Hadron Collider Physics
Summer School 2012

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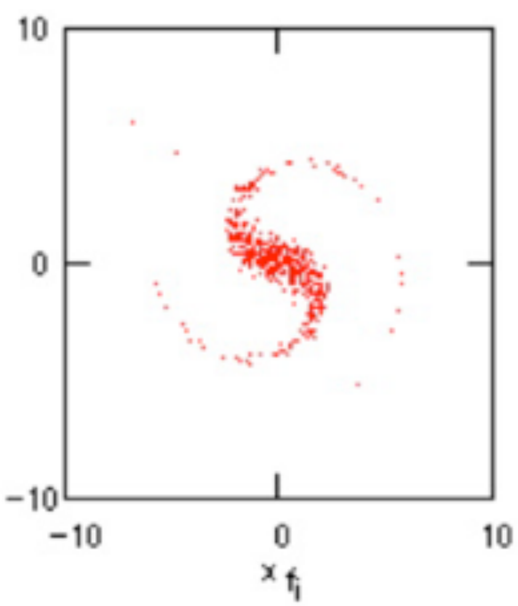
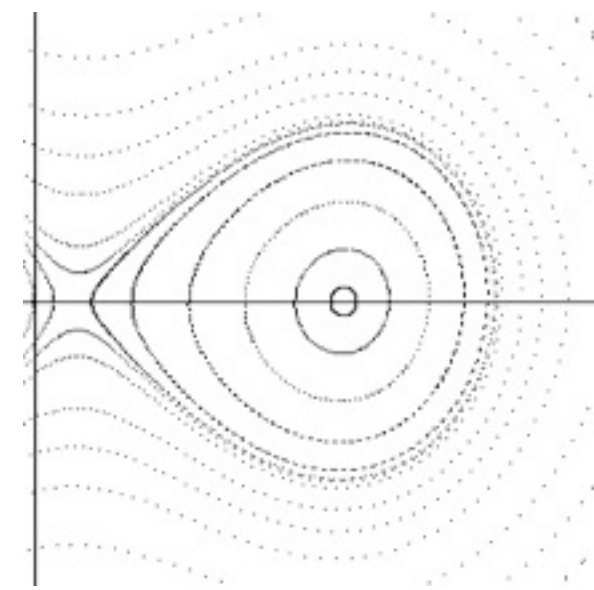
An Introduction to Hadron Colliders





Mike Syphers

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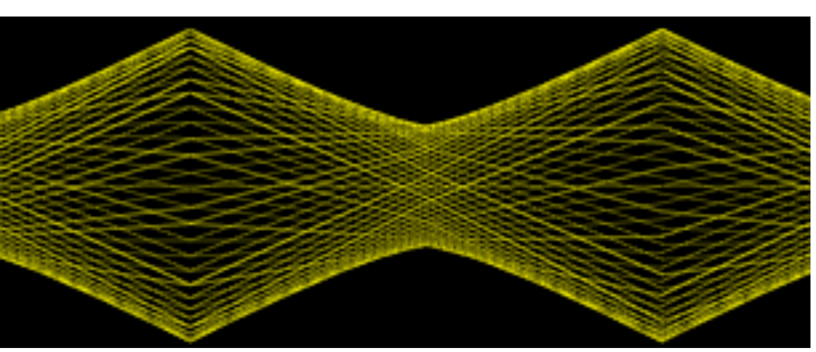

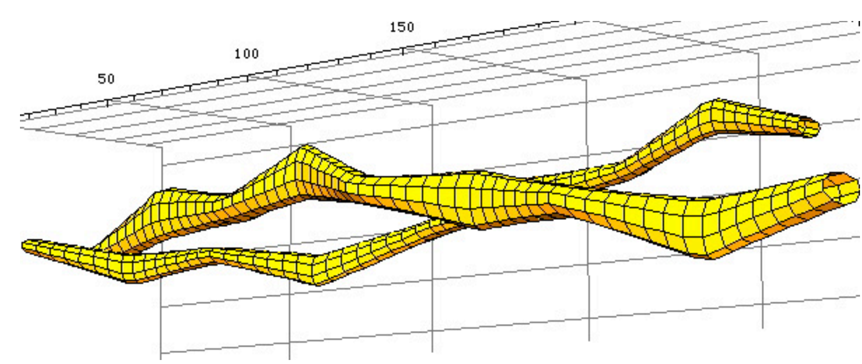

MICHIGAN STATE UNIVERSITY

Department of Physics and Astronomy

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Outline

- Day One:
  - luminosity
  - a little history -- the modern synchrotron
  - magnets and cavities
  - longitudinal dynamics
  - transverse dynamics
- Day Two:
  - Courant-Snyder variables (the ‘beta’ function)
  - transverse emittance
  - momentum dispersion and chromaticity
  - linear errors and adjustments
- Day Three:
  - beam-beam interactions, hour glass and crossing angles
  - synchrotron radiation; stored energy; diffusion and emittance growth
  - luminosity optimization
  - future directions

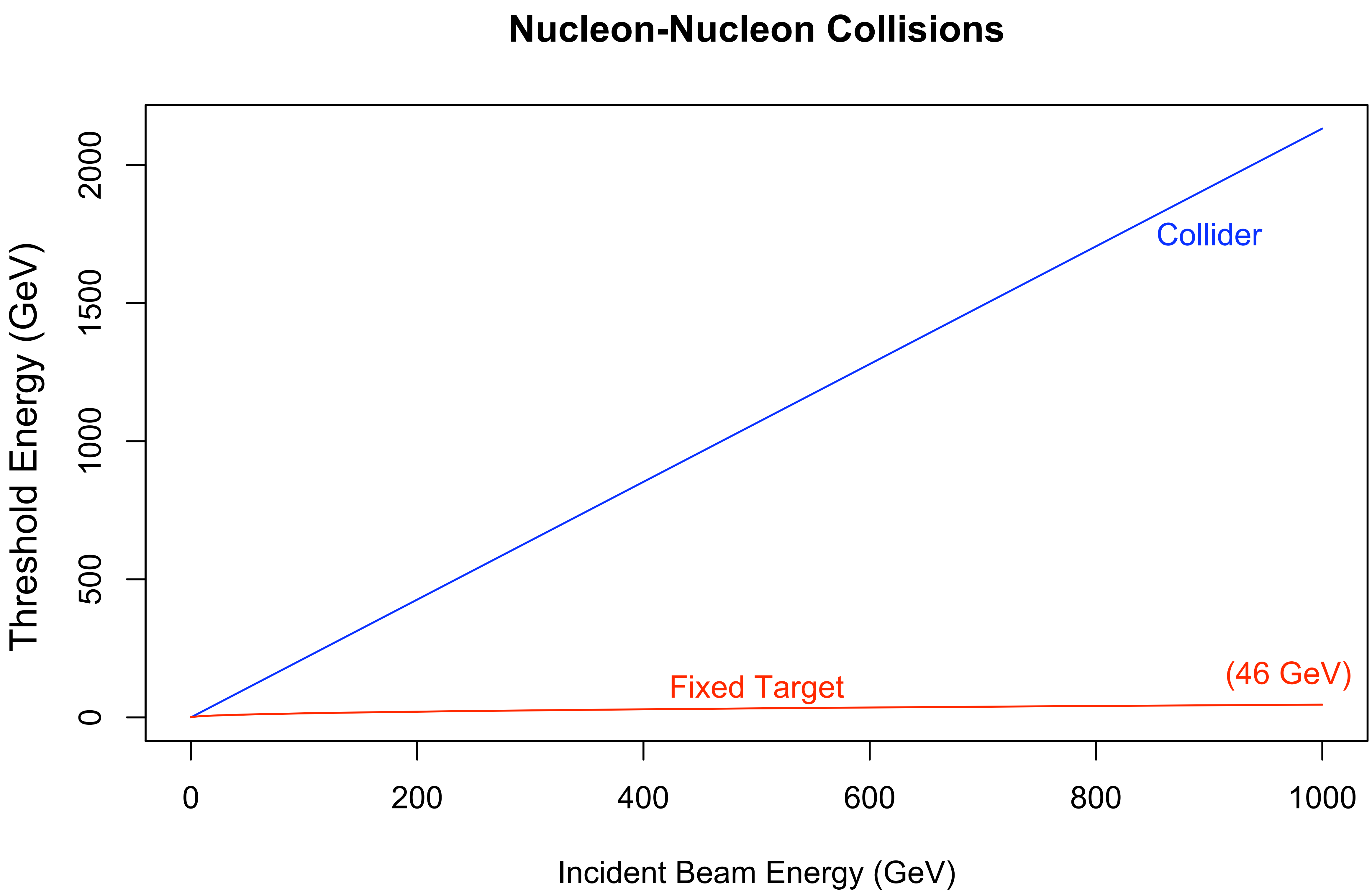
Introduction

- Will touch on technology, but mostly discuss the ***physics*** of particle accelerators, especially relevant to hadron colliding beams synchrotrons

- Will attempt to cover:
  - luminosity; how to meet the requirements?
  - basic principles of synchrotrons (in particular)
  - "the jargon" used in the accelerator/collider environment
  - a few major issues encountered at high energy, luminosity

- Will also have two discussion sections for further Q&A, but ask any time!

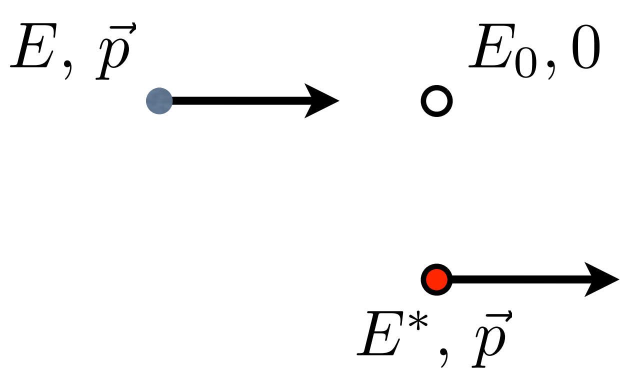
Fixed Target Energy vs. Collider Energy



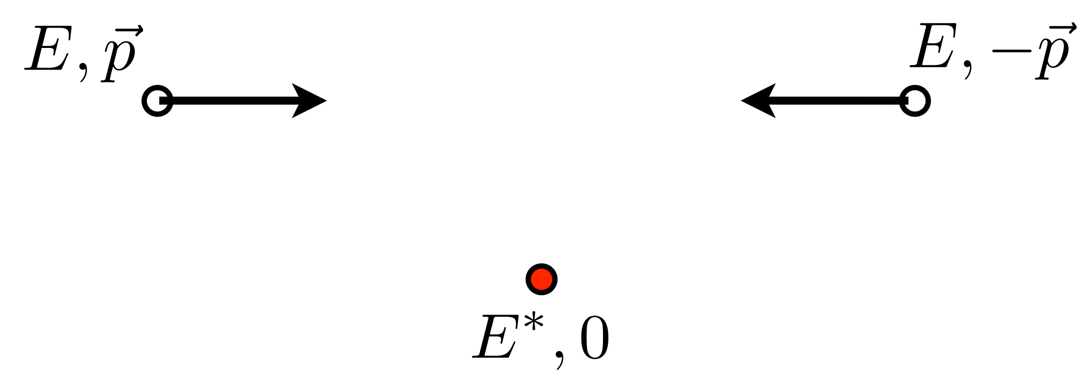
Fixed Target Energy vs. Collider Energy

- Beam/target particles: $E_0 \equiv m_p c^2$

Fixed Target



Collider



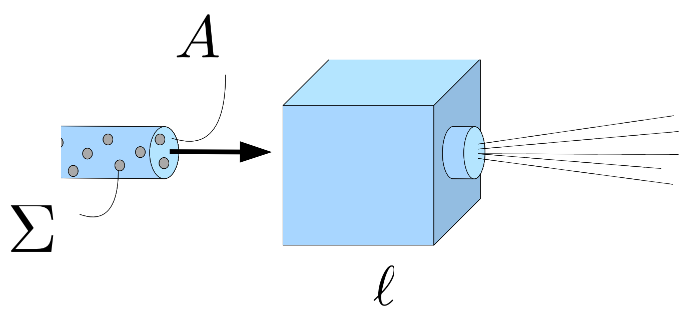
$$
\begin{aligned}
E^{*2} = (m^* c^2)^2 + (pc)^2 &= [\ E_0 + E\ ]^2 \\
&= E_0^2 + 2E_0 E + (E_0^2 + (pc)^2) \\
m^* c^2 &= \sqrt{2}\ E_0\ [1 + \gamma_{FT}]^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
m^* c^2 &= 2E \\
&= 2E_0 \gamma_{coll}
\end{aligned}
$$

100,000 TeV FT synch. == 14 TeV LHC

Luminosity

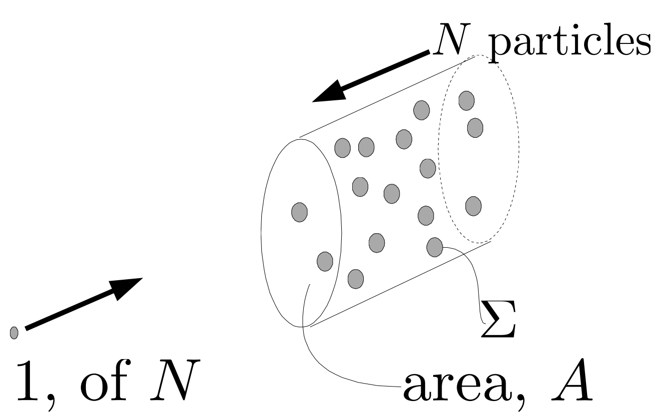
- Experiments want "collisions/events" -- rate?
- Fixed Target Experiment:



$$\begin{aligned} \mathcal{R} &= \left(\frac{\Sigma}{A}\right) \cdot \rho \cdot A \cdot \ell \cdot N_A \cdot \dot{N}_{beam} \\ &= \rho N_A \ell \dot{N}_{beam} \cdot \Sigma \\ &\equiv \mathcal{L} \cdot \Sigma \end{aligned}$$

ex. : $\mathcal{L} = \rho N_A \ell \dot{N}_{beam} = 10^{24}/\text{cm}^3 \cdot 100 \text{ cm} \cdot 10^{13}/\text{sec} = \boxed{10^{39}\text{cm}^{-2}\text{sec}^{-1}}$

- Bunched-Beam Collider:



$$\begin{aligned} \mathcal{R} &= \left(\frac{\Sigma}{A}\right) \cdot N \cdot (f \cdot N) \\ &= \frac{f\ N^2}{A} \cdot \Sigma \\ \mathcal{L} &\equiv \frac{f\ N^2}{A} \end{aligned}$$

$\boxed{(10^{34}\text{cm}^{-2}\text{sec}^{-1} \text{ for LHC})}$

Integrated Luminosity

- Bunched beam is natural in collider that "accelerates" (more later)

$$\mathcal{L} = \frac{f_0 B N^2}{A}$$

$f_0 = \text{rev. frequency}$
$B = \text{no. bunches}$

- In ideal case, particles would be "lost" only due to "collisions"
- So, in this ideal case,

$$B\dot{N} = -\mathcal{L}\,\Sigma\,n$$

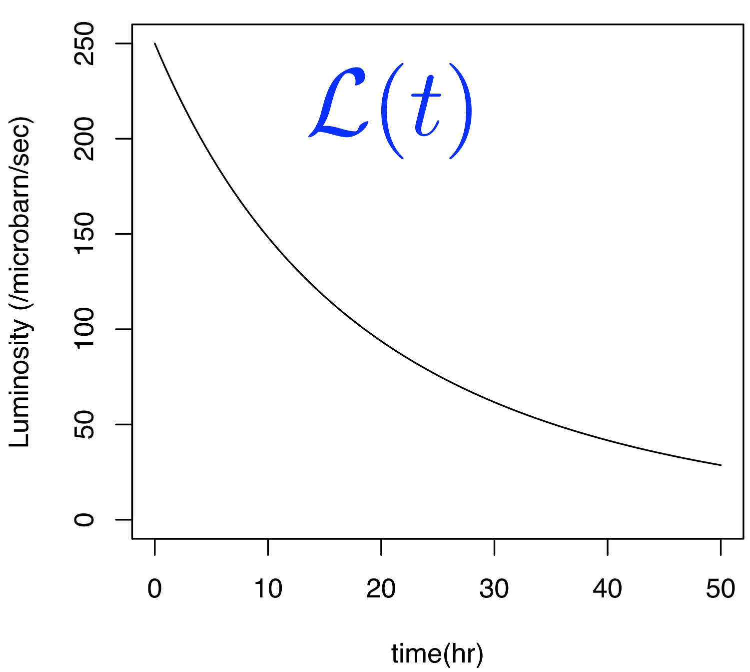
($n = $ no. of detectors receiving luminosity $\mathcal{L}$)

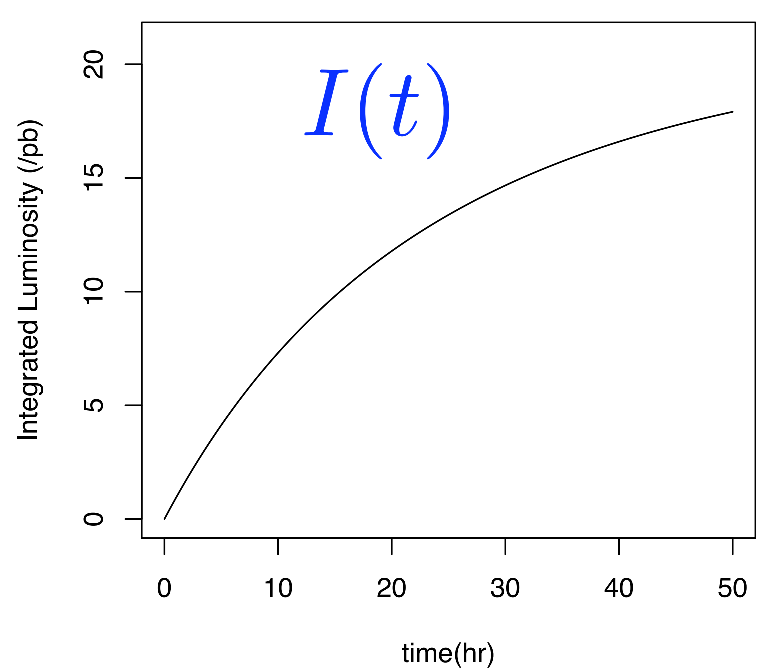
$$\mathcal{L}(t) = \frac{\mathcal{L}_0}{\left[1 + \left(\frac{n\mathcal{L}_0\Sigma}{BN_0}\right)t\right]^2}$$

Ultimate Number of Collisions

- Since $\mathcal{R} = \mathcal{L} \cdot \Sigma$ then, #events = $\int \mathcal{L}(t)dt \cdot \Sigma$
- So, our integrated luminosity is

$$I(T) \equiv \int_0^T \mathcal{L}(t)dt = \frac{\mathcal{L}_0 T}{1 + \mathcal{L}_0 T(n\Sigma/BN_0)} = I_0 \cdot \frac{\mathcal{L}_0 T/I_0}{1 + \mathcal{L}_0 T/I_0}$$





asymptotic limit:

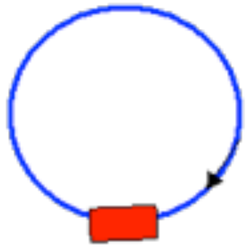
$$\boxed{I_0 \equiv \frac{BN_0}{n\Sigma}}$$

so, ...

$$\mathcal{L} = \frac{f_0 B N^2}{A}$$

How to Make Collisions?

- Simple Model of Synchrotron:
  - Accelerating device + magnetic field to bring particle back to accelerate again



- Field Strength -- determines size, ultimate energy of collider
  - ex:

$$\boxed{\rho = \frac{p}{e\,B}\ ; \quad R = \rho/f \quad (f \approx 0.8 - 0.9)}$$

"packing fraction"

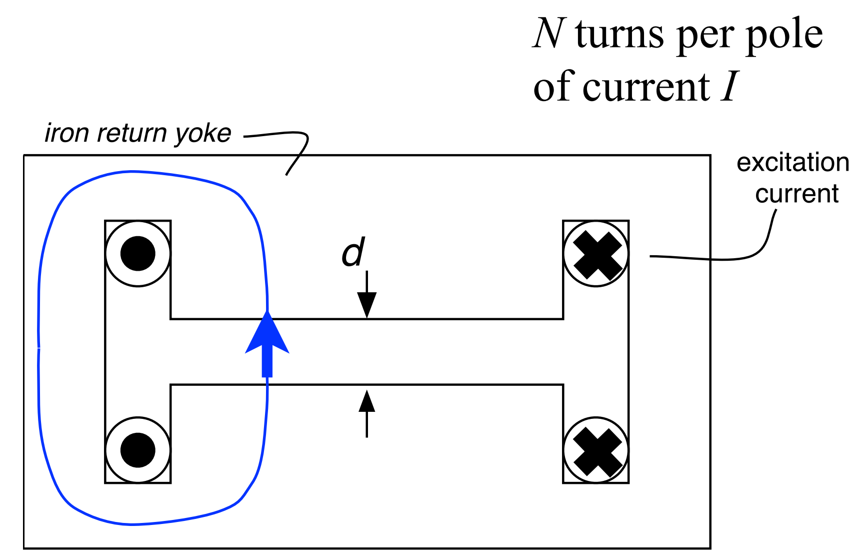
$$B = 1.8\ \mathrm{T}, \quad p = 450\ \mathrm{GeV/c} \quad f = 0.85 \rightarrow R \approx 1\ \mathrm{km}$$

Magnets

- iron-dominated magnetic fields

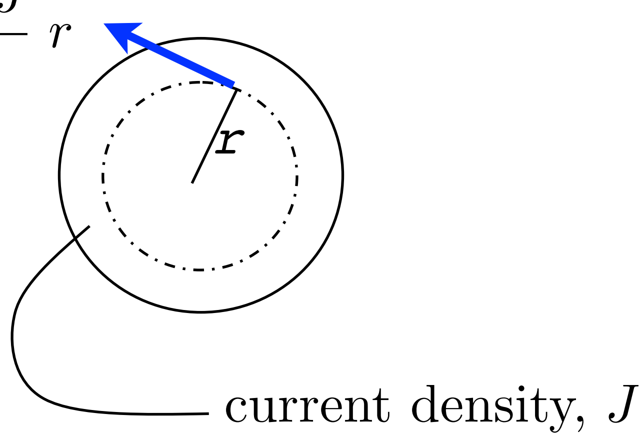
  - iron will “saturate” at about 2 Tesla

$$B = \frac{2\mu_0 N \cdot I}{d}$$



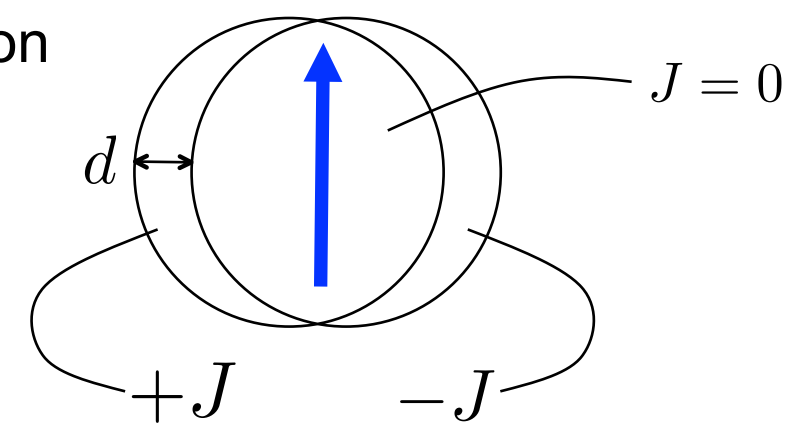
- Superconducting magnets
  - field determined by distribution of currents

$$B_\theta = \frac{\mu_0 J}{2}\, r$$



“Cosine-theta” distribution

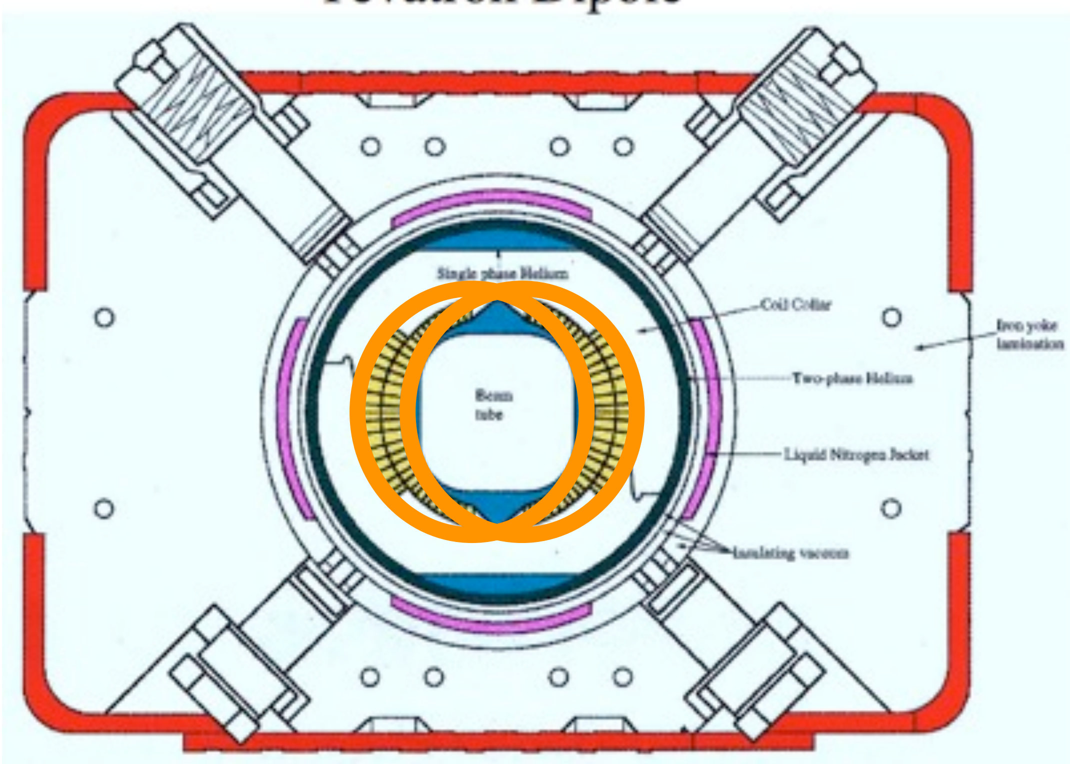
$$\boxed{B_x = 0, \quad B_y = \frac{\mu_0 J}{2}\, d}$$



Superconducting Designs

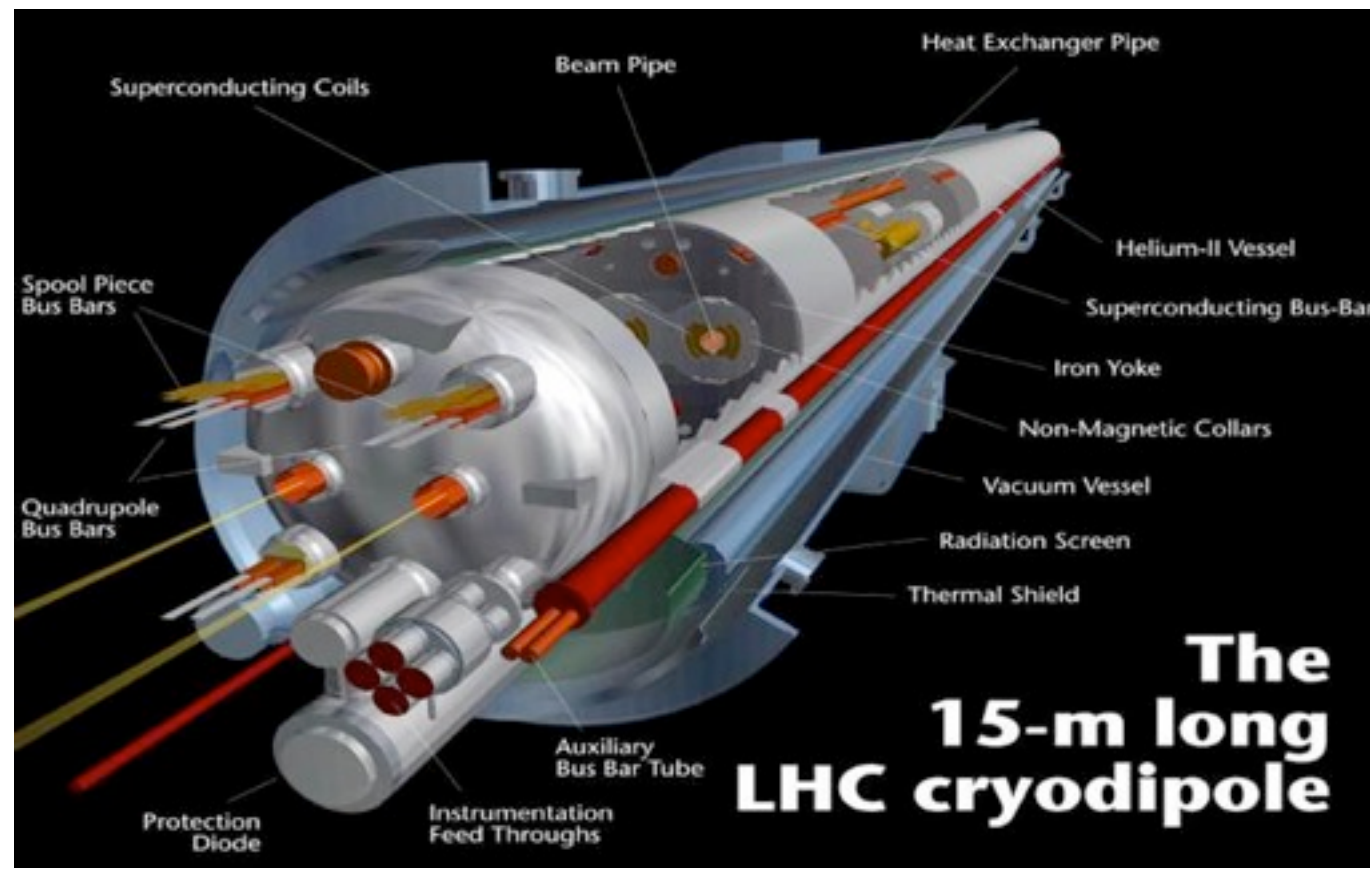
- Tevatron
  - 1^{st} SC synchrotron
  - 4.4 T; 4^oK

Tevatron Dipole



Numerical Example:

$$
\begin{aligned}
B &= \frac{\mu_0 J}{2} d \\
&= \frac{4\pi \text{ T m/A}}{10^7} \frac{1000 \text{ A/mm}^2}{2} \cdot (10 \text{ mm}) \cdot \frac{10^3 \text{mm}}{\text{m}} \\
&= 6 \text{ T}
\end{aligned}
$$

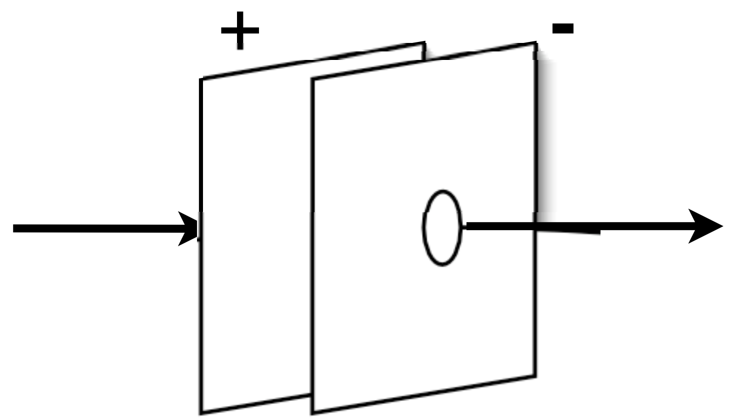


- LHC -- 8 T; 1.8^oK

The Route to Higher Energies

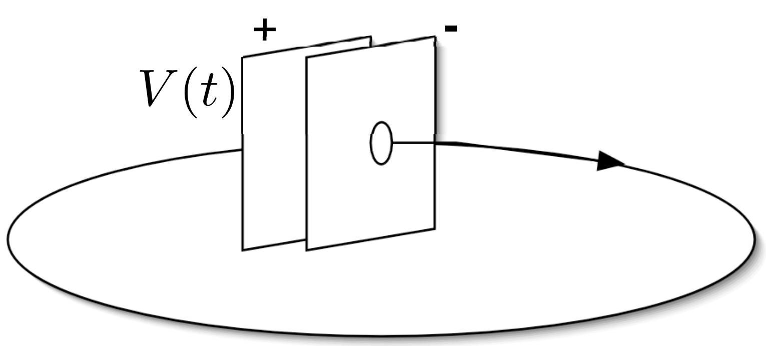
- Static electric fields break down at a few MV; need oscillating EM fields

$$\text{energy gain} = q \cdot V$$

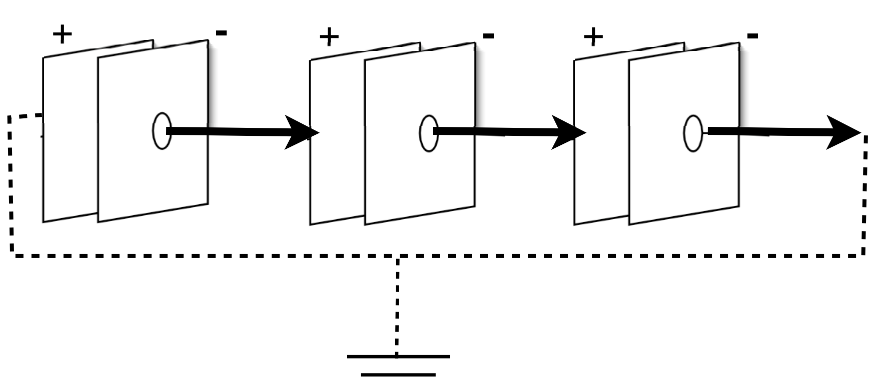


DC systems limited to a few MV

Circular Accelerator



Linear Accelerator



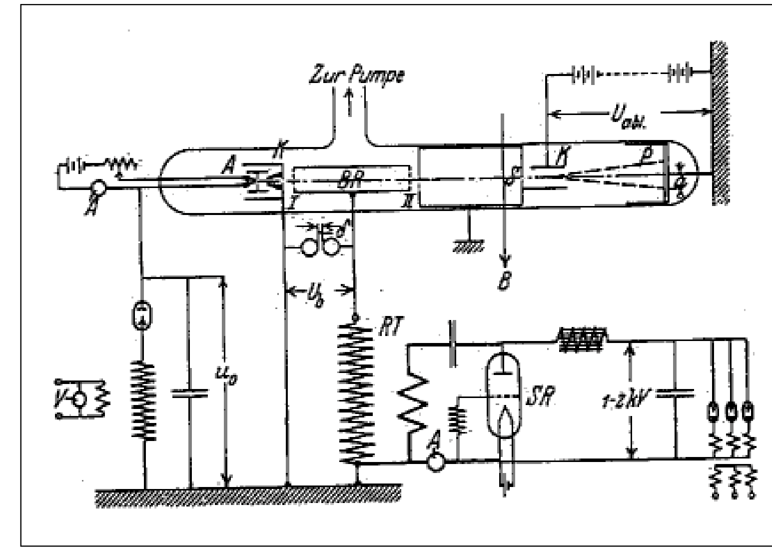
$$\oint (q\vec{E}) \cdot d\vec{s} = work = \Delta(energy)$$

To gain energy, a time-varying field is required:

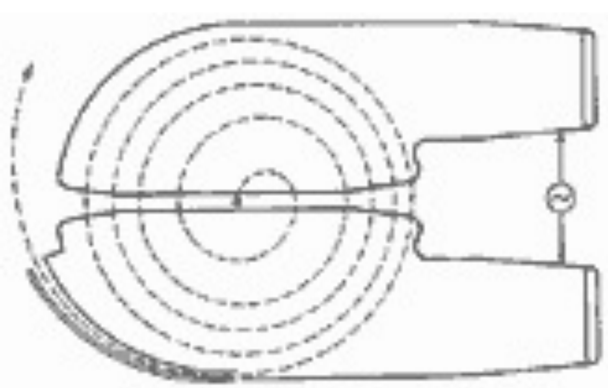
$$\oint \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \oint \vec{B} \cdot d\vec{A}$$

Oscillating Fields

- The linear accelerator (linac) -- 1928-29
  - Wideroe (U. Aachen; grad student!)
    - » Dreamt up concept of “Ray Transformer” (later, called the “Betatron”); thesis advisor said was “sure to fail,” and was rejected as a PhD project. Not deterred, illustrated the principle with a “linear” device, which he made to work -- got his PhD in engineering
  - 50 keV; accelerated heavy ions (K^+, Na^+)
  - utilized oscillating voltage of 25 kV @ 1 MHz



- The Cyclotron -- 1930's, Lawrence (U. California)
  - read Wideroe’s paper (actually, looked at the pictures!)
  - an extended “linac” unappealing -- make it more compact:

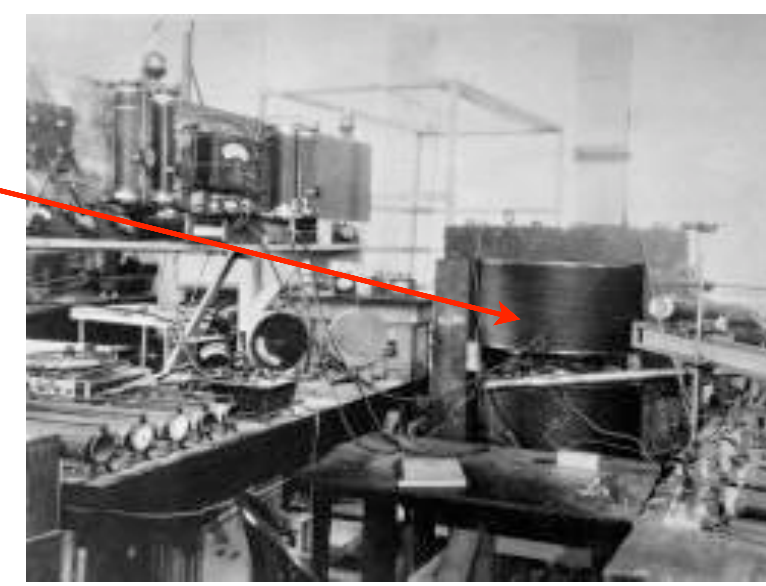


V

$$\frac{1}{T} = \frac{q \cdot B}{2\pi m}$$

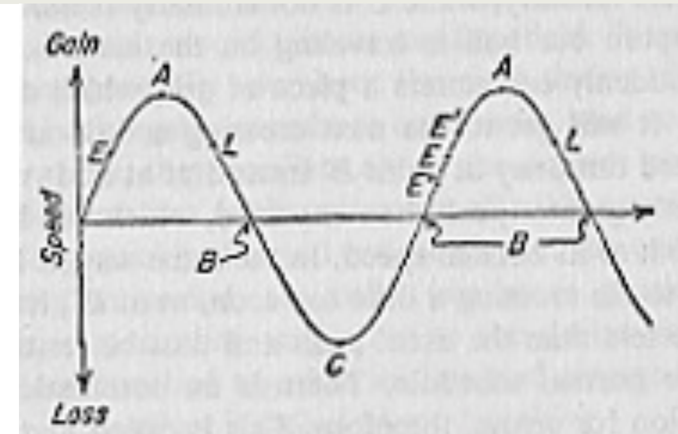
4.5 inch diameter!

11 inch diameter



Stable Acceleration

- Principal of phase stability
  - McMillan (U. California) and Veksler (Russia)



- Imagine: particle circulating in field, *B*; along orbit, arrange particle to pass through a cavity with max. voltage *V*, oscillating at frequency $h \times f_{rev}$ (where *h* is an integer); suppose particle arrives near time of zero-crossing
  - net acceleration/deceleration = $eV \sin(\omega \Delta t)$

- If arrives late, more voltage is applied; arrives early, gets less
  - thus, a restoring force --> energy oscillation

"Synchrotron Oscillations"

- In general, lower momentum particles take longer, arrive late gain extra momentum
- Next, slowly raise the strength of *B*; if raised adiabatically, oscillations continue about the "synchronous" momentum, defined by $p = eB \cdot R$ for constant *R*
- This is the principle behind the synchrotron, used in all major HEP accelerators today

Longitudinal Motion

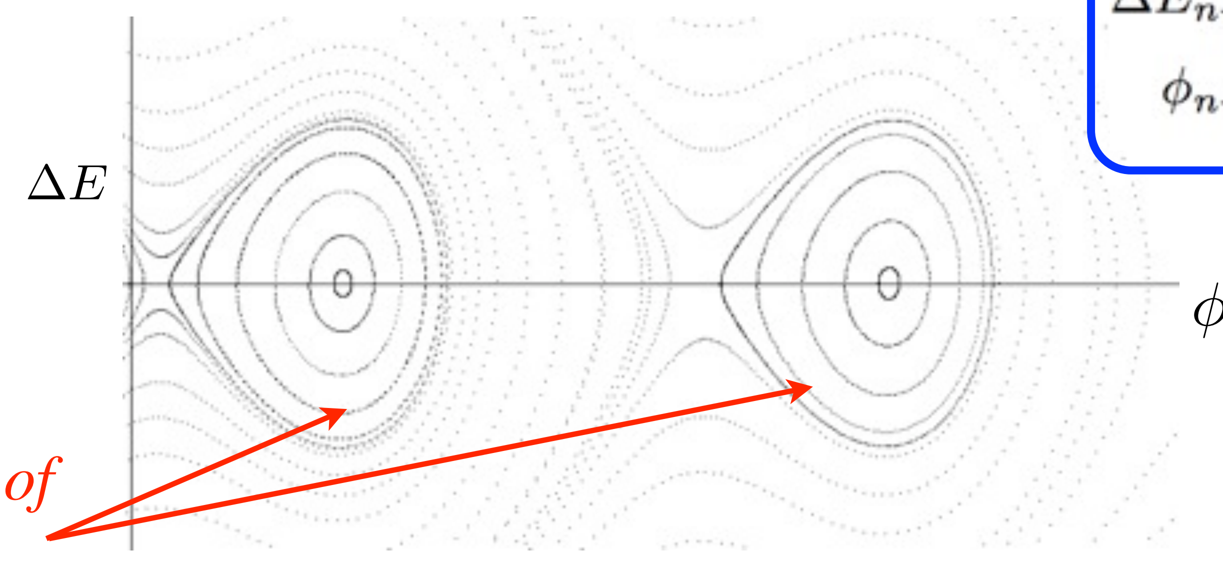
- Say ideal particle arrives at cavity with phase φ_s:

$$\boxed{\frac{dE_s}{dt} = f_0 eV \sin\phi_s \sim \frac{dB}{dt}}$$

- Particles arriving nearby in phase, and nearby in energy will oscillate about these ideal conditions ...

$$\begin{aligned} \Delta E_{n+1} &= \Delta E_n + eV(\sin\phi_n - \sin\phi_s) \\ \phi_{n+1} &= \phi_n + 2\pi h \cdot \frac{\eta}{\beta^2 E_s} \Delta E_{n+1} \end{aligned}$$

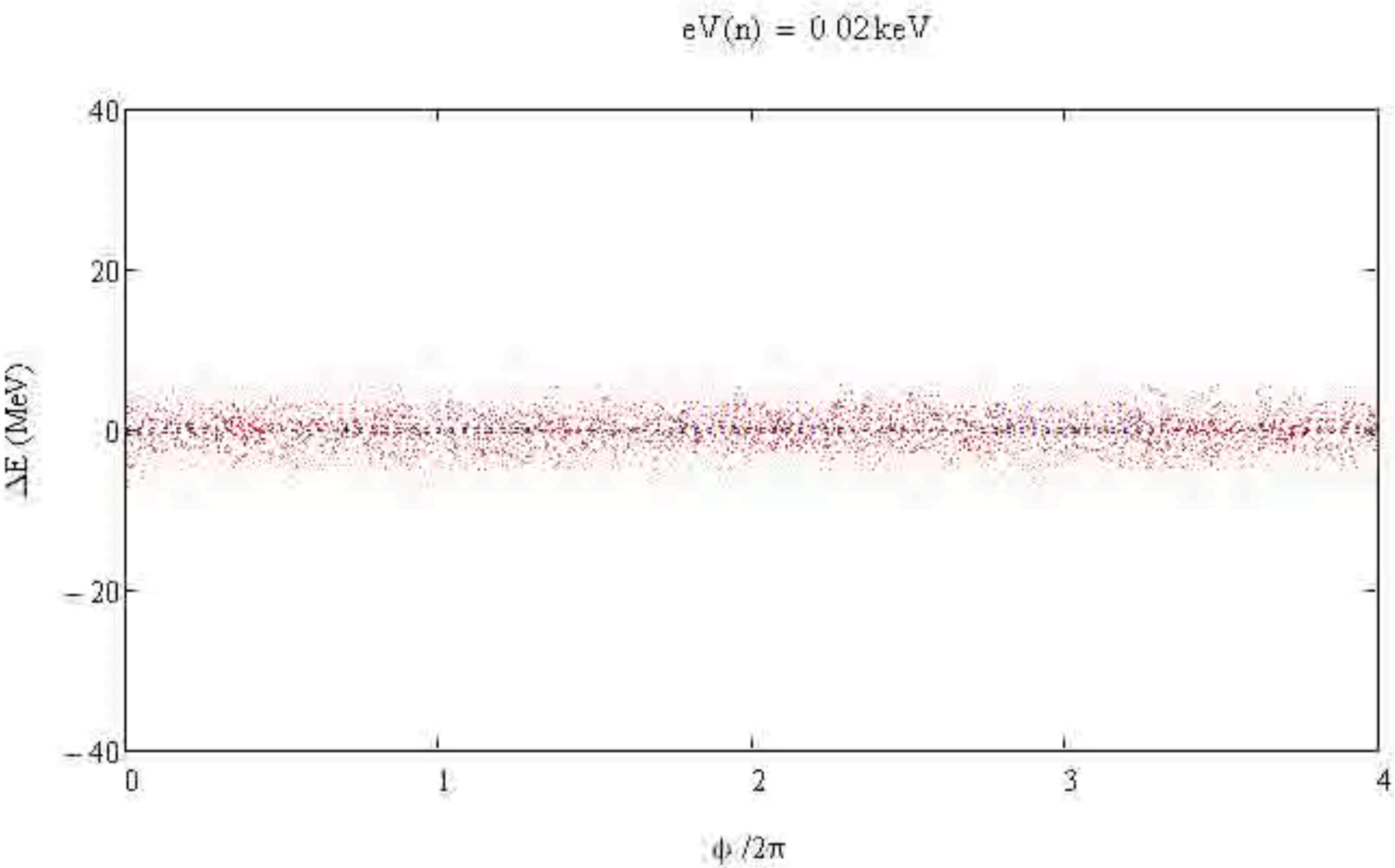
  - Phase Space plot:



- Adiabatic increase of bend field generates stable phase space regions; particles oscillate, follow along
- "bunched" beam; $h = f_{rf} / f_{rev}$ = # of possible bunches

Bunched Beam

- Create a bunched beam in an upstream injector
- Ex: inject a DC beam and adiabatically raise voltage of RF cavities
  - here, $h = f_{rf} / f_{rev} = 4$



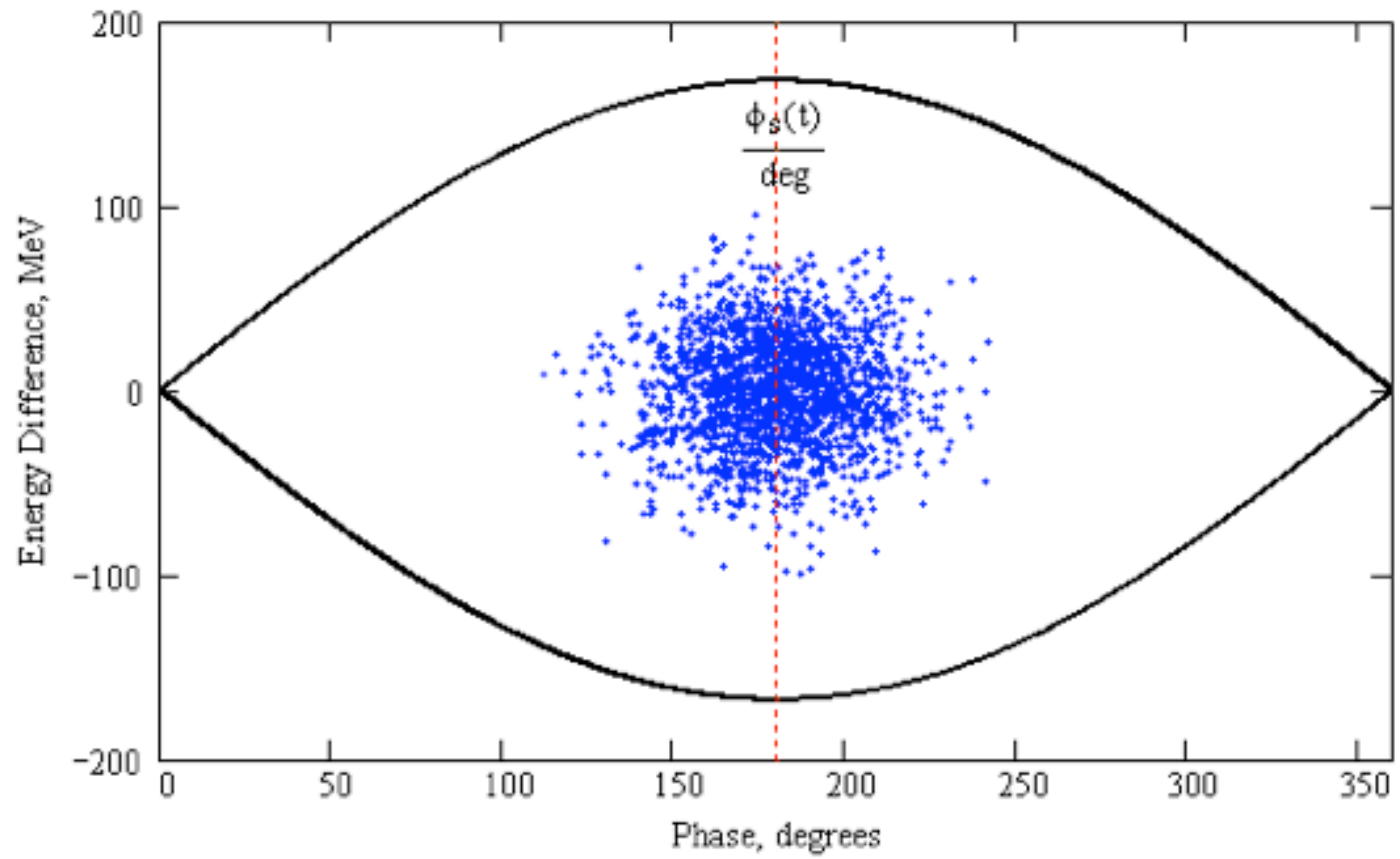
Buckets, Bunches, Batches, ...

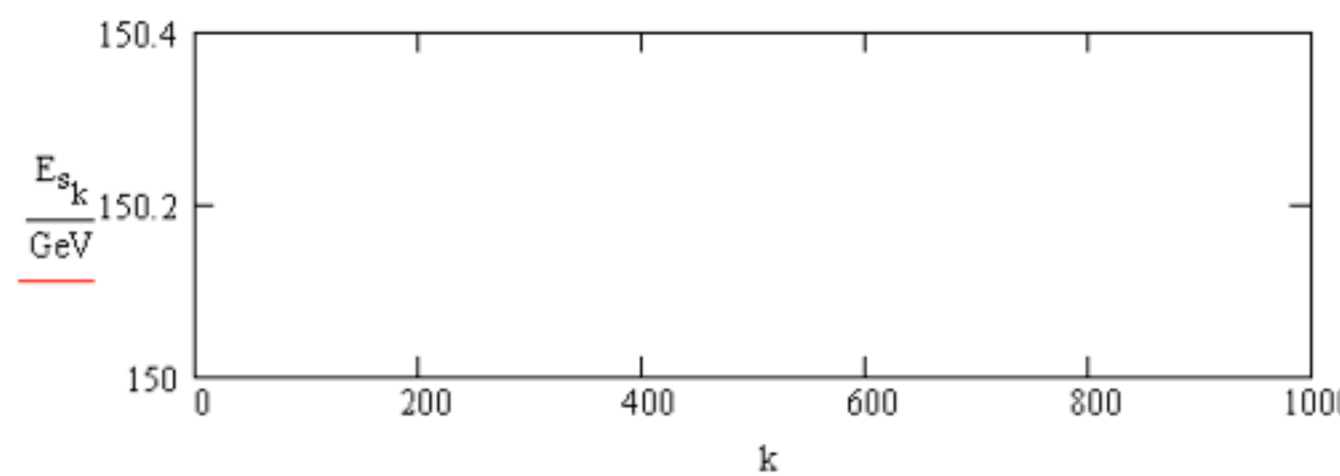
- Stable phase space region is called a bucket.
  - Boundary is the *separatrix*; only an approximation
  - $\varphi_s = 0, \pi$ -- particles outside bucket remain in accelerator
    - "*DC beam*"
  - For other values of φ_s -- particles outside bucket are lost
    - DC beam from injection is lost upon acceleration
- Bunches of particles occupy buckets; but not all buckets need be occupied.
- Batches (or, bunch trains) are groupings of bunches formed in specific patterns, often from upstream accelerators

Acceleration

- Stable regions shrink as begin to accelerate

- If beam phase space area is too large (or if DC beam exists), can lose particles in the process





Acceleration

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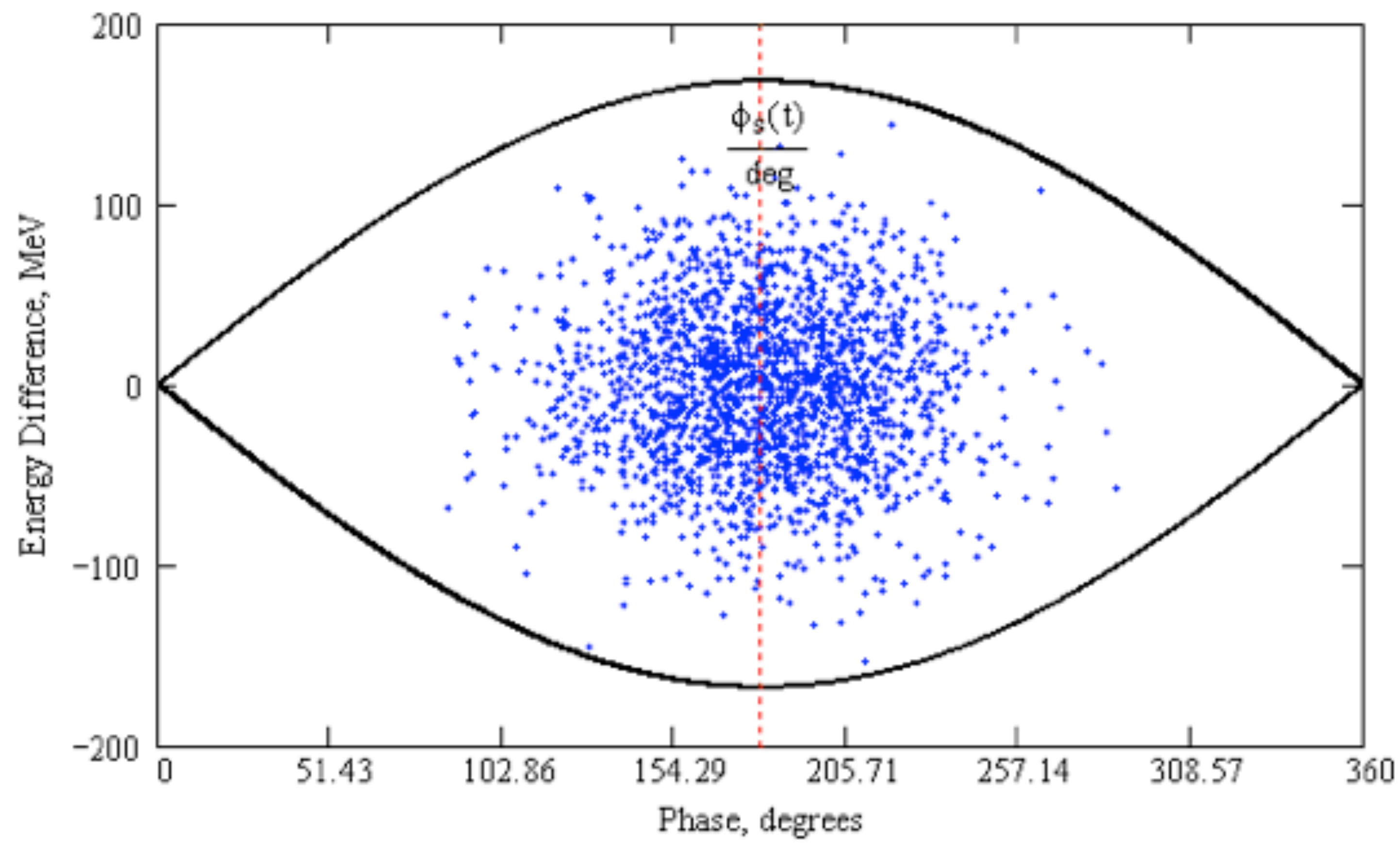
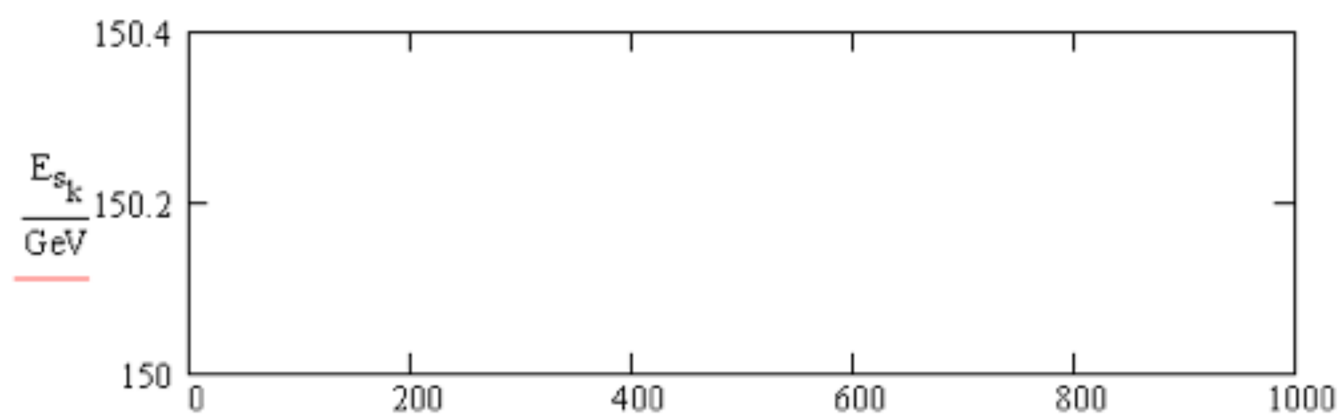




Longitudinal Dynamics

- The longitudinal (*Δz-Δp_z*, or *Δt-ΔE*) dynamics through the accelerator is essentially the same in synchrotrons and linacs

- Time between cavities: $\tau = L/v$
  - particles arrive at a phase with respect to the RF period which depends upon time to travel between cavities
  - this time can depend upon their speed *as well as* the path the particles take
    - » path length may depend upon speed as well -- through bending regions, for instance

Equations of motion:

$$\boxed{\begin{aligned} \Delta E_{n+1} &= \Delta E_n + eV(\sin\phi_n - \sin\phi_s) \\ \phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2 E_s}\,\Delta E_{n+1} \end{aligned}}$$

TRACK, or approximate w/ diff eq's ...

$$\frac{d\Delta E}{dn} = eV(\sin\phi - \sin\phi_s)$$

$$\frac{d\phi}{dn} = \frac{2\pi h\eta}{\beta^2 E_s}\,\Delta E$$

particle will take more/less time to go around, depending upon E (or, p)

$$T = \frac{C}{v} \qquad \frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v}$$

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\frac{\Delta p}{p} \qquad \frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E} \qquad \frac{\Delta C}{C} \equiv \frac{1}{\gamma_t^2}\frac{\Delta p}{p}$$

$$\Rightarrow \frac{\Delta T}{T} = \underbrace{\left(\frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\right)}_{\eta}\frac{\Delta p}{p}$$

(η = slip factor)

Stability and Synchrotron Tune

- Combining the above difference equations, and taking the differential limit leads us to the phase equation

$$\frac{d^2\Delta\phi}{dn^2} = \left(\frac{2\pi h\eta}{\beta^2 E_s} eV \cos\phi_s\right) \sin\Delta\phi$$

where $\Delta\phi = \phi - \phi_s$.

- For small angles we have simple harmonic motion, with "frequency"

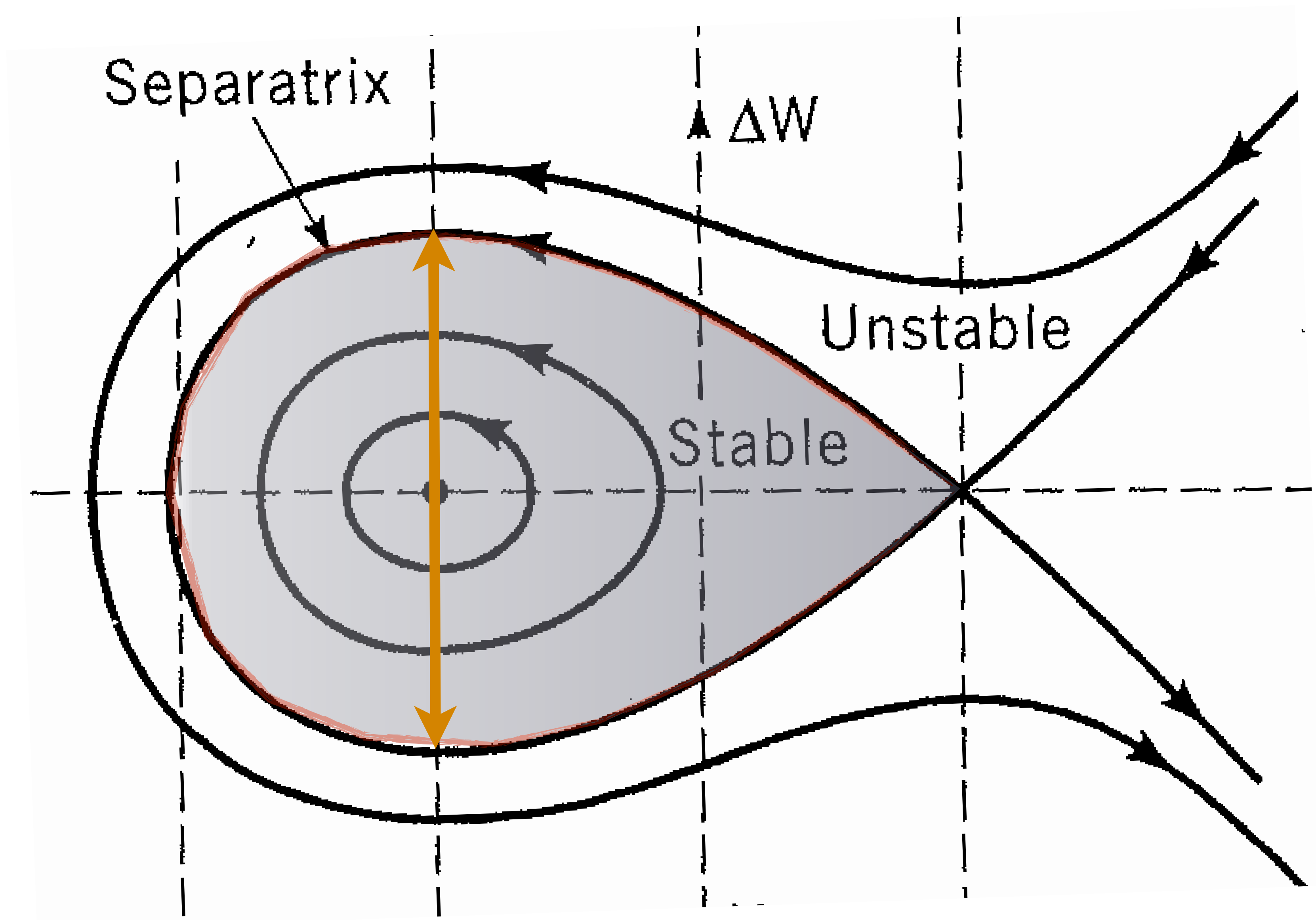
$$2\pi\nu_s = \sqrt{-\frac{2\pi h\eta}{\beta^2 E_s} eV \cos\phi_s}$$

- The quantity ν_s is the *synchrotron tune* and is the number of *synchrotron oscillations* which occur per revolution. At 150 GeV in the Tevatron, for example, we get

$$\nu_s = \sqrt{-\frac{h\eta}{2\pi\beta^2 E_s} eV \cos\phi_s} = \sqrt{-\frac{1113 \cdot 1/18^2}{2\pi(150 \times 10^3)}(1)(-1)} \approx \frac{1}{500}$$

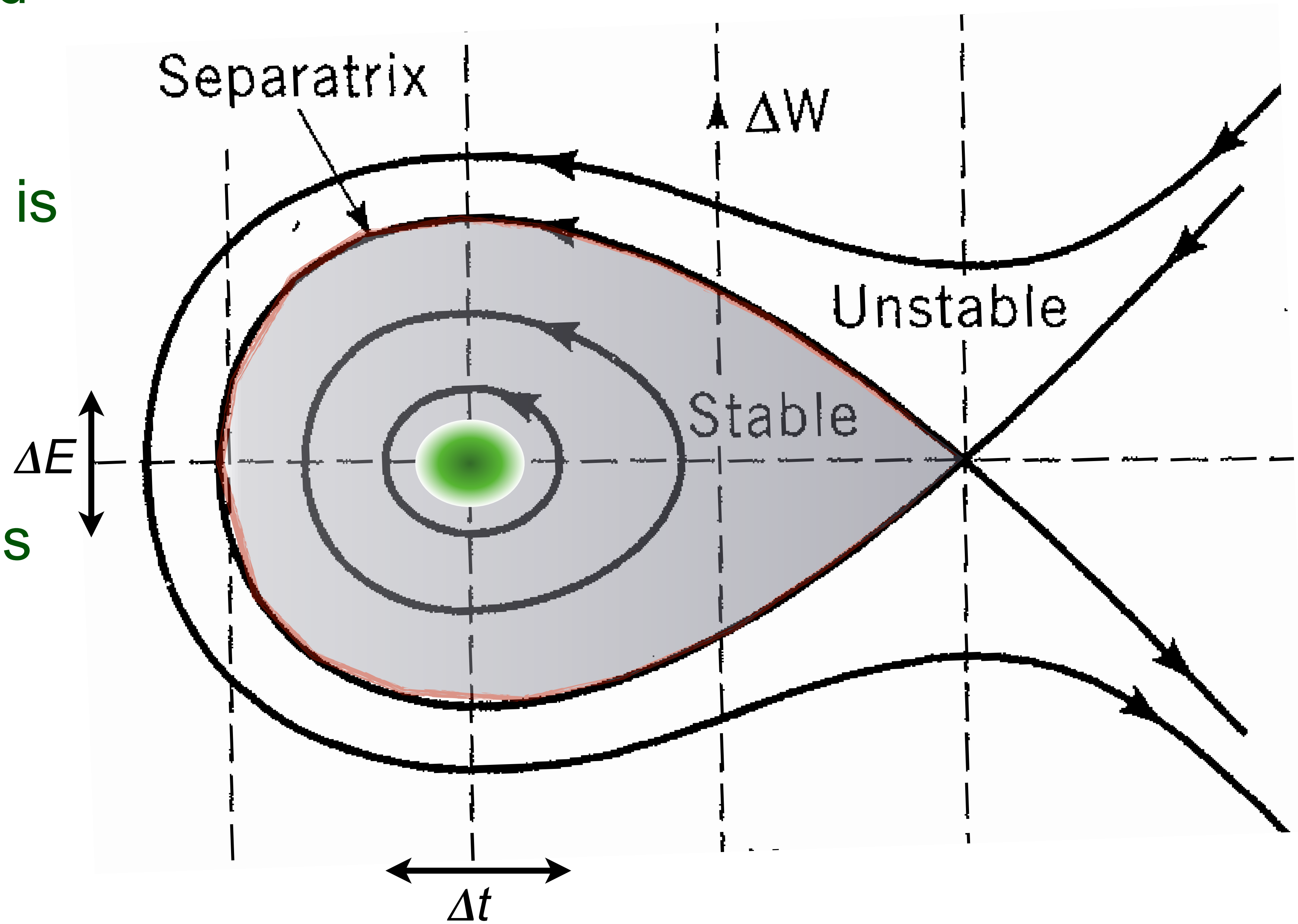
Acceptance and Emittance

- Stable region often called an RF “bucket”
  - “contains” the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system



Acceptance and Emittance

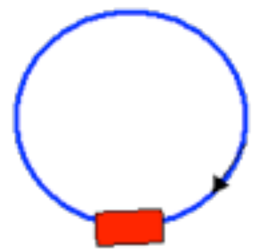
- Stable region often called an RF “bucket”
  - “contains” the particles
- Maximum vertical extent is the maximum spread in energy that can be accelerated through the system
- Desire the beam particles to occupy much smaller area in the phase space



area: “eV-sec”
Note: *E, t canonical*

Summary of Longitudinal Motion

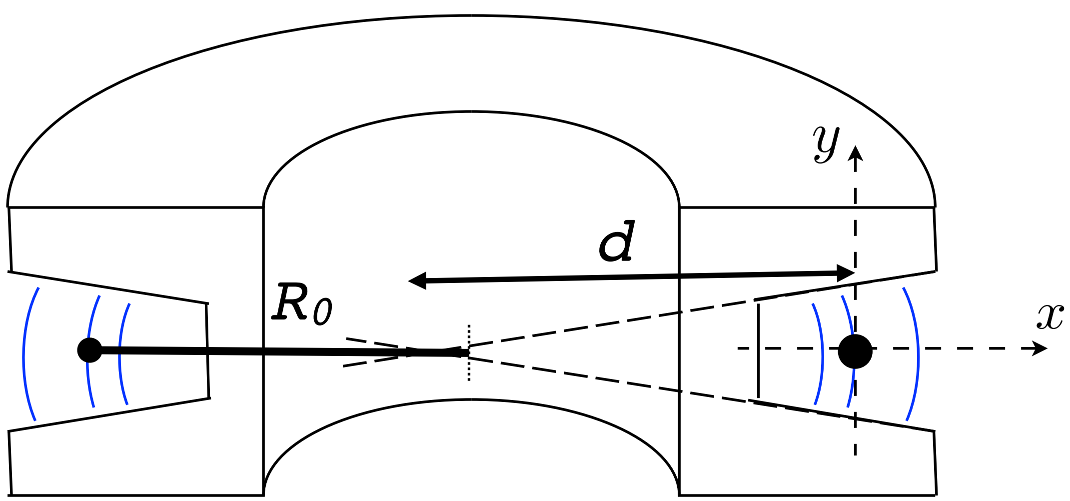
- So, we have our simple picture of a synchrotron in which the cavity or set of cavities -- operating at an ‘RF’ frequency -- form *h* bunches about the circumference



- The particles are guided around the accelerator by a (vertical) bending field of strength *B* corresponding to the bending radius *R* and the central momentum *p* of the particle distribution
- The maximum voltage of the RF system will determine the possible energy spread that can be contained within the accelerator
- Particles nearby the ideal energy will oscillate about the ideal energy (synchrotron oscillations) with the “synchrotron frequency”

Keeping Focused

- In addition to increasing the particle's energy, must keep the beam focused transversely along its journey
- Early accelerators employed what is now called "weak focusing"



$$n \approx \frac{R_0}{d}$$

must have
$0 \leq n \leq 1$
for stability

$$B = B_0 \left(\frac{R_0}{r}\right)^n$$

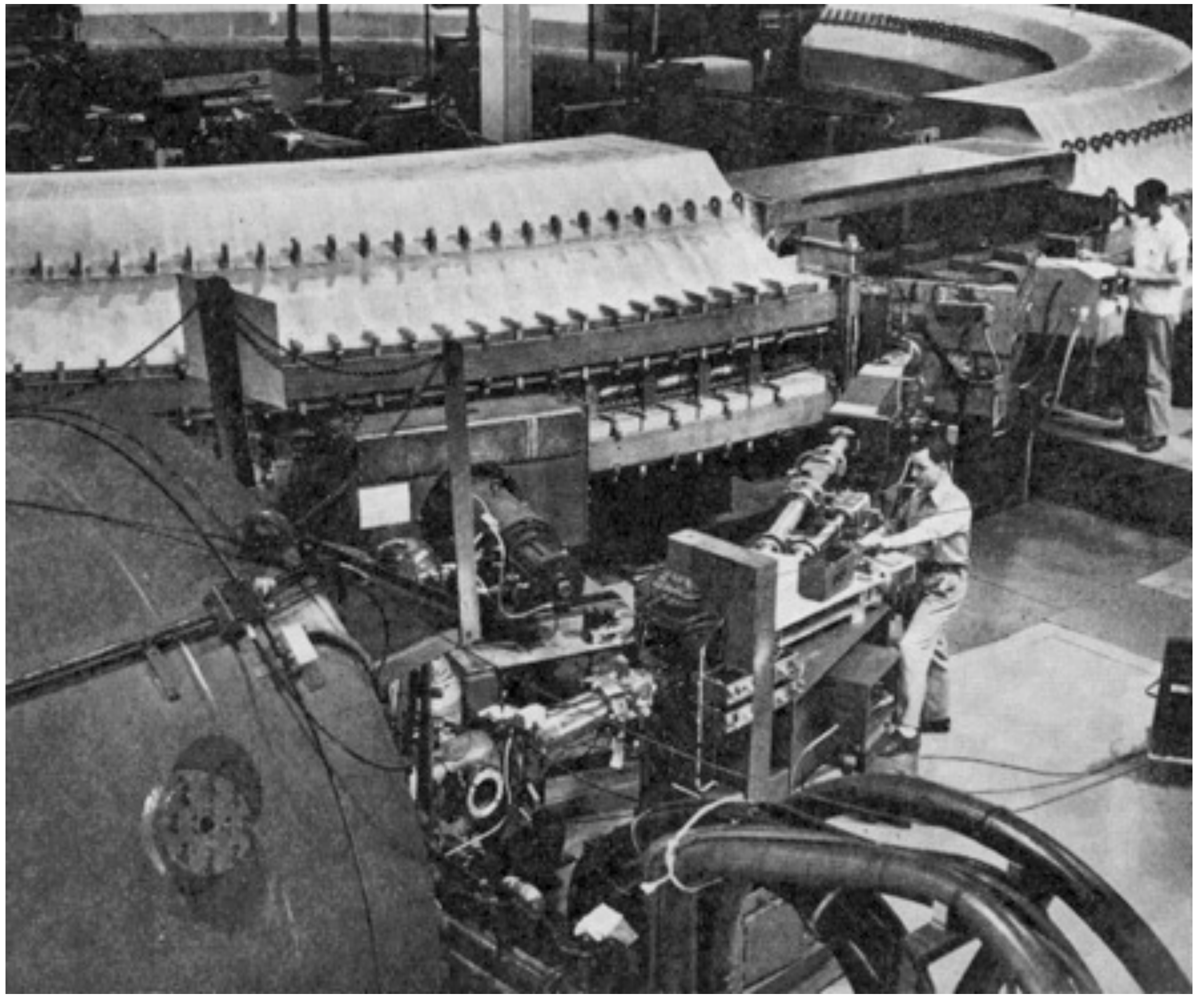
n is determined by adjusting the opening angle between the poles

$$d = \infty,\ n = 0$$

$$d = R_0,\ n = 1$$

Room for improvement...

- With weak focusing, for a given transverse angular deflection, $x_{max} \sim \frac{R_0}{\sqrt{n}}\,\theta$
- Thus, aperture ~ radius ~ energy



Cosmotron (1952)

(3.3 GeV)

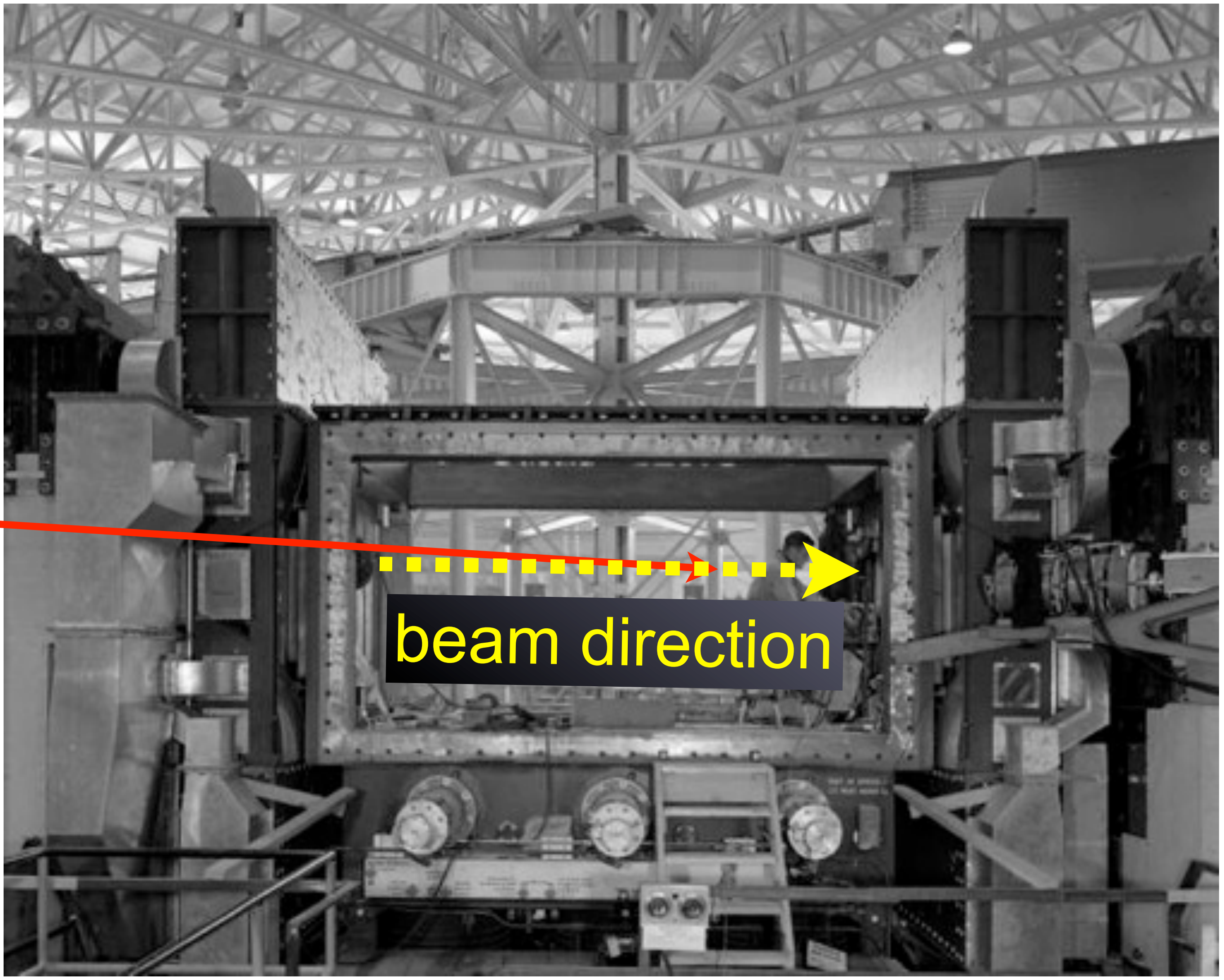
Room for improvement...

- With weak focusing, for a given transverse angular deflection, $x_{max} \sim \frac{R_0}{\sqrt{n}}\,\theta$
- Thus, aperture ~ radius ~ energy

Bevatron (1954)

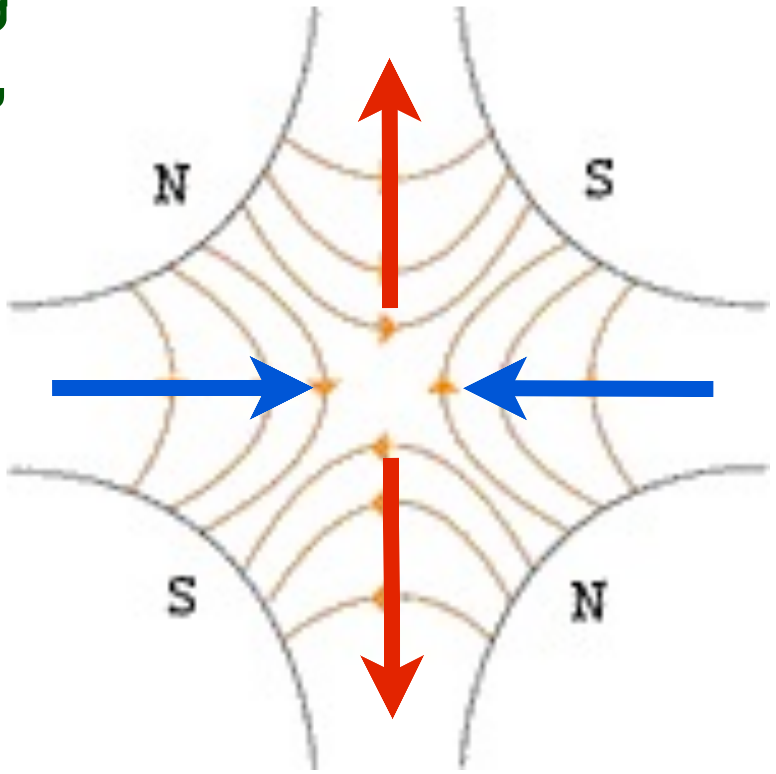
(6 GeV)

Could actually sit inside the vacuum chamber!!



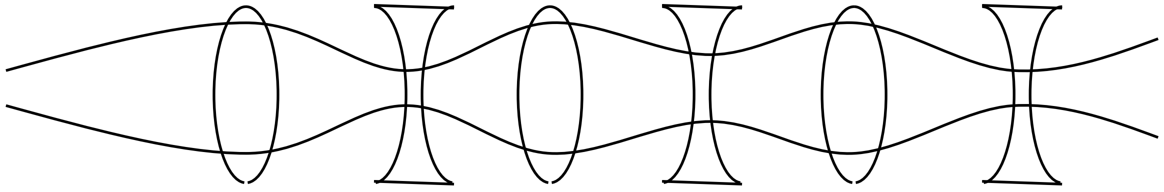
Strong Focusing

- Think of standard focusing scheme as alternating system of focusing and defocusing lenses (today, use quadrupole magnets)



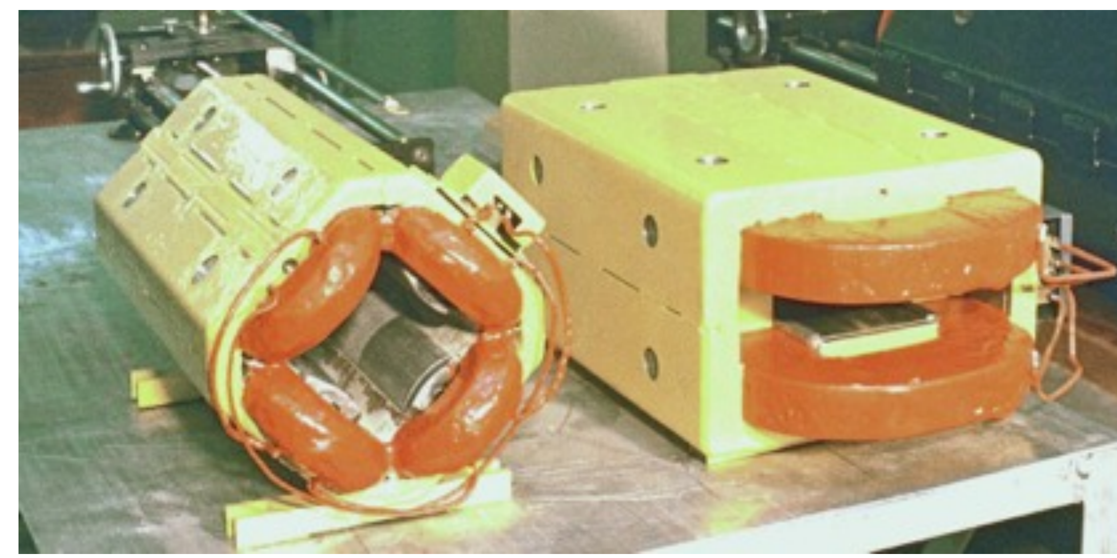
- Quadrupole will focus in one transverse plane, but defocus in other; if alternate, can have net focusing in both

- FODO cells:

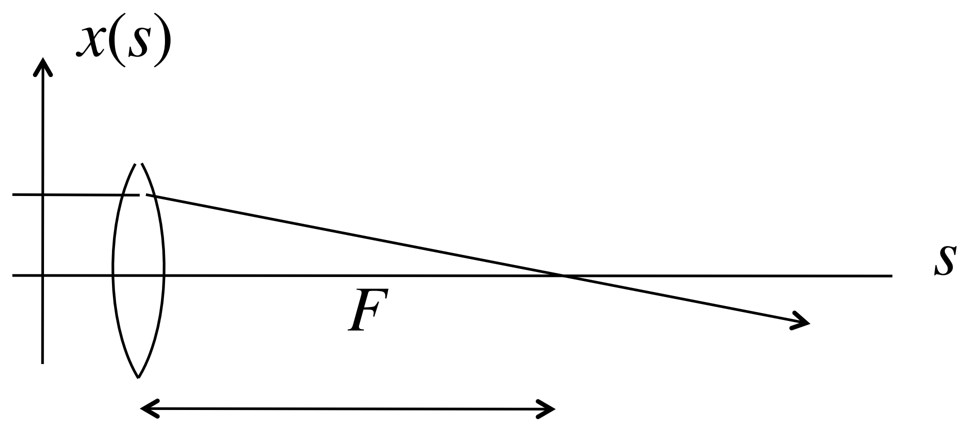


Separated Function

- Until late 60's, synchrotron magnets (wedge-shaped variety) both focused and steered the particles in a circle. ("combined function")
- With Fermilab Main Ring and CERN SpS, use "dipole" magnets to steer, and use "quadrupole" magnets to focus
- Quadrupole magnets, with alternating field gradients, "focus" particles about the central trajectory -- act like lenses
- Thin lens focal length:

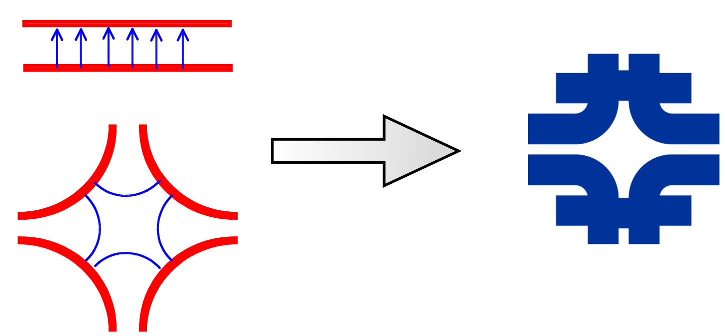


$$\Delta x' = eB_y\ell/p = (eB'\ell/p)\,x \rightarrow 1/F = eB'\ell/p$$



Tevatron: $B' = 77\ \mathrm{T/m}, \quad \ell = 1.7\ \mathrm{m} \rightarrow F = 25\ \mathrm{m}$

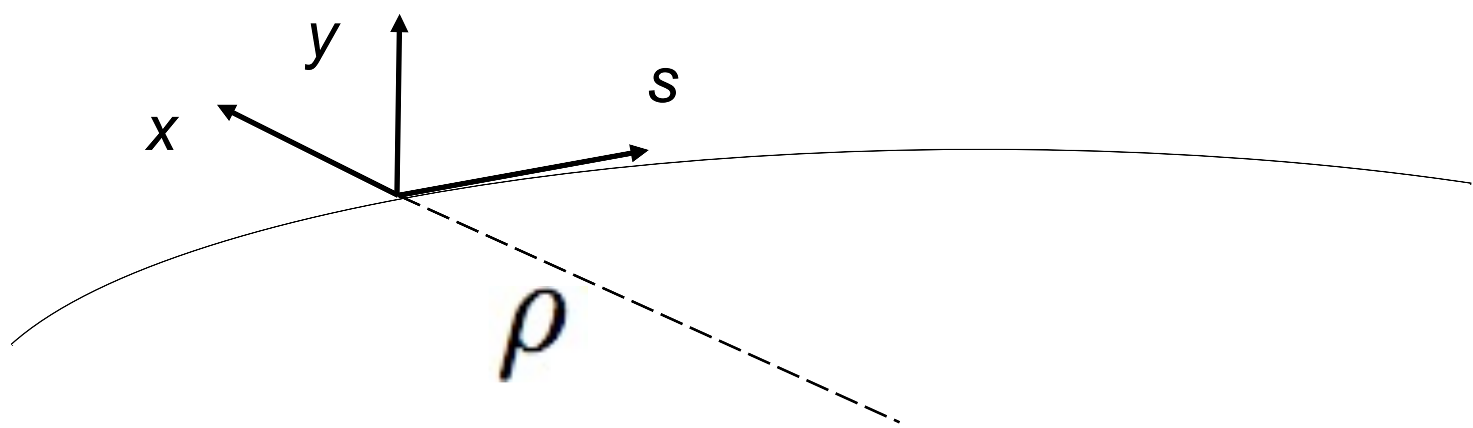
and $L = 30$ m



Fermilab Logo

Transverse Equations of Motion

- Reference Trajectory and Local Coordinate System
  - Electrostatic -- low energy
  - Magnetic -- high energy



- Lorentz Force $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$

- Magnetic Rigidity

$$B\rho \approx \left(\frac{10}{3} \; \frac{\mathrm{T \cdot m}}{\mathrm{GeV/c}} \right) \cdot p$$

Linear Restoring Forces

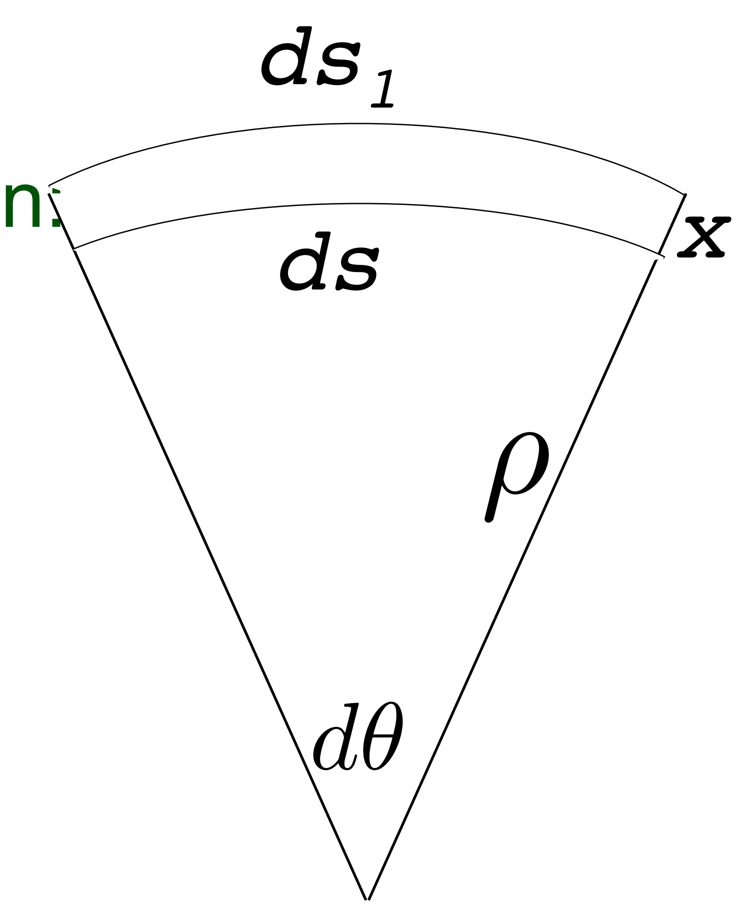
- Wish to look at motion "near" the ideal trajectory of the accelerator system

- Assume linear guide fields: --

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

- $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = x' v_s$ notion:

$$\frac{ds_1}{\rho + x} = \frac{ds}{\rho}$$



$$\begin{aligned}
\gamma m \frac{d^2(X_d)}{dt^2} &= -e v_s B_0 \\
\gamma m \frac{d^2(X_d + x)}{dt^2} &= -e v_{s1} B_y(X) \\
\gamma m (X_d'' + x'') v_s^2 &= -e v_{s1} B_y(X) \\
\gamma m v_s x'' &= -e \frac{v_{s1}}{v_s} B_y + e B_0 \\
\gamma m v_s x'' &= -e \left[B_y \left(1 + \frac{x}{\rho} \right) - B_0 \right] \\
x'' &= -\frac{e}{p} \left[(B_y - B_0) + B_y \frac{x}{\rho} \right] \\
&\approx -\frac{1}{B\rho} \left[B'x + B_0 \frac{x}{\rho} \right]
\end{aligned}$$

Hill's Equation

- Now, for vertical motion:

$$B_y = B_0 + B'x$$
$$B_x = B'y$$

$$\begin{aligned}
\gamma m \frac{d^2(Y_d)}{dt^2} &= 0 \\
\gamma m \frac{d^2(Y_d + y)}{dt^2} &= e v_{s1} B_x(Y) \\
\gamma m v_s^2 y'' &= e v_{s1} B_x(Y) \\
\gamma m v_s y'' &= e \frac{v_{s1}}{v_s} B_x \\
\gamma m v_s y'' &= e B_x \left(1 + \frac{x}{\rho}\right) \\
y'' &= \frac{e}{p}\left[B_x(1 + \frac{x}{\rho})\right] \\
&\approx \left(\frac{B'}{B\rho}\right) y
\end{aligned}$$

- So we have,

 to lowest order,

$$\boxed{\begin{aligned}
x'' + \left(\frac{B'}{B\rho} + \frac{1}{\rho^2}\right) x &= 0 \\
y'' - \left(\frac{B'}{B\rho}\right) y &= 0
\end{aligned}}$$

Hill's Equation

General Form: $\longrightarrow$ $x'' + K(s)x = 0$

- As accelerate, scale *K* with momentum; becomes purely geometrical

Piecewise Method of Solution

- Hill's Equation: $$x'' + K(s)x = 0$$

- Though K(s) changes along the design trajectory, it is typically constant, in a piecewise fashion, through individual elements (drift, sector mag, quad, edge, ...)

- K = 0: drift $$x'' = 0 \longrightarrow x(s) = x_0 + x_0' s$$

- K > 0: $$x(s) = x_0 \cos(\sqrt{K}\ s) + \frac{x_0'}{\sqrt{K}} \sin(\sqrt{K}\ s)$$

Quad, Gradient Magnet, edge, ...

- K < 0: $$x(s) = x_0 \cosh(\sqrt{|K|}s) + \frac{x_0'}{\sqrt{|K|}} \sinh(\sqrt{|K|}s)$$

Here, x refers to horizontal or vertical motion, with relevant value of K

Piecewise Method -- Matrix Formalism

- Write solution to each piece in matrix form
  - for each, assume K = *constant* from s = 0 to $s = L$

- K = 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- K > 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

- K < 0:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cosh(\sqrt{|K|}L) & \frac{1}{\sqrt{|K|}}\sinh(\sqrt{|K|}L) \\ \sqrt{|K|}\sinh(\sqrt{|K|}L) & \cosh(\sqrt{|K|}L) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

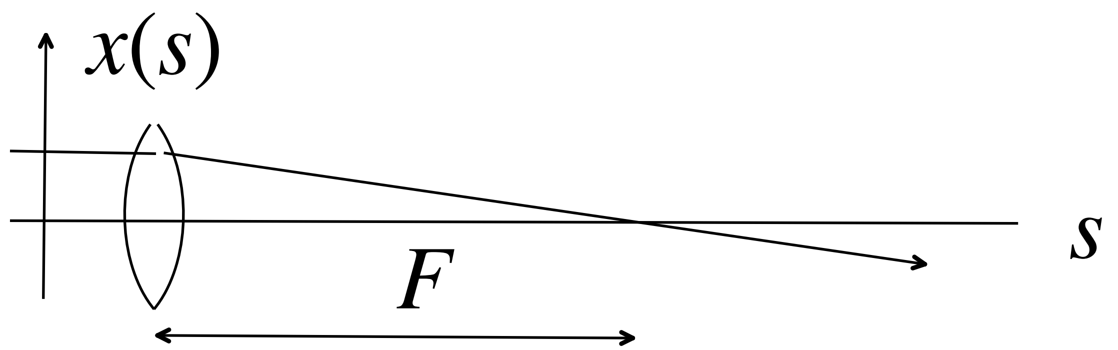
"Thin Lens" Quadrupole

- If quadrupole magnet is short enough, particle's offset through the quad does not change by much, but the slope of the trajectory does -- acts like a "thin lens" in geometrical optics
- Take limit as L --> 0, while KL remains finite

$$\begin{pmatrix} \cos(\sqrt{K}L) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}L) \\ -\sqrt{K}\sin(\sqrt{K}L) & \cos(\sqrt{K}L) \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -KL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{F} & 1 \end{pmatrix}$$

  - (similarly, for defocusing quadrupole)

- Valid approx., if F >> L

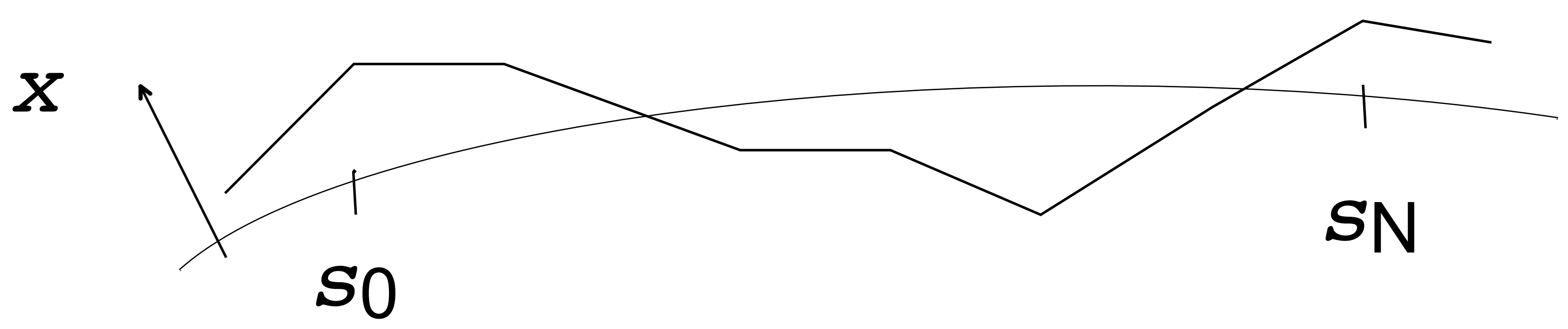


$$KL = \frac{B'L}{B\rho} = \frac{1}{F}$$

Piecewise Method -- Matrix Formalism

- Arbitrary trajectory, relative to the design trajectory, can be computed via matrix multiplication

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$



Stability Criterion

- For single pass through a short system of elements, above may be all we need to know to describe the system. But, suppose the “system” is very long and made of many repetitions of the same type of elements (or, perhaps the “repetition” is a complete circular accelerator, for instance) -- how to show that the motion is stable for many (infinite?) passages?
- Look at matrix describing motion for one passage:

$$M = M_N M_{N-1} \cdots M_2 M_1$$

- We want:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_k = M^k \begin{pmatrix} x \\ x' \end{pmatrix}_0 \text{ finite as } k \to \infty \text{ for arbitrary } \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

Stability Criterion

V = eigenvector
λ = eigenvalue

$$X_k = M^k X_0 = M^k(AV_1 + BV_2) = A\lambda_1^k V_1 + B\lambda_2^k V_2$$

$$\det M = 1 = \lambda_1\lambda_2 \rightarrow \lambda_2 = 1/\lambda_1 \rightarrow \lambda = e^{\pm i\mu}$$

If μ is imaginary, then repeated application of *M* gives exponential growth; if μ real, gives oscillatory solutions…

$$\text{characteristic equation: } \det(M - \lambda I) = 0$$

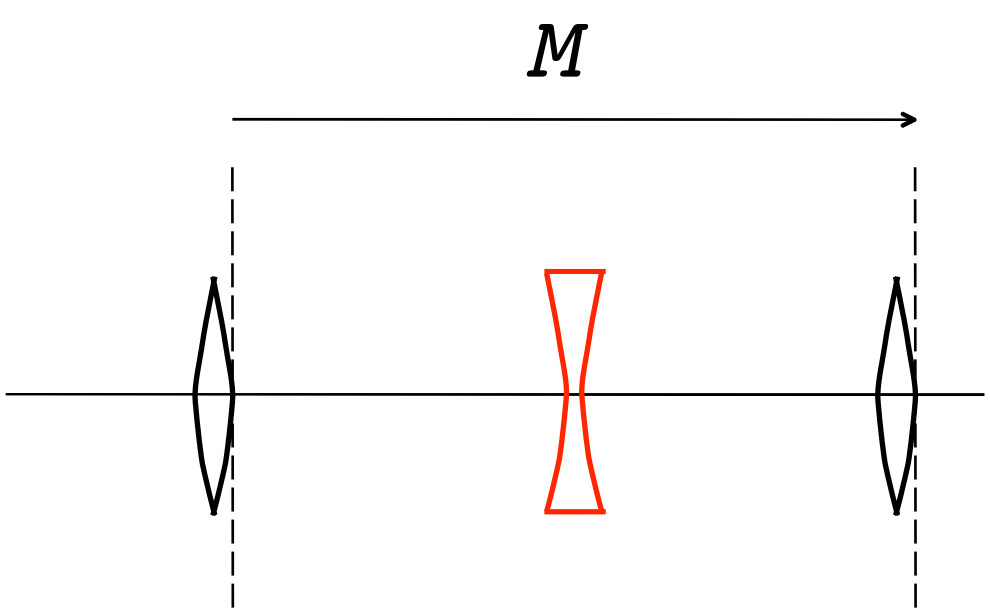
$$\text{if } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ then } (a-\lambda)(d-\lambda) - bc = 0$$

$$\begin{aligned}
&\lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\
&\lambda^2 - tr M\lambda + 1 = 0 \\
&\lambda + 1/\lambda = tr M \\
&e^{i\mu} + e^{-i\mu} = 2\cos\mu = tr M
\end{aligned}$$

So, μ real (stability)
$\rightarrow |tr M| < 2$

Example: Application to FODO system

$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$



and repeat…

So, $tr M = 2 - L^2/F^2$ and thus, for stability,

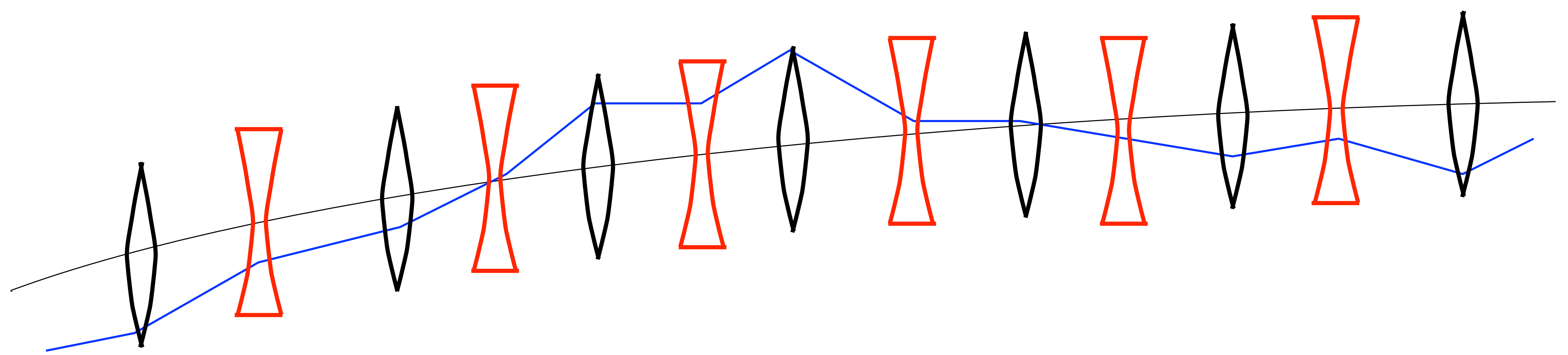
$$-2 < 2 - L^2/F^2 < 2$$

$$-4 < -L^2/F^2 < 0$$

$$F > L/2$$

Can now make LARGE accelerators!

- Since the lens spacing can be made arbitrarily short, with corresponding focusing field strengths, then in principal can make beam transport systems (and linacs and synchrotrons, for instance) of arbitrary size



- Instrumental in paving the way for very large accelerators, both linacs and especially synchrotrons, where the bending and focusing functions can be separated into distinct magnet types

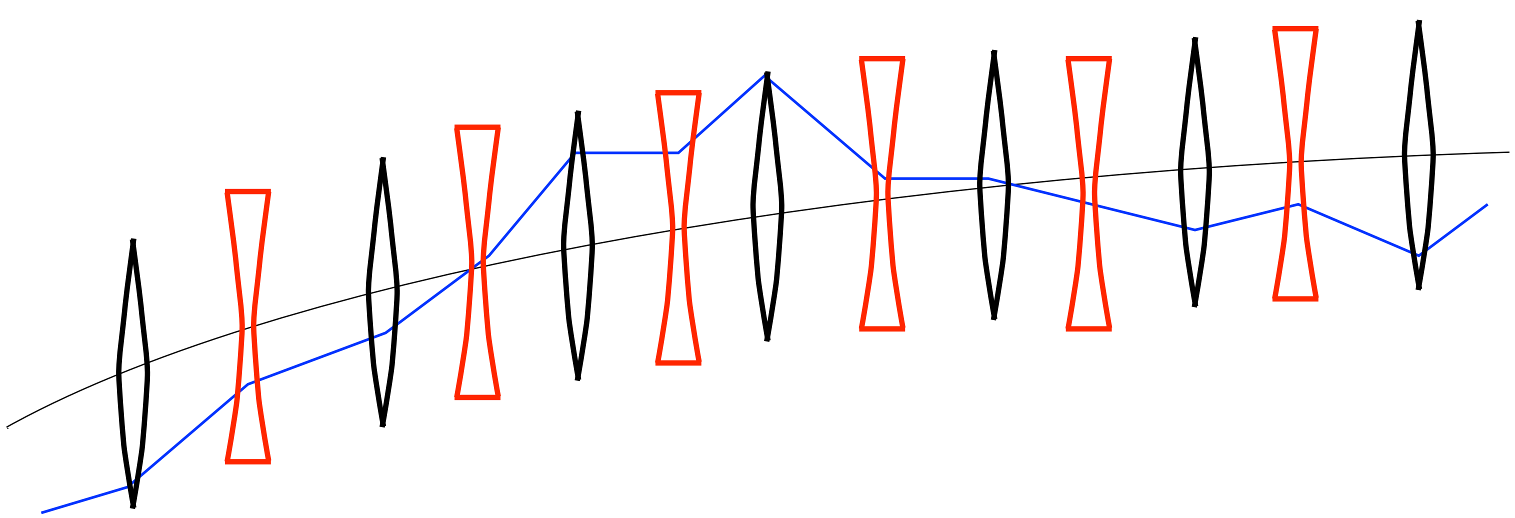
Strong Focusing -- what it means

- Essentially, if focus (positive gradient, say), and then defocus (negative gradient), with appropriate "lens" spacing, then can control beam size over great distances

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

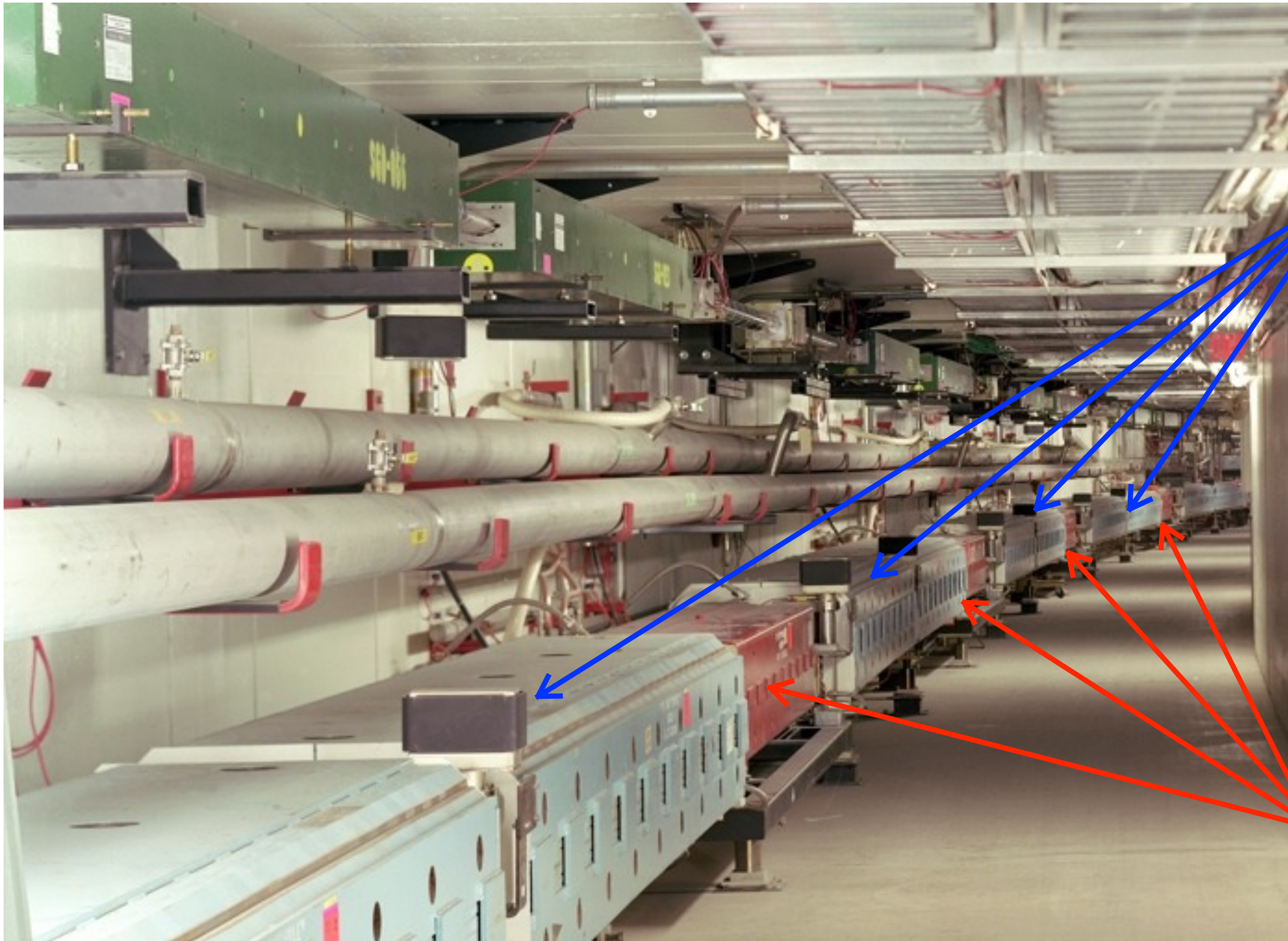
- Ex: simple system of lenses, spaced by d: $f_2 = -f_1 \longrightarrow F = \frac{f_1^2}{d} > 0$

So, can in principle generate arbitrarily long focusing system:



AGS construction, Brookhaven, New York

Example: Fermilab Main Injector



Bending Magnets

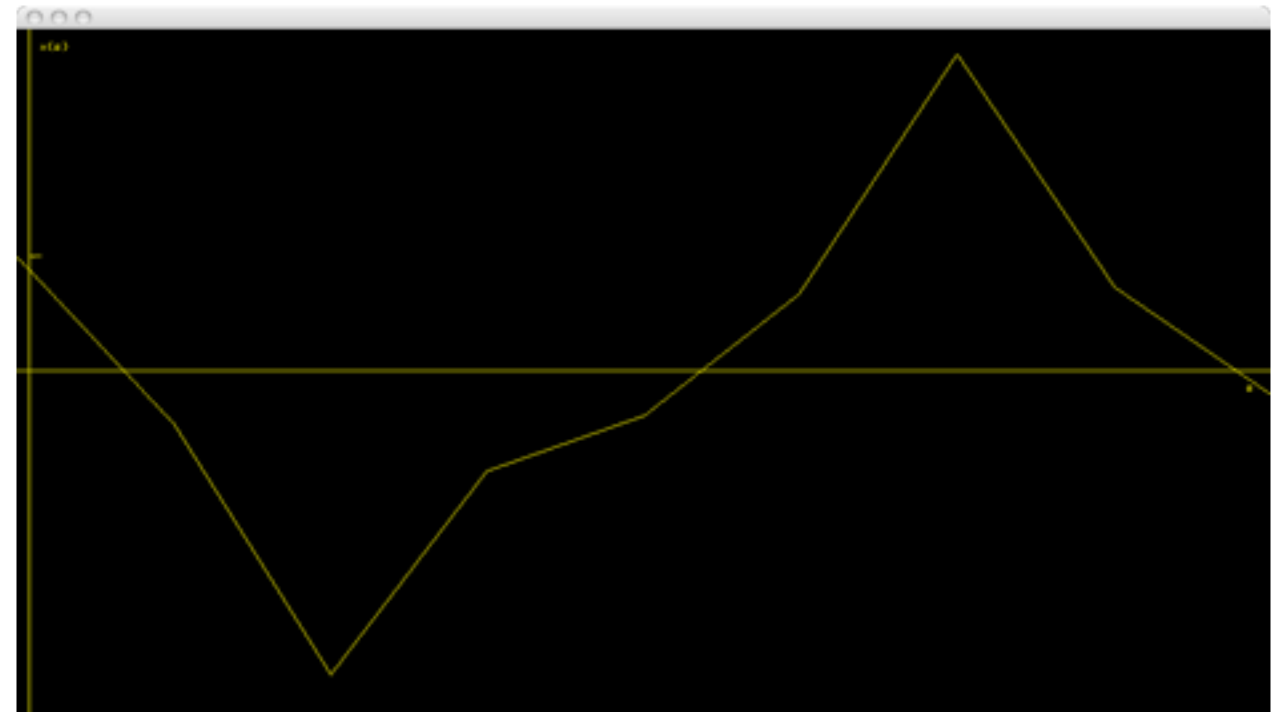
~ 2 mile circumference

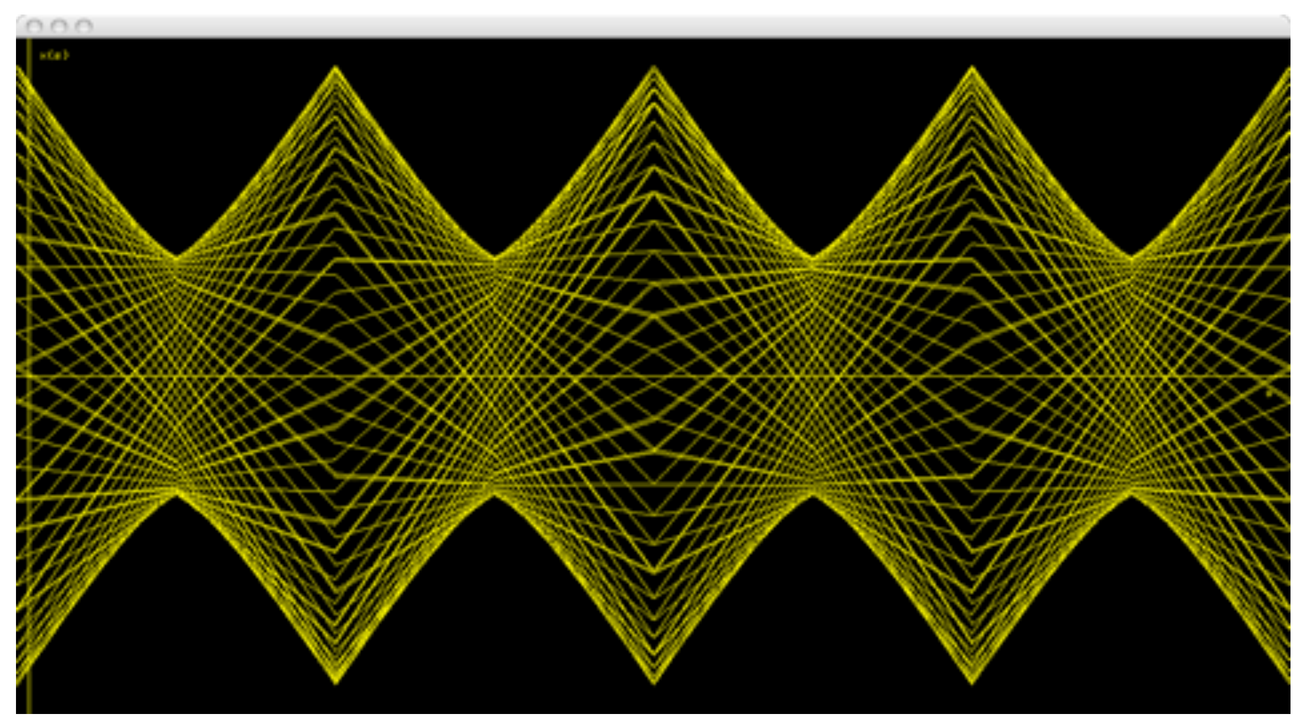
Focusing Magnets

The Notion of an Amplitude Function…

- Track single particle(s) through a periodic system

- Can represent either
  - multiple passages around a circular accelerator, or
  - multiple particles through a beam line





Can we describe the maximum amplitude of particle excursions in analytical form?

of course! ***coming up…***

Break till Day Two...

- Tomorrow:
  - Analytical formulation of beam optics and transverse oscillations
    - the "beta function"
  - Transverse beam emittance and phase space
  - Chromatic effects, momentum dispersion
  - Accelerator "lattice"
  - Errors and adjustments
    - orbits, tunes, and resonances