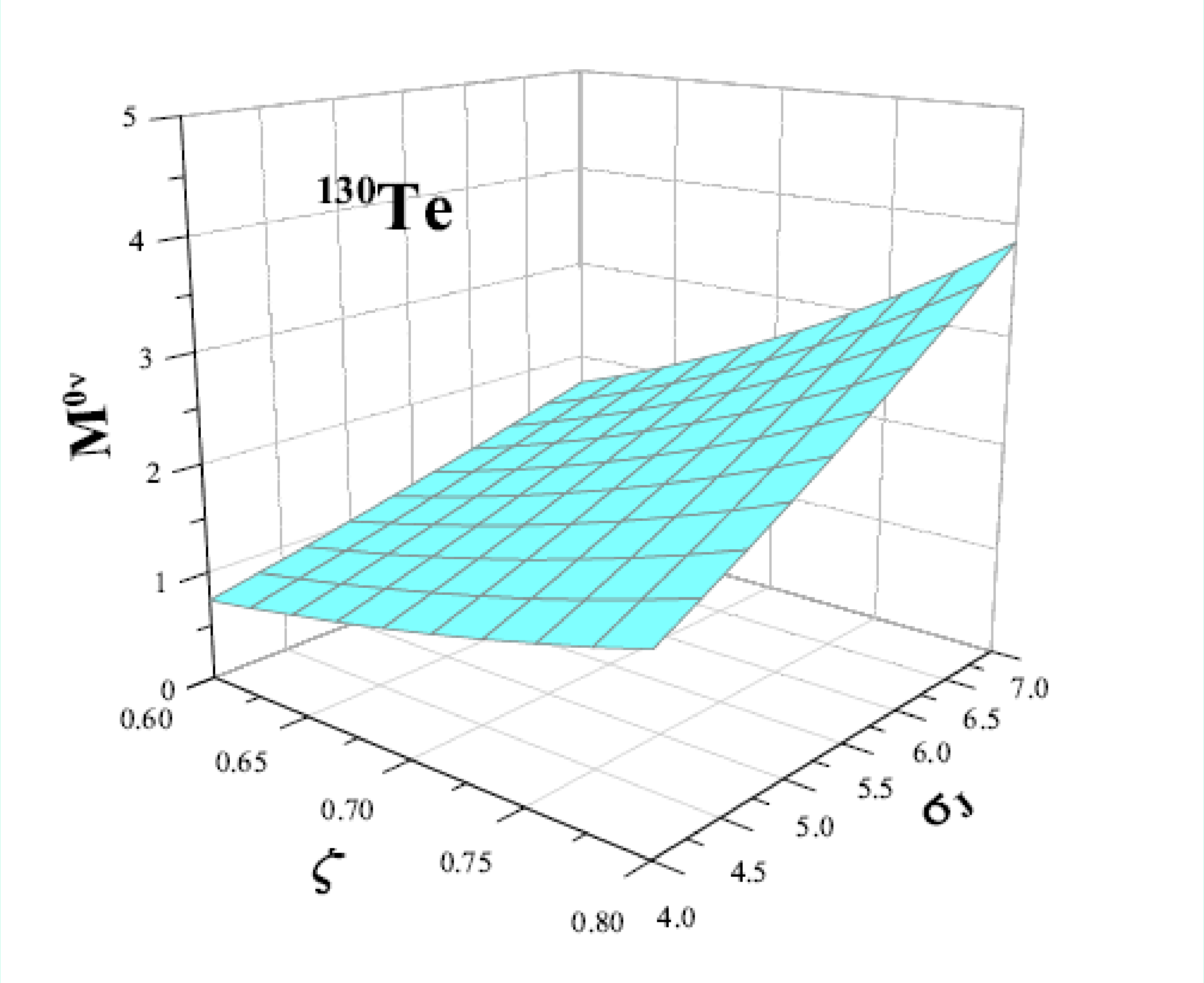
(viii) ground state ~ $-3\,\sigma$ **from the lowest configuration centroid**

(ix) total strength = 2195



data from Schiffer et al:
PRC 87 (2013) 011302(R)
PRC 93 (2016) 064312

$$\left\langle n_{\alpha}^{p(n)} \right\rangle^{E} = \sum_{\left(\widetilde{m_p},\widetilde{m_n}\right)} \frac{I_G^{\left(\widetilde{m_p},\widetilde{m_n}\right)}(E)}{I^{(m_p,m_n)}(E)} \left\langle n_{\alpha}^{p(n)} \right\rangle^{\left(\widetilde{m_p},\widetilde{m_n}\right)}$$



with σ_J ~ 4-5 and ζ ~ 0.7-0.75 will give $M^{0\nu}$ ~ 1.1-1.9

SDM for ^{136}Xe$\rightarrow$ ^{136}Ba NDBD NTME: first results

(i) no. of TBME = 327; no. of SPE = 5 [same SPE and TBME as used for ^{130}Te NTME calculations]

(ii) for ^{136}Xe, (E_R, J_R, N_R)= (*1.892* MeV,6^+,*28*) → *for gs*

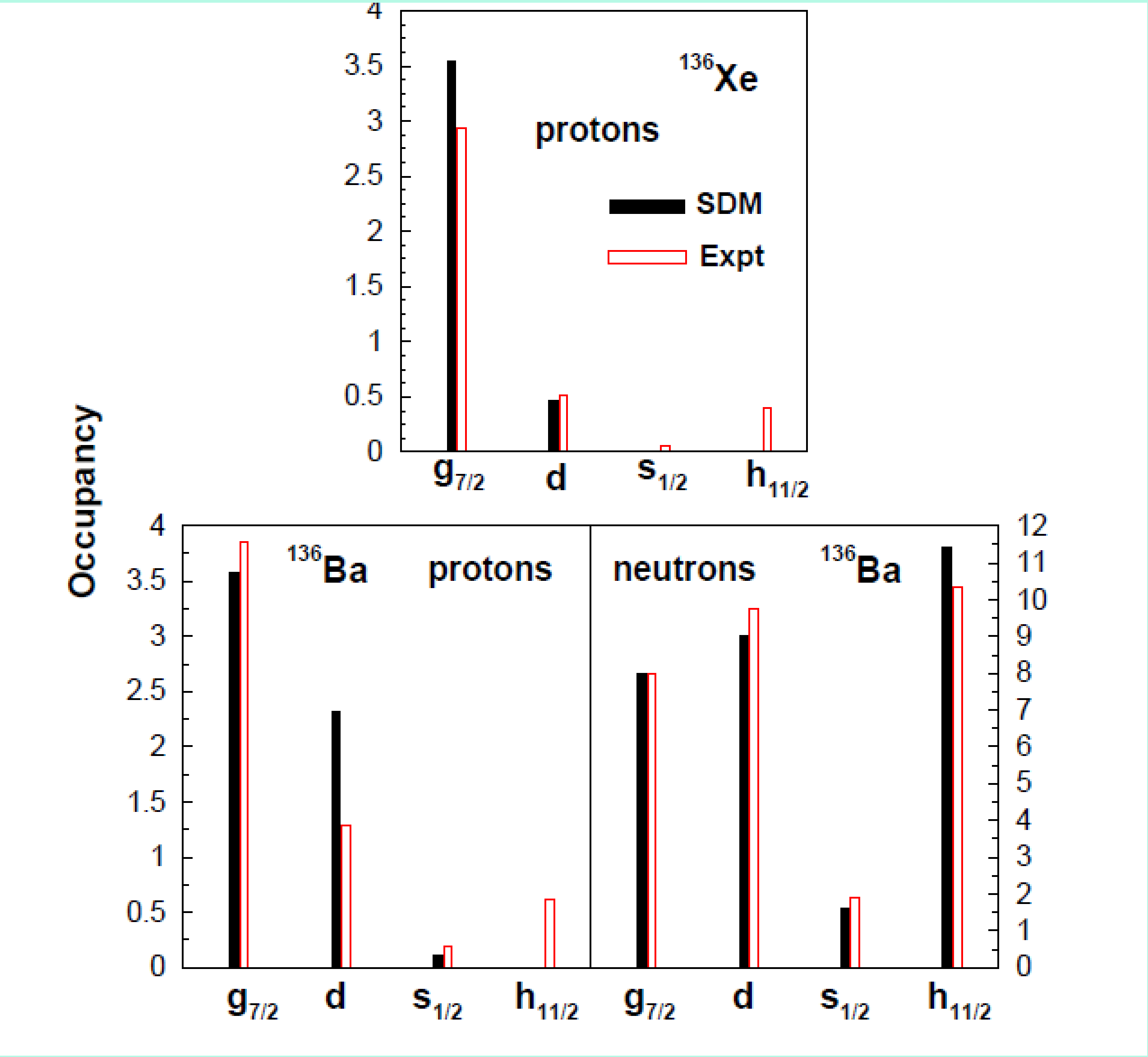
for ^{136}Ba, (E_R, J_R, N_R)= (*2.141* MeV,0^+,*41*) → *for gs*

(iii) +ve parity configurations for ^{136}Xe and ^{136}Ba are 42 and 1354 respectively

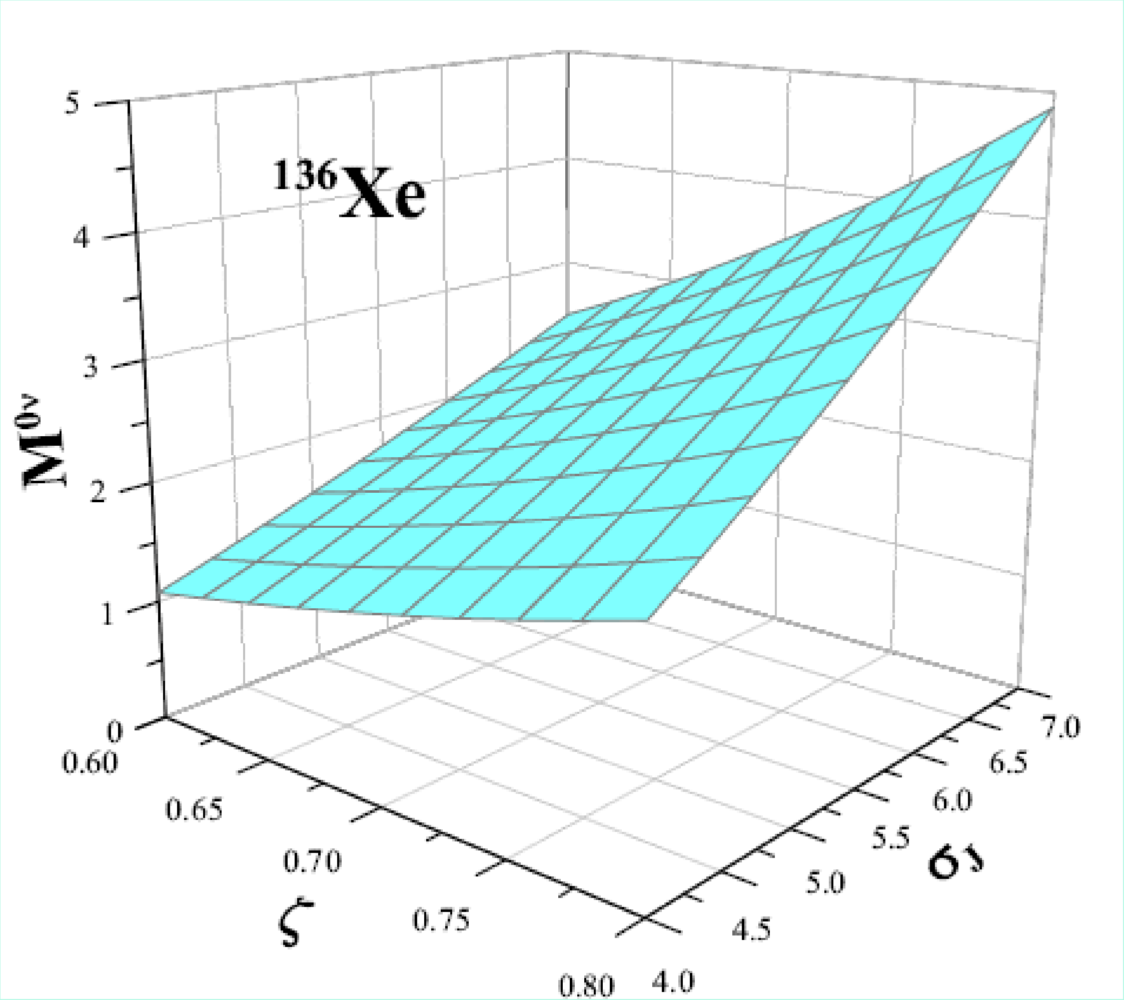
(iv) average width ~ 0.82 MeV with 12% fluctuation for ^{136}Xe and 2.05 MeV with 7% fluctuation for ^{136}Ba

(v) ground state ~ *-3.6 to 4* σ from the lowest configuration centroid

(vi) total strength = 2195

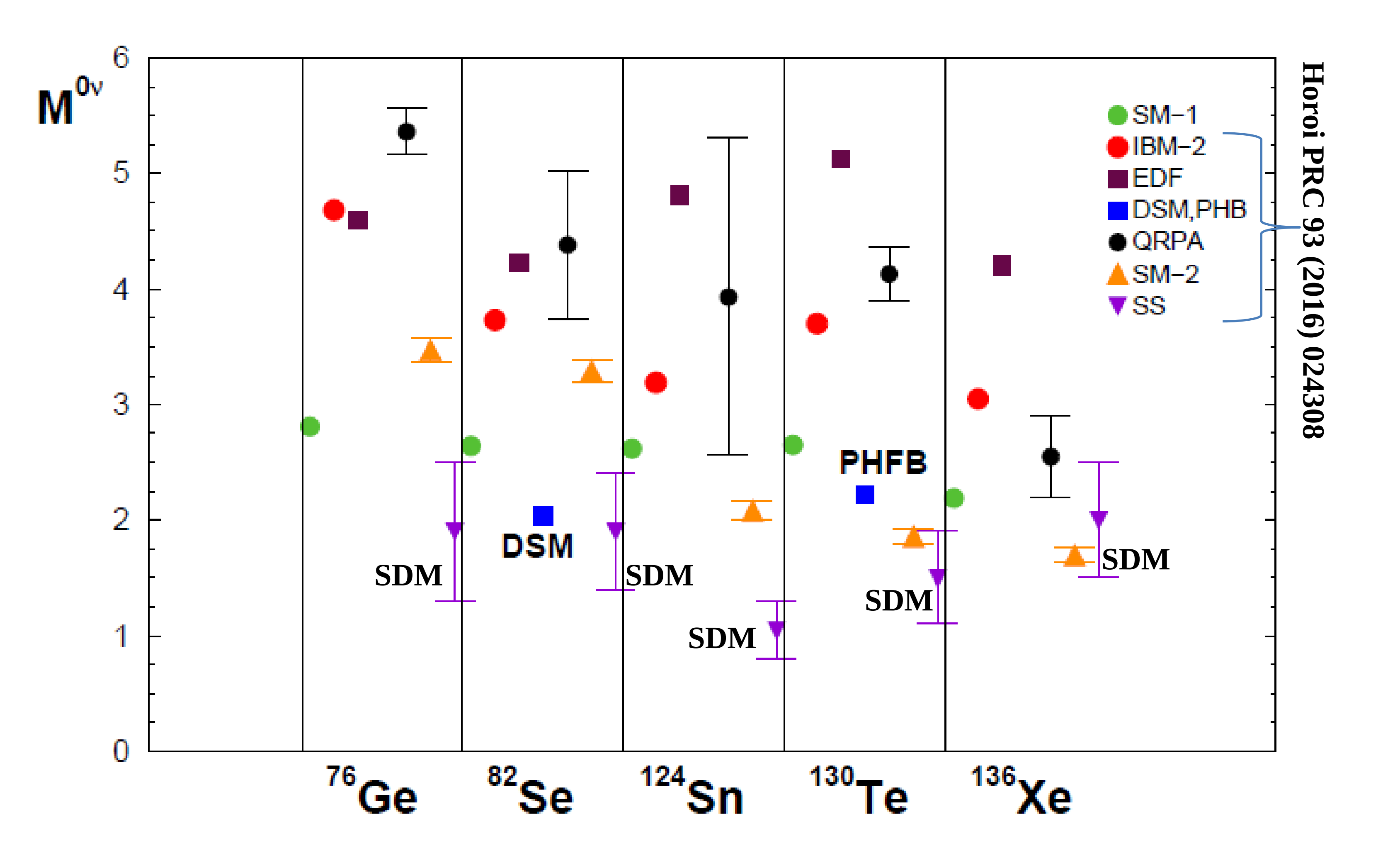


data from : PRC 94 (2016) 054314; PRC 93 (2016) 064312



with σ_J ~ 4-5 and ζ ~ 0.7-0.75 will give $M^{0\nu}$ ~ 1.5-2.5

SDM $\Rightarrow$ VKBK and R.U. Haq, arXiv:1608.08785; in preparation



5. Open Questions in RMT-EE Theory

Further studies using the present formulation:

▪**better *J*-projection – understand more about four variable distributions – we may have to use J_Z operator for initial and final spaces plus Edgeworth expansion in four variables**

▪***J*-projection by using exact fixed-*J* averages: configuration centroids and variances possible (Senkov *et al.* codes) but formula for the rms matrix elements with fixed-*J* need to be derived:**

$$\left|\left\langle\left(\widetilde{m_p},\widetilde{m_n}\right)_f J_{f=0}\left|\mathcal{O}(2:0\nu)\right|\left(\widetilde{m_p},\widetilde{m_n}\right)_i J_{i=0}\right\rangle\right|^2$$

▪**better treatment of ζ by calculating it using the definition involving $\left\langle O^\dagger V O V\right\rangle^{(m_p,m_n)}$ rather than using it as a free parameter**

▪**testing the formulation using a full shell model example**

▪**estimate errors as the theory is applied in the ground state region**

▪**study sum rules for transition strengths and this is possible**

Extensions of the present formulation:

•corrections to the convolution form with terms involving products of *h*,*V* and *O* cross correlations

• extending the analytical EGUE(2) results to EGUE(2) with spin and numerical EGOE(1+2) results to EGOE(1+2)-s: these will establish the generality of the biv-G form with internal quantum numbers

• a definition of ζ with partitioning (condition: $-1 \leq \zeta \leq +1$)?

•in larger spaces m $\rightarrow \Sigma\,\Gamma \oplus$ and then we neeed to define proper positive definite partial strength densities. In special situations this is possible as already discussed - they are due to FKPT, Flambaum, VKBK+(MV,NDC,RS).

•transition strengths with multi $\hbar\omega$ excitations?

Issues with partitioning:

(1)

$$\left|\left\langle E_f \middle| O \middle| E_i \right\rangle\right|^2 = \left|\sum_{\Gamma_i,\Gamma_f} C_{E_f}^{\Gamma_f} C_{E_i}^{\Gamma_i} \left|\left\langle \Gamma_f \middle| O \middle| \Gamma_i \right\rangle\right|\right|^2 = \sum_{\Gamma_i,\Gamma_f} \left|C_{E_f}^{\Gamma_f}\right|^2 \left|C_{E_i}^{\Gamma_i}\right|^2 \left|\left\langle \Gamma_f \middle| O \middle| \Gamma_i \right\rangle\right|^2$$

diagonal term: ζ=0

$$+ \sum_{\Gamma_f^1 \neq \Gamma_f^2, \Gamma_i^1 \neq \Gamma_i^2,} C_{E_f}^{\Gamma_f^1} C_{E_f}^{\Gamma_f^2} C_{E_i}^{\Gamma_i^1} C_{E_i}^{\Gamma_i^2} \left\langle \Gamma_f^1 \middle| O \middle| \Gamma_i^1 \right\rangle \left\langle \Gamma_f^2 \middle| O \middle| \Gamma_i^2 \right\rangle$$

off-diagonal term: ζ≠0

(2)

$$\left\langle\left\langle O^\dagger \delta(H - E_f) O (H - E_i) \right\rangle\right\rangle^{m_i} = I_O^{(m_i,m_f)}(E_i, E_f) \qquad O\left|m_i\alpha\right\rangle = \sum_\beta \left|m_f\beta\right\rangle$$

$$= \sum_{\Gamma_i \in m_i, \Gamma_f \in m_f} \left[\sum_{\alpha \in \Gamma_i, \beta \in \Gamma_f} \left\langle \Gamma_i \alpha \middle| O^\dagger \underset{\sum_{\Gamma'_f,\beta'} \left|\Gamma'_f\beta'\right\rangle\left\langle\Gamma'_f\beta'\right|}{\downarrow} \delta(H - E_f) \middle| \Gamma_f \beta \right\rangle \left\langle \Gamma_f \beta \middle| O \underset{\sum_{\Gamma'_i,\alpha'} \left|\Gamma'_i\alpha'\right\rangle\left\langle\Gamma'_i\alpha'\right|}{\downarrow} (H - E_i) \middle| \Gamma_i \alpha \right\rangle \right]$$

$$\Uparrow \; I_O^{\Gamma_i,\Gamma_f}(E_i, E_f)$$

$H's$ and $O's$ will not in general preserve Γ and $[H, O]_- \neq 0$

$I_O^{\Gamma_i,\Gamma_f}(E_i, E_f)$ **need not be +ve and ζ may not be within ±1**

(also other moments may not be proper moments plus we have a conditional density)

Thank you all