$(g-2)$ in QED and Beyond

1 A brief introduction

Anomalous magnetic moment of the electron is one of the crowning achievements of Quantum Electrodynamics (QED) in particular, and of Quantum Field Theory in general. Experimental and theoretical calculations agree to about one part in a trillion [1].

The anomalous magnetic moment of the muon is sensitive to states beyond the Standard Model. These states could be heavier than what we can produce at colliders; they might couple dominantly to leptons, making harder to produce at hadron colliders. The muon $(g-2)$ measurement then provides an interesting way to probe these new states, by measuring their tiny effect with a highly sensitive experiment.

To compare theory and experiment, highly technical calculations are involved. QED calculations have to be performed at high order, where the number of Feynman diagrams increase to hundreds. Contributions from hadronic physics becomes important at such high precision. These cannot be calculated using Feynman diagrams and require input from either experimental measurement, or novel theoretical techniques (a lot of progress is being made in this direction using lattice QCD, which will be covered in detail in forthcoming lectures in this series).

In this note we will focus on the calculation of first radiative corrections to the lepton $g-2$, and briefly comment on estimating one-loop corrections from new physics.

2 What to calculate?

Let's start with a classical system. The magnetic moment of a classical current carrying loop of wire is given by,

$$\vec{\mu} = I\vec{s} \tag{1}$$

where I is the current through the loop, and $\vec{s}$ is the vector area. In presence of magnetic field, the loop experiences a torque, and there is a potential energy associated with the orientation of the loop relative to the magnetic field,

$$U = -\vec{\mu}.\vec{B} \tag{2}$$

That is, the configuration where the loop is aligned with the magnetic field has lower potential energy.

For simplicity, let us imagine a circular loop with a charge q traveling along the loop with angular velocity ω, as shown in figure 1. Then,

$$\vec{\mu} = \frac{q}{2\pi/\omega}\pi a^2 \hat{z} \tag{3}$$

since the current is given by a charge q traversing the loop per revolution. We can rewrite this as,

$$\vec{\mu} = \frac{q}{2m}\omega m a^2 \hat{z} = \frac{q}{2m}\vec{L} \tag{4}$$

where L is the angular momentum of the charged particle about the origin of the loop.

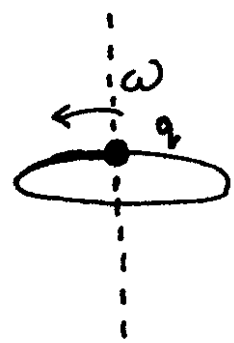

Figure 1: A charge running around a loop of wire.

We see that classically, the magnetic dipole moment is proportional to the angular momentum of charged particles. This continues to be true in quantum mechanics (including the spin angular momentum for particles), with a small difference. For example, for an electron at rest, the magnetic dipole moment is given by,

$$\vec{\mu}_e = g\frac{e}{2m_e}\vec{S} \tag{5}$$

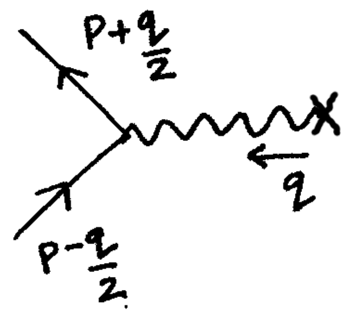

Figure 2: Tree level diagram for a lepton scattering off a background classical field.

where $\vec{S}$ is the spin operator. Analogously, in presence of a classical magnetic field, the spin of the electron prefers to be aligned with the magnetic field. The interaction Hamiltonian is,

$$H_{int} = -\vec{\mu} \cdot \vec{B} \tag{6}$$

In order to calculate the QED prediction for the muon (electron) magnetic moment, we need to compare the QED interaction of the electron with a classical background field.

Let us focus on the tree level contribution first (figure 2). The interaction Lagrangian density at tree level is given by,

$$\mathcal{L}_{int} = -e\bar{\ell}(x)\gamma^{\mu}\ell(x)\mathtt{A}_{\mu}(x) \tag{7}$$

where we have denoted the classical field by $\mathtt{A}$, and the lepton (electron/muon) by ℓ.

The way to connect this relativistic Lagrangian density with the non-relativistic quantum mechanical potential is to compare the scattering amplitudes. The tree-level scattering amplitude in momentum space is given by,

$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^{\mu})u(k)\,\widetilde{\mathtt{A}}_{\mu}(q) \tag{8}$$

where $u, \bar{u}$ are plane-wave solutions to the Dirac equation, and $\widetilde{\mathtt{A}}_{\mu}(q)$ is the Fourier-transform of the classical field, and $q = p' - p$. Since we are interested in time-independent field configurations,

$$\widetilde{\mathtt{A}}_{\mu}(q) = \widetilde{\mathtt{A}}_{\mu}(\vec{q})\delta(q^0) \tag{9}$$

The vector potential corresponding to a static magnetic field is given by $\mathtt{A}_\mu = (0, \vec{\mathtt{A}})$, with the magnetic field given by

$$\vec{B}(x) = \nabla \times \vec{\mathtt{A}} \tag{10}$$

$$\Rightarrow \widetilde{B}_k(\vec{q}) = -i\epsilon^{ijk} q^i \mathtt{A}_j(\vec{q}) \tag{11}$$

We now set the scattering amplitude from the non-relativistic theory equal to the one above, obtained from the relativistic theory. The electron couples in two different ways to the magnetic field (or the 3-vector potential): one via its spin, and the other via its motion ($\vec{p}$, which can also be thought of as orbital angular momentum). To isolate the spin coupling, we set $\vec{p} = 0$. Since the magnetic field is linear in $\vec{q}$, we retain a non-zero $\vec{q}$, but work to linear order.

The non-relativistic answer is easily obtained using the Born approximation. Comparing the two,

$$-i\langle \mu^k \rangle \widetilde{B}_k(\vec{q}) = +ie\,\bar{u}(p')\gamma^k u(k)\,\widetilde{\mathtt{A}}_k(q) \tag{12}$$

with $p' = (m, \vec{q}/2), k = (m, -\vec{q}/2)$. (Note $p'^2 = k^2 = m^2$ is satisfied to linear order in q.)

Therefore, in order to pick out the magnetic moment, we need to pick out the term proportional to $\vec{q}$ from the spinor,

$$-i\langle \mu^k \rangle \left[-i\epsilon^{ijk} q^i\, \widetilde{\mathtt{A}}_j(\vec{q}) \right] = +ie\,\bar{u}(p')\gamma^j u(k)\,\widetilde{\mathtt{A}}_j(q) \tag{13}$$

Let us expand the spinor on the r.h.s in the non-relativistic limit

$$u(k) = \begin{pmatrix} \sqrt{k \cdot \sigma}\xi \\ \sqrt{k \cdot \bar{\sigma}}\xi \end{pmatrix} \tag{14}$$

where as usual, the square root of a matrix is understood to pick out the square root of eigenvalue when acting on an eigenvector.

In the non-relativistic limit,

$$\sqrt{k \cdot \sigma} = \sqrt{m - \left(-\frac{1}{2}\right) \vec{q} \cdot \vec{\sigma}} \approx \sqrt{m}(1 + \frac{1}{2}\vec{q} \cdot \vec{\sigma}/2m) \tag{15}$$

$$\sqrt{p' \cdot \sigma} = \sqrt{m - \left(\frac{1}{2}\right) \vec{q} \cdot \vec{\sigma}} \approx \sqrt{m}(1 - \frac{1}{2}\vec{q} \cdot \vec{\sigma}/2m) \tag{16}$$

Therefore,

$$\bar{u}(p')\gamma^k u(k) = \begin{pmatrix} \sqrt{p'\cdot\sigma}\xi' \\ \sqrt{p'\cdot\bar{\sigma}}\xi' \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \sqrt{k\cdot\sigma}\xi \\ \sqrt{k\cdot\bar{\sigma}}\xi \end{pmatrix} \tag{17}$$

$$= m\,\xi'^\dagger \left((1 - \frac{1}{4m}\vec{q}\cdot\vec{\sigma})\bar{\sigma}^k(1+\frac{1}{4m}\vec{q}\cdot\vec{\sigma}) + (1+\frac{1}{4m}\vec{q}\cdot\vec{\sigma})\sigma^k(1-\frac{1}{4m}\vec{q}\cdot\vec{\sigma})\right)\xi \tag{18}$$

$$\approx m\xi'^\dagger \left(\bar{\sigma}^k + \sigma^k + \frac{1}{4m}\vec{q}\cdot\vec{\sigma}(\sigma^k - \bar{\sigma}^k) - (\sigma^k - \bar{\sigma}^k)\frac{1}{4m}\vec{q}\cdot\vec{\sigma}\right)\xi \tag{19}$$

$$= \frac{1}{2}\xi'^\dagger q_i \left(\sigma^i\sigma^k - \sigma^k\sigma^i\right)\xi \tag{20}$$

$$= \frac{1}{2}\xi'^\dagger \left[2i\epsilon^{ikj}\sigma^j\right] q_i\xi \tag{21}$$

$$= 2i\epsilon^{ikj} q_i \xi'^\dagger \frac{\sigma^j}{2}\xi \tag{22}$$

$$= -\frac{i}{m}\epsilon^{ijk} q_i \langle S^j\rangle \tag{23}$$

remembering that the relativistic states are normalized as,

$$\langle S^j\rangle = 2m\,\xi^\dagger S^j \xi \tag{24}$$

due to the relative factor of $2m$ in the normalization of relativistic states.

Comparing with the non-relativistic result,

$$-i\langle\mu^k\rangle\left[-i\epsilon^{ijk}q^i\,\widetilde{\mathtt{A}}_j(\vec{q})\right] = -ie\frac{1}{m}\langle S^k\rangle\left[-i\epsilon^{ijk}q_i\,\widetilde{\mathtt{A}}_j(q)\right] \tag{25}$$

$$\langle\mu^k\rangle = \frac{e}{m}\langle S^k\rangle = g\frac{e}{2m}\langle S^k\rangle \tag{26}$$

which is to say, that the leading order prediction of QED is $g = 2$.

2.1 Higher order corrections

The radiative corrections to the calculation above can be compactly written as,

$$i\mathcal{M} = -ie\,\bar{u}(p')\Gamma^\mu u(k)\mathtt{A}_\mu \tag{27}$$

Using equations of motion and gauge invariance, the most general form of Γ^μ can be deduced to be,

$$\Gamma^\mu(p,q) = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2m} \tag{28}$$

Remember we need to work to linear order in q_i.

$$\frac{i}{2m}\bar{u}(p')\sigma^{k\nu}q_\nu u(k) = \frac{i}{2m}\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi \tag{29}$$

Therefore, we only need the form factors at 0 momentum transfer,

$$\bar{u}(p')\Gamma^\mu(p,q)u(k) = \frac{i}{2m}\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi[F_1(0)+F_2(0)] \tag{30}$$

From above, it is clear that,

$$g = 2(F_1(0)+F_2(0)) \tag{31}$$

Due to gauge invariance, to all orders $F_1(0) = 1$. Therefore, the contribution from F_1 is always $g = 2$. The correction, $g-2$ then arises solely from F_2. The anomalous magnetic moment is written as,

$$a = \frac{g-2}{2} = F_2(0) \tag{32}$$

2.2 One loop diagrams

The one loop diagrams relevant are shown in figure 3. To calculate the one loop contribution to the matrix element, we use the following master formula,

$$\langle S\rangle = (\sqrt{Z_2})^2\bar{u}(p')[\text{Amputated one loop diagram}]u(k) \tag{33}$$

where Z_2 is the wavefunction renormalization, calculated at one loop, for the electron. Note that there is no wavefunction renormalization included for the external classical field.

> `The wavefunction renormalization cancels the renormalization of the` $F_1(0)$ `form factor of the vertex function to every order in perturbation theory.`

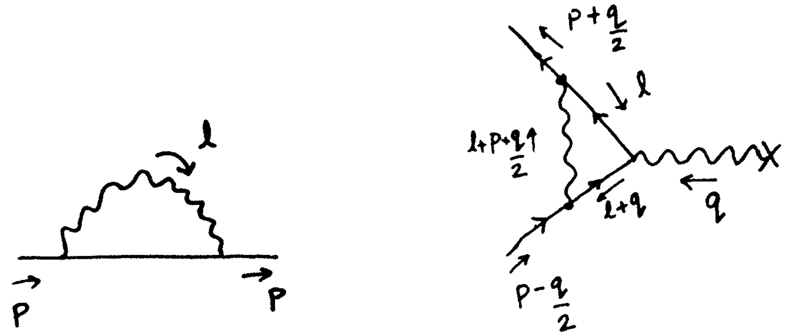

Figure 3: One loop corrections to the magnetic moment. Wavefunction renormalization diagram for the lepton (left) and amputated vertex correction.

3 Feynman diagram calculation

The tree-level contribution to F_2 is absent. Therefore, the wavefunction renormalization does not contribute. The entire one-loop correction to $g-2$ then involves calculation of the amputated vertex correction, and extracting the form factor $F_2(0)$.

In this section we will focus on the Feynman diagram calculation: the calculation of of the lowest order radiative correction to the muon magnetic moment. This is a famous calculation by Schwinger [4, 5], first performed for the electron (although he did not use Feynman diagrams for his calculation). As we shall see, at this order, the magnetic moments of the electron, muon and the tau lepton are identical. The Feynman diagram is shown in figure 4.

This particular calculation involves three aspects which we will try to isolate:

1. Symmetry: The result of the diagram is constrained by gauge invariance, and the charge conjugation and the parity symmetries of the QED Lagrangian.

2. Gamma matrix algebra: Much of the calculation involves simplification of Dirac gamma matrices. We will adopt a method which is modular, easily generalized, and easily implemented on a computer.

3. Loop integration: As with any loop diagram, the loop momentum is integrated over. We will isolate all these integrations, and we will find that there are a only a few distinct integrations we ever need to do (the so-called scalar integrals) and those have been done and tabulated at e.g. [2].

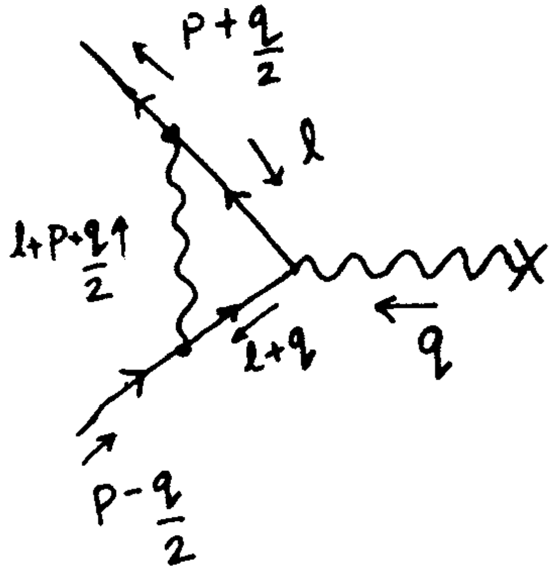

Figure 4: One loop correction to the electron/muon $g - 2$.

The method adopted will hopefully help in extending this calculation to an arbitrary 1-loop contribution to g-2 (and perhaps other processes, or higher loop order).

Our strategy is to first expand the amplitude in terms of all structures allowed by symmetry. We pick out the terms which would survive in the limit we are interested in. Finally, all such terms are cast in terms of scalar integrals i.e. loop integrals with only denominator terms. We will find that the calculation converges nicely and we only need 2 or 3 simple expressions for the integrals.

The scattering amplitude is written as,

$$i\mathcal{M} = \bar{u}\left(p + \frac{q}{2}\right)\left[\int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}(-ie\gamma_\alpha)\frac{i(-\not{l} + m)}{(l^2 - m^2)}(-ie\gamma^\mu)\frac{i(-\not{l} - \not{q} + m)}{((l+q)^2 - m_\mu^2)}(-ie\gamma_\beta)\frac{-ig^{\alpha\beta}}{(l + p + \frac{q}{2})^2}\right]u\left(p - \frac{q}{2}\right) \tag{34}$$

$$\equiv -ie\bar{u}\left(p + \frac{q}{2}\right)\Gamma^\mu u\left(p - \frac{q}{2}\right) \tag{35}$$

We can expand Γ^μ in a basis of Dirac matrices,

$$\Gamma^\mu = c_1^\mu \mathbb{1} + C_5^\mu \gamma_5 + C_v^{\mu\beta}\gamma_\beta + C_a^{\mu\beta}\gamma_\beta\gamma_5 + C_s^{\mu\alpha\beta}\sigma_{\alpha\beta} \tag{36}$$

where each co-efficient depends on p^μ and q^μ. The co-efficients have a Lorentz index, and so we can expand them in terms of p^μ and q^μ.

We can use symmetries of the theory (here QED) to restrict the most general dependence. Gauge invariance implies the Ward identity,

$$q_\mu \Gamma^\mu = 0 \tag{37}$$

with or without $q^2 = 0$. Since this is a matrix equation, this holds for each co-efficient separately. Note that $p \cdot q = 0$, and $p^2 = m^2 - \frac{1}{4}q^2$, so the only scalar variable apart from the masses we have is q^2.

In a theory respecting C and P, C_5 should vanish (and indeed does). Further, the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$ also does not appear, except when multiplying a $\gamma_\mu\gamma_5$, i.e. for C_a. Therefore, all other co-efficients must be formed out of the four-vectors p^μ, q^α and the metric tensor $g^{\mu\nu}$.

Thus, the only non-zero co-efficients are,

$$C_1^\mu = c_1(q^2)\, p^\mu \tag{38}$$

$$C_v^{\mu\beta} = c_{v1}(q^2)\, p^\mu p^\beta + c_{v2}(q^2)\, p^\mu q^\beta + c_{v3}(q^2)\, g^{\mu\beta} \tag{39}$$

$$C_a^{\mu\beta} = c_{a4}(q^2)\, \epsilon^{\mu\alpha\delta\beta} p_\alpha q_\delta \tag{40}$$

$$C_s^{\mu\alpha\beta} = c_{s1}(q^2)\, p^\mu(p^\alpha q^\beta - p^\beta q^\alpha) + c_{s2}(q^2)\, (g^{\mu\alpha} p^\beta - g^{\mu\beta} p^\alpha) + c_{s3}(q^2)\, (g^{\mu\alpha} q^\beta - g^{\mu\beta} q^\alpha) \tag{41}$$

It maybe looks like a hopeless mess to calculate all these coefficients, each of which is a loop integral. Keep in mind that thus far, this expression applies to any contribution to $g-2$, as long as the symmetries are respected. If not, there can be additional terms, but it is easy to generalize this method.

The good news is, there are only a couple of distinct loop integrations we need to do, and further, only a subset of these co-efficients contributes to F_2. Recall that we have not yet used the equations of motion, and by doing so, we can cast the entire expression as,

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu} q_\nu}{2m} \tag{42}$$

The contribution from F_1 is only at the tree level, and all corrections to $F_1(0)$ vanish. Therefore, we are only interested in the co-efficients above which contribute to $F_2(0)$.

The simplification of various co-efficients can be found in section 7 in equation 121. From there, we conclude that,

$$F_2(0) = -(2mi)\left(-\frac{i}{2}(c_1(0) + mc_{v1}(0)) + 2c_{s3}(0)\right) - im^2 c_{a4}(0) \tag{43}$$

3.1 Evaluating the coefficients

The co-efficients themselves can be easily evaluated in terms of the loop integral by using traces (the equivalent of inner product for γ matrices). The one subtlety is to keep track on the dependence number of dimensions we are working in, i.e. $d = 4 - 2\epsilon$. We should always start with well-defined expressions, or in other words regulated integrals. In this note we assume that all integrals will be carried out in $4 - 2\epsilon$ dimensions.

3.1.1 Extracting $c_1(0)$

$$Tr.\left[\Gamma^\mu\right] = 4c_1(q^2)\, p^\mu = \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{-8ie^2(-2+\epsilon)m(2l^\mu - q^\mu)}{(l^2-m^2)((l+q)^2-m_\mu^2)(l+p+\frac{q}{2})^2} \tag{44}$$

To extract c_1, we can take dot product with p_μ, and divide by p^2,

$$c_1(q^2) = \frac{1}{p^2} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{-2ie^2(-2+\epsilon)m(2l\cdot p)}{(l^2-m^2)((l+q)^2-m_\mu^2)(l+p+\frac{q}{2})^2} \tag{45}$$

Now, since we are interested only in $c_1(0)$, we can set $q = 0$ in the above expression, simplifying it immensely.

$$c_1(q^2) = \frac{1}{p^2} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{-2ie^2(-2+\epsilon)m(2l\cdot p)}{(l^2-m^2)^2(l+p)^2} \tag{46}$$

Just to simplify notation, let us label $(l+p) = Q$. Then, c_1 can be

$$c1(0) = \int \frac{1}{p^2} \frac{-2ie^2(-2+\epsilon)m(Q^2-l^2-p^2)}{(l^2-m^2)(l^2-m_\mu^2)(l+p)^2} \tag{47}$$

$$= \frac{-2ime^2(-2+\epsilon)}{p^2}\left[\mathcal{I}_{21}^{Q^2} - \mathcal{I}_{21}^{l^2} - p^2\mathcal{I}_{21}\right] \tag{48}$$

where the short-hand $\mathcal{I}_{21}$ denotes the denominator, with 2 powers of $l^2 - m^2$, and one of Q^2. The superscript denotes the numerator, which we always recast as Q^2, l^2 or powers thereof.

3.1.2 Extracting $c_{v1}(0)$

$$Tr.\left[\Gamma^\mu\gamma^\beta\right] = 4C_v^{\mu\beta} = 4c_{v1}(q^2)\, p^\mu p^\beta + 4c_{v2}(q^2)\, p^\mu q^\beta + 4c_{v3}(q^2)\, g^{\mu\beta} \tag{49}$$

Notice that, dotting this with $(4-2\epsilon)p_\beta p_\mu - g_{\mu\beta}\, p^2$ picks out c_{v1} (remember, $p\cdot q = 0$ and $g^{\alpha\beta}g_{\alpha\beta} = d = 4-2\epsilon$).

$$\left[(4-2\epsilon)p_\beta p_\mu - g_{\mu\beta}\, p^2\right] Tr.\left[\Gamma^\mu\gamma^\beta\right] = 4p^4(3-2\epsilon)c_{v1}(q^2) \tag{50}$$

$$Tr.\left[\Gamma^\mu\gamma^\beta\right] = \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{-8ie^2(-1+\epsilon)\left[(m^2-l^2+l\cdot q)g^{\mu\beta} + l^\mu q^\beta + l^\beta(2l^\mu + q^m u)\right]}{(l^2-m^2)((l+q)^2-m_\mu^2)(l+p+\frac{q}{2})^2} \tag{51}$$

Therefore,

$$c_{v1}(0) = \frac{1}{4p^4}\frac{1}{3-2\epsilon}\left[(4-2\epsilon)p_\beta p_\mu - g_{\mu\beta}\, p^2\right]\int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{-8i\left[(m^2-l^2)g^{\mu\beta} + 2l^\beta l^\mu\right]}{(l^2-m^2)^2(l+p)^2} \tag{52}$$

$$= \frac{-4ie^2(1-\epsilon)}{(3-2\epsilon)p^4}\int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{l^2m^2 + 2(-2+\epsilon)(l\cdot p)^2}{(l^2-m^2)(l^2-m_\mu^2)(l+p)^2} \tag{53}$$

$$= -\frac{2ie^2}{9m^4}\left((-6+5\epsilon)\left[\mathcal{I}_{21}^{l^4} - 2\mathcal{I}_{21}^{Q^2l^2} + \mathcal{I}_{21}^{Q^4} + \mathcal{I}_{21}^{l^2}m^2 - 2\mathcal{I}_{21}^{Q^2}m^2 + \mathcal{I}_{21}m^4\right] + 3\epsilon\mathcal{I}_{21}^{l^2}m^2\right) \tag{54}$$

where we have only retained terms to $\mathcal{O}(\epsilon)$.

3.1.3 Extracting $c_{s3}(0)$

In this case, this co-efficient only contributes at $\mathcal{O}(\epsilon)$,

$$Tr.\left[\Gamma^\mu\sigma^{\lambda\rho}\right] = 4(C_s^{\mu\lambda\rho} - C_s^{\mu\rho\lambda}) = 8\epsilon\, m(g^{\mu\rho}q^\lambda - g^{\mu\lambda}q^\rho) \tag{55}$$

Compare this with our general expansion,

$$C_s^{\mu\alpha\beta} = c_{s1}(q^2)\, p^\mu(p^\alpha q^\beta - p^\beta q^\alpha) + c_{s2}(q^2)\,(g^{\mu\alpha}p^\beta - g^{\mu\beta}p^\alpha) + c_{s3}(q^2)\,(g^{\mu\alpha}q^\beta - g^{\mu\beta}q^\alpha) \tag{56}$$

Thus,

$$c_{s3} = -e^2\epsilon\, m \tag{57}$$

3.1.4 Extracting $c_{a4}(0)$

$$Tr.\left[\Gamma^\mu\gamma^\beta\gamma^5\right] = 4C_a^{\mu\beta} = 4c_{a4}(q^2)\,\epsilon^{\mu\alpha\delta\beta}p_\beta q_\delta \tag{58}$$

In our case

$$Tr.\left[\Gamma^\mu\gamma^\beta\gamma^5\right] = 8e^2\epsilon^{\mu\alpha\delta\beta}\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{l_\alpha q_\delta}{(l^2-m^2)^2(l+p)^2} \tag{59}$$

Or,

$$c_{a4}(0)p_\alpha = 2e^2\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{l^\alpha}{(l^2-m^2)^2(l+p)^2} \tag{60}$$

This is simple to evaluate,

$$c_{a4}(0) = \frac{e^2}{p^2}\int\frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{Q^2-l^2-p^2}{(l^2-m^2)^2(l+p)^2} \tag{61}$$

$$= \frac{e^2}{m^2}\left[\mathcal{I}_{21}^{Q^2} - \mathcal{I}_{21}^{l^2} - m^2\mathcal{I}_{21}\right] \tag{62}$$

3.1.5 Collecting terms

Recalling,

$$F_2(0) = -(2mi)\left(-\frac{i}{2}(c_1(0)+mc_{v1}(0)) + 2c_{s3}(0)\right) - im^2c_{a4}(0) \tag{63}$$

we find that,

$$\begin{aligned} F_2(0) = 2ie^2 \Big[&(-6+5\epsilon)\left(\mathcal{I}_{21}^{l^4} - 2\mathcal{I}_{21}^{Q^2l^2} + \mathcal{I}_{21}^{Q^4}\right) \\ &+3\mathcal{I}_{21}^{l^2}m^2 + 8\epsilon\,\mathcal{I}_{21}^{l^2}m^2 + 3\mathcal{I}_{21}^{Q^2}m^2 - 10\epsilon\,\mathcal{I}_{21}^{Q^2}m^2 + 3\mathcal{I}_{21}m^4 + 23\epsilon\,\mathcal{I}_{21}m^4\Big] \end{aligned} \tag{64}$$

We still seem far from the answer. However, we have cast the integrations required in a convenient form, so that we can algebraically reduce them to only 2 or 3 different integrals. The reduction is shown in detail in section 6.3. After this reduction, we find,

$$F_2(0) = 2ie^2\left(-\mathcal{I}_{11} + \frac{1}{3}\mathcal{I}_2 + \frac{2}{3}\mathcal{I}_1\right) + 2ie^2\,\epsilon\left(\mathcal{I}_{11} - \frac{1}{9}\mathcal{I}_2 - \frac{5}{9}\mathcal{I}_1\right) \tag{65}$$

Note that finite terms multiplying ϵ obviously do not contribute, and can be dropped.

Thus we have reduced the calculation to these three integrals. They are all divergent in 4 dimensions. However, we can see that the divergent piece cancels — this is easy to see, since each of them has the same coefficient for the $\frac{1}{\epsilon}$ divergence, which clearly cancels in the first term.

We also see that we have not been carrying the second term needlessly. The $1/\epsilon$ piece does not cancel in the second term, and hence upon multiplying the ϵ out front, gives a finite addition to the first term. Finally, putting together the result,

$$\boxed{\begin{aligned} F_2(0) &= \frac{e^2}{8\pi^2} && (66) \\ &= \frac{\alpha}{2\pi} && (67) \end{aligned}}$$

4 Estimating contributions from heavier states

In principle, given a Lagrangian it is possible to calculate the $g-2$ contributions explicitly. It is much more convenient to be able to estimate a contribution quickly, however. The estimate will of course not get factors of 2 (or even the sign, which in principle is crucial for these calculations), but it can certainly give us an order of magnitude estimate whether it is worth calculating the diagram in detail or not. It can also give us an idea of the relative contribution to the electron and muon $g-2$.

Here, we consider the correction to $g-2$ from Z boson running in the loop instead of a photon. This toy process will hopefully help clarifying the kind of tricks employed in estimation of these diagrams.

Since we want to keep track of dependence on masses, we work in the approximation that the electron/muon are massless, and include their mass terms as perturbations (insertions in Feynman diagrams).

If we write the 4-component Dirac spinor as,

$$\ell = \begin{pmatrix} \ell_L \\ \ell_R \end{pmatrix} \tag{68}$$

, we can see that $\bar{\ell}\sigma^{\mu\nu}\ell$ couples ℓ_L to ℓ_R. This coupling is proportional to the mass for Standard Model leptons. We have our first factor in the estimate: we need one lepton mass insertion in the loop diagram.

Consider next the Z-propagator,

$$\frac{g_{\mu\nu} - \frac{Q_\mu Q_\nu}{m_Z^2}}{Q^2 - m_Z^2} \tag{69}$$

Since we know that the full calculation yield finite answers (i.e. no divergence), we can expand the momentum in the loop in powers of Q^2/m_Z^2. The leading contribution is,

$$\frac{1}{m_Z^2} \tag{70}$$

This is our second factor in the estimate. Since there are no other scales in the problem, this factor of m_Z^2 has to be made up by the mass of the lepton as well.

Thus, so far our estimate of the diagram reads,

$$m_\ell \frac{m_\ell}{m_Z^2} \tag{71}$$

We can put in a factor of $g^2 e/16\pi^2$ to account for the loop-factor and the couplings.

Finally, we inspect figure 5 again. Even though the full calculation was convergent, by removing the Q^2 from the denominator, we have made the integral logarithmically divergent. This is to say that the integral picks up equal contributions between the higher mass scale (m_Z) and the lower mass scale (m_ℓ). Since this is a large hierarchy of scales, the logs involved can be numerically significant. Therefore, our final estimate is,

$$m_\ell \frac{m_\ell^2}{m_Z^2} \frac{g^2 e}{16\pi^2} \log\left[\frac{m_Z^2}{m_\ell^2}\right] \tag{72}$$

to be compared with the tree-level estimate for the same operator: $\sim m_\ell$ (due to the mass insertion).

Therefore, $a_\ell \sim (g_\ell - 2)$ is approximately,

$$a_\ell \sim \frac{m_\ell^2}{m_Z^2} \frac{g^2 e}{16pi^2} \log\left[\frac{m_Z^2}{m_\ell^2}\right] \tag{73}$$

In general the contribution of heavy states appears as the following operator in the low energy theory,

$$\frac{m_\ell}{\Lambda^2} \bar{\ell} \sigma^{\mu\nu} \ell F_{\mu\nu} \tag{74}$$

where F is the field strength tensor, and we have inserted a lepton mass term explicitly as discussed above.

We know that the operator

$$\frac{1}{2m_\ell}\bar{\ell}\sigma^{\mu\nu}\ell F_{\mu\nu} \tag{75}$$

produces $a_\ell \sim \alpha/\pi$. Therefore, generically the higher dimensional operator contribution is,

$$\frac{m_\ell^2}{\Lambda^2}\frac{1}{2m}\bar{\ell}\sigma^{\mu\nu}\ell F_{\mu\nu} \approx \frac{\alpha}{\pi}\frac{m_\ell^2}{\Lambda^2} \tag{76}$$

It is clear that the electroweak (and by the same token, new physics) contributions to the electron $g-2$ are severely suppressed due to its small mass. The τ lifetime is short, so the measurement of $g_\tau - 2$ is difficult. Consequently, the best place to look for new physics effects is the measurement of $g_\mu - 2$. This is why the measurement of the muon $g-2$ is interesting.

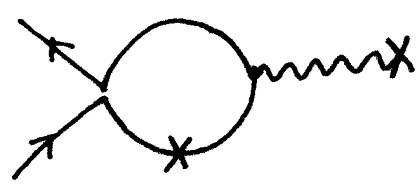

Figure 5: Feynman diagram for $g-2$ of the electron/muon with the Z boson "integrated out".

5 Numerical values

We report some numerical values taken from [1].

5.1 Electron $g-2$

The electron $g-2$ is measured to be

$$(a_e)_{exp} = 1.15965218073(28) \times 10^{-3} \tag{77}$$

$$(a_e)_{SM} = 1.15965218188(78) \times 10^{-3} \tag{78}$$

where the prediction uncertainty being dominated by the measurement uncertainty for α.

5.2 Muon $g - 2$

The experimentally measured value of a_μ is

$$(a_\mu)_{\exp} = 11659208.9(5.4)(3.3) \times 10^{-10} \tag{79}$$

Various contributions to the theoretical calculation can be broken down as,

$$(a_\mu)_{SM} = (a_\mu)_{QED} + (a_\mu)_{EW} + (a_\mu)_{Had} \tag{80}$$

where,

$$(a_\mu)_{QED} = 116584718.09(0.15) \times 10^{-11} \tag{81}$$

using the value of α from a_e (i.e. assuming no significant new contributions to a_e.

$$(a_\mu)_{EW} = 154(1)(2) \times 10^{-11} \tag{82}$$

$$(a_\mu)_{Had}[LO] = 6923(42)(3) \times 10^{-11} \tag{83}$$

$$(a_\mu)_{Had}[NLO] = 7(26) \times 10^{-11} \tag{84}$$

Combining,

$$(a_\mu)_{SM} = 116591802(2)(42)(26) \times 10^{-11} \tag{85}$$

The errors are due to the electroweak, lowest-order hadronic, and higher-order hadronic contributions. There is a discrepancy between this prediction and the measured value,

$$\Delta a_\mu = (a_\mu)_{\exp} - (a_\mu)_{SM} = 287(63)(49) \times 10^{-11} \tag{86}$$

If this discrepancy becomes statistically more significant, this might be a signal of new physics.

6 Useful identities

6.1 Fermion spinors

These are the momentum space Dirac equation,

$$\not{p}u(p) = mu(p) \tag{87}$$
$$\bar{u}(p)\not{p} = m\bar{u}(p) \tag{88}$$

The plane wave solutions have spinor coefficients, $u(p)$ given by,

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} \tag{89}$$

Note that we have chosen the outgoing momentum, $p' = p + \frac{1}{2}q$, and the incoming momentum, $k = p - \frac{1}{2}q$.

$$\bar{u}(p')\not{p}'u(k) = \bar{u}(p')\left(\not{p} + \frac{1}{2}\not{q}\right)u(k) = m\bar{u}(p')\bar{u}(k) \tag{90}$$
$$\bar{u}(p')\not{k}u(k) = \bar{u}(p')\left(\not{p} - \frac{1}{2}\not{q}\right)u(k) = m\bar{u}(p')\bar{u}(k) \tag{91}$$

Thus,

$$\bar{u}(p')\not{p}u(k) = m\bar{u}(p')\bar{u}(k) \tag{92}$$
$$\bar{u}(p')\not{q}u(k) = 0 \tag{93}$$

Also note that since $p'^2 = k^2 = m^2$, $p \cdot q = 0$.

Gordon's identity is a particularly useful identity. It can be used to reduce the spinor structures we need to worry about.

$$\bar{u}(p')\gamma_\mu u(k) = \bar{u}(p')\left[\frac{(k+p')_\mu}{2m} + \frac{i\sigma_{\mu\nu}(p'-k)_\mu}{2m}\right]u(k) \tag{94}$$

6.2 Gamma matrices

The Dirac gamma matrices satisfy,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu} \tag{95}$$

In the chiral basis,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{96}$$

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \tag{97}$$

where $\sigma^\mu = (1, \vec{\sigma}), \bar{\sigma}^\mu = (1, -\vec{\sigma})$.

Traces and contraction identities (in d-dimensions) can be found in the appendices of Peskin and Schroeder [3].

6.3 Loop integrals

We use QCDloop developed here at Fermilab [2] for our loop calculation. The final result we want is finite (i.e. it contains no divergence), but to get to that result we want a carefully regulated integral, and see that divergences cancel.

For scalar integrals, analytic expressions are available. We only need expressions for "bubble" and "triangle" diagrams (diagrams with 2 and 3 propagator terms in the denominator).

The integrals are notated as,

$$\mathcal{I}_{pq}^{\mathcal{N}} = \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{\mathcal{N}}{[l^2 - m^2]^p \, [(l+p)^2]^q} \tag{98}$$

6.3.1 Reduction of integrals to scalar integrals

It is pretty simple to reduce all integrals we encounter to scalar integrals,

$$\begin{aligned}
\mathcal{I}_{21}^{l^4} &= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{l^4}{(l^2 - m^2)^2 Q^2} && (99) \\
&= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{l^2(l^2 - m^2) + l^2 m^2}{(l^2 - m^2)^2 Q^2} && (100) \\
&= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{l^2}{(l^2 - m^2) Q^2} + \mathcal{I}_{21}^{l^4} && (101) \\
&= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{Q^2} + m^2 \mathcal{I}_{11} + m^2 \mathcal{I}_{21}^{l^2} && (102) \\
&= \mathcal{I}_{01} + m^2 \mathcal{I}_{11} + m^2 \mathcal{I}_{21}^{l^2} && (103)
\end{aligned}$$

Note also that,

$$\mathcal{I}_{21}^{Q^4} = \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{Q^4}{(l^2-m^2)^2 Q^2} \tag{104}$$
$$= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{Q^2}{(l^2-m^2)^2} \tag{105}$$
$$= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{l^2+m^2+2l\cdot p}{(l^2-m^2)^2} \tag{106}$$
$$= \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{l^2+m^2}{(l^2-m^2)^2} \tag{107}$$
$$= 2m^2\mathcal{I}_2 + \mathcal{I}_1 \tag{108}$$

where we have used the property the $l \cdot p$ is odd, while the rest of the integrand is even in l.

In summary,

$$\mathcal{I}_{21}^{l2} = \mathcal{I}_{11} + m^2\mathcal{I}_{21} \tag{109}$$
$$\mathcal{I}_{21}^{Q^2} = \mathcal{I}_2 \tag{110}$$
$$\mathcal{I}_{21}^{Q^2 l^2} = m^2\mathcal{I}_{21}^{Q^2} + \mathcal{I}_1 \tag{111}$$
$$\mathcal{I}_{21}^{l^4} = m^2(\mathcal{I}_{11} + \mathcal{I}_{21}^{l^2}) + \mathcal{I}_{01} \tag{112}$$
$$\mathcal{I}_{21}^{Q^4} = 2m^2\mathcal{I}_2 + \mathcal{I}_1 \tag{113}$$

6.4 Integrals required

These are the scalar integrals needed. All integrals are performed in $4-2\epsilon$ dimensions [2]. As before, the scalar integrals are notated as,

$$\mathcal{I}_{pq} = \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{[l^2-m^2]^p\,[(l+p)^2]^q} \tag{114}$$

Thus,

$$\mathcal{I}_{11} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} + 2 - \log(m^2) - \gamma + \log(4\pi)\right) \tag{115}$$

$$\mathcal{I}_{2} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} - \log(m^2) - \gamma + \log(4\pi)\right) \tag{116}$$

$$\mathcal{I}_1 = \frac{im^2}{16\pi^2}\left(\frac{1}{\epsilon} + 1 - \log(m^2) - \gamma + \log(4\pi)\right) \tag{117}$$

The Euler-Mascheroni constant γ drops out of physical answers, as always.

6.5 Discrete symmetries of QED

The C, P and T transformation properties of various fermion bilinears are tabulated in Peskin and Schroeder[3], Chapter 3. The classical electromagnetic field $\mathtt{A}^\mu$ transforms exactly like $\bar{\psi}\gamma^\mu\psi$, and therefore in QED the interaction $\bar{\psi}\gamma^\mu\psi\mathtt{A}_\mu$ preserves all three symmetries above separately. Consequently, QED loop corrections only produce terms respecting these symmetries, a fact we have used to restrict the terms we investigate.

7 Co-efficient reduction

We use Gordon's identity and the Dirac equation to reduce the co-efficients from their most general form into F_1 and F_2 form factors. Since we have stated (without proof) that loop contributions to F_1 cancel, we are only interested in co-efficients which contribute to the F_2 form factor.

The co-efficients which do contribute are,

$$c_1: \quad c_1 p^\mu \bar{u}(p')u(k) = c_1(m\bar{u}\gamma^\mu u - \frac{i}{2}\bar{u}\sigma^{\mu\nu}q_\nu u) \tag{118}$$

$$c_{v1}: \quad c_{v1}\bar{u}(p')p^\mu \not{p} u(k) = c_{v1}m\bar{u}(p')p^\mu u(k) = c_{v1}m(m\bar{u}\gamma^\mu u - \frac{i}{2}\bar{u}\sigma^{\mu\nu}q_\nu u) \tag{119}$$

$$c_{s3}: \quad c_{s3}\bar{u}(p')\sigma_{\mu\beta}u(k)q_\beta \tag{120}$$

$$c_{a4}: \quad c_{a4}\bar{u}(p')\gamma_\beta\gamma^5 u(k)p_\alpha q_\delta \epsilon^{\mu\alpha\delta\beta} = mc_{a4}\bar{u}(p')\sigma^{\mu\beta}q_\beta u(k) \tag{121}$$

The simplification for the co-efficient c_{a4} requires a bit of algebra,

$$\begin{aligned}
c_{a4}\bar{u}(p')\gamma_\beta\gamma^5 u(k)p_\alpha q_\delta \epsilon^{\mu\alpha\delta\beta} &= \frac{1}{m}c_{a4}\bar{u}(p')\not{p}'\gamma_\beta\gamma^5 u(k)p_\alpha q_\delta\epsilon^{\mu\alpha\delta\beta} && (122)\\
&= \frac{1}{m}c_{a4}\bar{u}(p')\gamma_\rho\gamma_\beta\gamma^5 u(k)p_\alpha q_\delta(p+\frac{q}{2})^\rho\epsilon^{\mu\alpha\delta\beta} && (123)\\
&= \frac{1}{m}c_{a4}\bar{u}(p')(g_{\rho\beta} - i\sigma_{\rho\beta})\gamma^5 u(k)p_\alpha q_\delta(p+\frac{q}{2})^\rho\epsilon^{\mu\alpha\delta\beta} && (124)\\
&= \frac{1}{2m}c_{a4}\bar{u}(p')\sigma^{\lambda\kappa}u(k)p_\alpha q_\delta(p+\frac{q}{2})^\rho\epsilon_{\lambda\kappa\rho\beta}\epsilon^{\mu\alpha\delta\beta} && (125)
\end{aligned}$$

where we have used $\sigma^{\mu\nu}\gamma^5 = \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}$. Continuing,

$$\frac{1}{2m}c_{a4}\bar{u}(p')(\sigma_{\rho\beta})u(k)p_\alpha q_\delta(p+\frac{q}{2})^\rho\epsilon^{\lambda\kappa\rho\beta}\epsilon^{\mu\alpha\delta\beta} = \frac{1}{m}c_{a4}p^2\bar{u}(p')(\sigma_{\mu\beta})q_\beta u(k) \tag{126}$$

$$= mc_{a4}\bar{u}(p')\sigma^{\mu\beta}q_\beta u(k) \tag{127}$$

The other co-efficients are irrelevant for this calculation. Some give 0 (identically, or at $q^2 = 0$),

$$c_{v2}\bar{u}(p')p^\mu \not{q} u(k) = 0 \tag{128}$$

$$c_{s1}\bar{u}(p')\sigma_{\alpha\beta}u(k)p^\alpha q^\beta = \frac{i}{2}c_{s1}\bar{u}(p')\left[\not{p}\not{q} - \not{q}\not{p}\right]u(k) \tag{129}$$

$$= \frac{i}{2}c_{s1}\bar{u}(p')\left[(m - \frac{1}{2}\not{q})\not{q} - \not{q}(m + \frac{1}{2}\not{q})\right]u(k) \sim q^2\bar{u}u \tag{130}$$

and others are prohibited by gauge invariance and other symmetries.

$$c_{s2}\bar{u}(p')\sigma_{\mu\beta}u(k)p_\beta = \frac{i}{2}c_{s2}\bar{u}(p')\left[\gamma^\mu\not{p} - \not{p}\gamma^\mu\right]u(k) \tag{131}$$

$$= \frac{i}{2}c_{s2}\bar{u}(p')\left[\gamma^\mu(m + \frac{1}{2}\not{q}) - (m - \frac{1}{2}\not{q})\gamma^\mu\right]u(k) \sim \bar{u}q^\mu u \tag{132}$$

Some co-efficients also only contribute to the F_1 form factor,

$$c_{v3}\bar{u}(p')\gamma^\mu u(k). \tag{133}$$

Finally, the contributions to the F_2 form factor can be summarized as,

$$F_2(0) = -(2mi)\left(-\frac{i}{2}(c_1(0) + mc_{v1}(0)) + 2c_{s3}(0)\right) - im^2c_{a4}(0) \tag{134}$$

References

[1] J. Beringer et al. Review of Particle Physics (RPP). *Phys.Rev.*, D86:010001, 2012.

[2] R. Keith Ellis and Giulia Zanderighi. Scalar one-loop integrals for QCD. *JHEP*, 0802:002, 2008.

[3] Michael E. Peskin and Daniel V. Schroeder. An Introduction to quantum field theory. 1995.

[4] Julian S. Schwinger. On Quantum electrodynamics and the magnetic moment of the electron. *Phys.Rev.*, 73:416–417, 1948.

[5] Julian S. Schwinger. Quantum electrodynamics. III: The electromagnetic properties of the electron: Radiative corrections to scattering. *Phys.Rev.*, 76:790–817, 1949.