The Most Compact Formulas for Neutrino Oscillation Probabilities in Matter

Hisakazu Minakata
U. Sao Paulo

Sorry for theorist contamination to the experimental session

- Does our formula useful to experimental analysis?
- Simple answer: NO
- Yet, its simplicity appeals to you, as a human being..

In vacuum probability formulas are simple

In vacuum P is simple

$$P(\nu_\beta \to \nu_\alpha) = -4 \sum_{j>i} \mathrm{Re}[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}] \sin^2 \left(\frac{\Delta m^2_{ji} L}{4E} \right)$$
$$-2 \sum_{j>i} \mathrm{Im}[U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}] \sin^1 \left(\frac{\Delta m^2_{ji} L}{4E} \right)$$

$$U_{MNS} = U_{23} U_{13} U_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13} e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13} \end{bmatrix},$$

In matter probability formulas are complicated

Kimura-Takamura-Yokomakura formula

PLB, PRD 2002

$$P(\nu_e \to \nu_\mu) = \tilde{A}_{e\mu} \cos\delta + \tilde{B} \sin\delta + \tilde{C}_{e\mu}$$

$$\tilde{\Delta}'_{ij} \equiv \frac{\tilde{\Delta}_{ij} L}{4E}.$$

$$\tilde{A}_{e\mu} = \sum_{(ijk)}^{\text{cyclic}} \frac{-8[J_r \Delta_{21}\Delta_{31}\lambda_k(\lambda_k - \Delta_{31}) + (\tilde{A}_{e\mu})_k]}{\tilde{\Delta}^2_{jk}\tilde{\Delta}^2_{ki}} \times \cos\tilde{\Delta}'_{ij} \sin\tilde{\Delta}'_{jk} \sin\tilde{\Delta}'_{ki},$$

$$\tilde{B} = \frac{8J_r \Delta_{12}\Delta_{23}\Delta_{31}}{\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} \sin\tilde{\Delta}'_{12} \sin\tilde{\Delta}'_{23} \sin\tilde{\Delta}'_{31},$$

$$\tilde{C}_{e\mu} = \sum_{(ij)}^{\text{cyclic}} \frac{-4[\Delta^2_{31} s^2_{13} s^2_{23} c^2_{13} \lambda_i \lambda_j + (\tilde{C}_{e\mu})_{ij}]}{\tilde{\Delta}_{ij}\tilde{\Delta}_{12}\tilde{\Delta}_{23}\tilde{\Delta}_{31}} \sin^2\tilde{\Delta}'_{ij}.$$

$$(\tilde{A}_{e\mu})_k = \Delta^2_{21} J_r \times [\Delta_{31}\lambda_k(c^2_{12} - s^2_{12}) + \lambda^2_k s^2_{12} - \Delta^2_{31} c^2_{12}], \quad \text{(A1)}$$

$$\begin{aligned}(\tilde{C}_{e\mu})_{ij} = {} & \Delta_{21} s^2_{13} \times [\Delta_{31}\{-\lambda_i(\lambda_j s^2_{12} + \Delta_{31} c^2_{12}) \\ & - \lambda_j(\lambda_i s^2_{12} + \Delta_{31} c^2_{12})\} s^2_{23} c^2_{13}] \\ & + \Delta^2_{21}[(\lambda_i - \Delta_{31})(\lambda_j - \Delta_{31}) s^2_{12} c^2_{12} c^2_{23} c^2_{13}] \\ & + \Delta^2_{21} s^2_{13}[(\lambda_i s^2_{12} + \Delta_{31} c^2_{12})(\lambda_j s^2_{12} + \Delta_{31} c^2_{12}) s^2_{23} c^2_{13}],\end{aligned}$$

$$\lambda_1 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t}[u + \sqrt{3(1-u^2)}],$$

$$\lambda_2 = \frac{1}{3}s - \frac{1}{3}\sqrt{s^2 - 3t}[u - \sqrt{3(1-u^2)}],$$

$$\lambda_3 = \frac{1}{3}s + \frac{2}{3}u\sqrt{s^2 - 3t},$$

$$s = \Delta_{21} + \Delta_{31} + a,$$

$$t = \Delta_{21}\Delta_{31} + a[\Delta_{21}(1 - s^2_{12}c^2_{13}) + \Delta_{31}(1 - s^2_{13})],$$

$$u = \cos\left[\frac{1}{3}\cos^{-1}\left(\frac{2s^3 - 9st + 27a\Delta_{21}\Delta_{31}c^2_{12}c^2_{13}}{2(s^2 - 3t)^{3/2}}\right)\right],$$

We need more human approach: Perturbation theory

People used 2 expansion parameters

- $\varepsilon = \Delta m_{21}^2 / \Delta m_{31}^2 = 0.031$
- $\sin\theta_{13} = 0.146$
- But $\sin^2\theta_{13} \sim 0.021\,\varepsilon$ (s_{13}~\sqrt{ε})

$$P_{e\mu}^{(1)} = 4s_{23}^2 s_{13}^2 \frac{1}{(1-r_A)^2} \sin^2 \frac{(1-r_A)\Delta L}{2},$$

$$P_{e\mu}^{(3/2)} = 8J_r \frac{r_\Delta}{r_A(1-r_A)} \cos\left(\delta - \frac{\Delta L}{2}\right) \sin\frac{r_A \Delta L}{2} \sin\frac{(1-r_A)\Delta L}{2},$$

$$P_{e\mu}^{(2)} = 4c_{23}^2 c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A}\right)^2 \sin^2 \frac{r_A \Delta L}{2}$$

$D = \Delta m^2_{31}/2E$
$r_A = a/\Delta m^2_{31}$
$(a = 2\sqrt{2} G_F N_e E)$

Cervera et al formula

$$-4s_{23}^2 \left[s_{13}^4 \frac{(1+r_A)^2}{(1-r_A)^4} - 2s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} \right] \sin^2 \frac{(1-r_A)\Delta L}{2}$$

$$+2s_{23}^2 \left[2s_{13}^4 \frac{r_A}{(1-r_A)^3} - s_{12}^2 s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} \right] (\Delta L) \sin(1-r_A)\Delta L.$$

K. Asano, HM, JHEP 2011 (before T2K, Daya Bay!)

$s_{13}=\sqrt{\varepsilon}$ vs. $s_{13}=\varepsilon$ perturbation theories

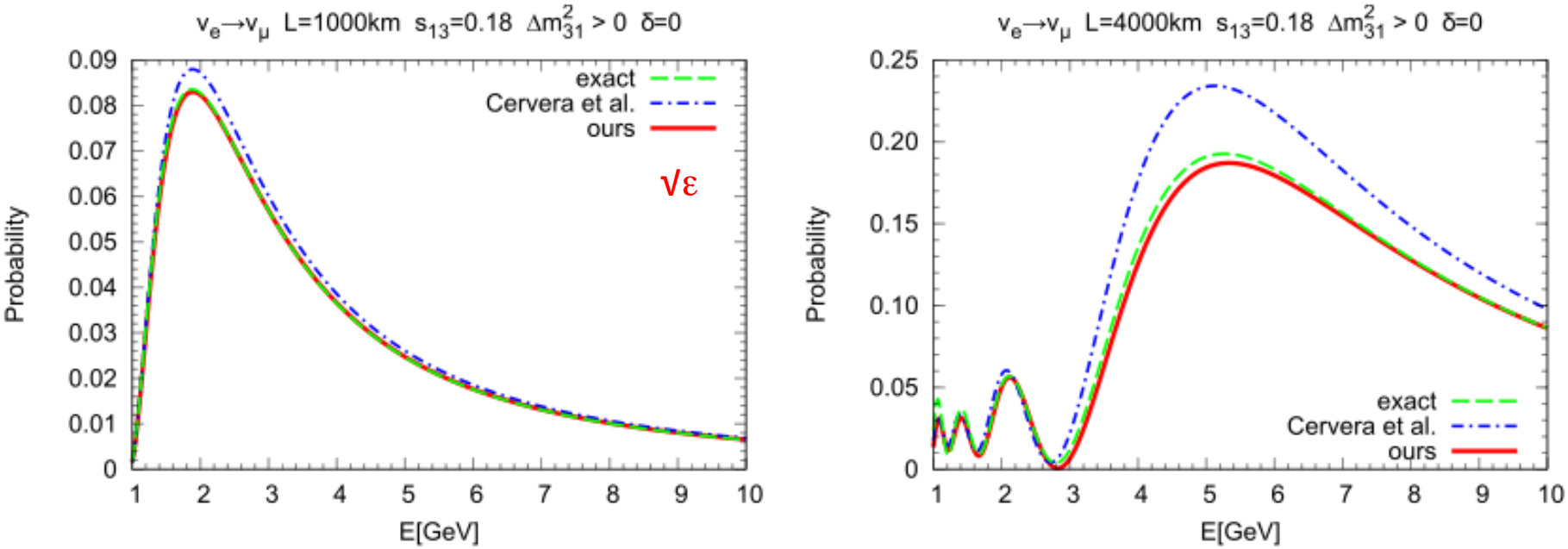


Figure 3. Comparison between the exact oscillation probability $P(\nu_e \to \nu_\mu)$ computed numerically as a function of energy (green dashed line), the one calculated by the Cervera *et al.* formula (blue dash-dotted line), and with our formula with large θ_{13} corrections (red solid line). The left and the right panels are for baselines $L = 1000$ km and $L = 4000$ km for which the matter density is taken as 2.8 g/cm^3 and 3.6 g/cm^3, respectively. θ_{13} is taken as $\sin\theta_{13} = 0.18$ and $\delta = 0$. The values of the remaining mixing parameters are the same as given in the caption of figure 1.

Nature is simple: unique expansion parameter ε!

$\varepsilon = \Delta m^2_{21} / \Delta m^2_{31} \sim 0.03$ as a unique expansion parameter

- J. Arafune, J. Sato, PRD 1997
- A. Cervera et al. NPB 2000
- M. Freund, PRD 2001
- E. K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson and T. Schwetz, JHEP 2004
- May be many more refs.

Simple and Compact formula for oscillation probability in matter

HM-S.J.Parke, arXiv: 1505.01826

What we did

- Use unique expansion parameter

$$\varepsilon = \Delta m^2_{21} / \Delta m^2_{31} \sim 0.03$$

- We restrict to first order in ε ($\varepsilon^2 = 10^{-3} << 1$)
- By a "magic" we make our formula extremely simple and compact:

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi \ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E}$$

Contains all order effect of s_{13} and matter

In a nutshell, why became so simple?

Simple expansion by ε

$$\Delta \equiv \frac{\Delta m_{31}^2 L}{4E}$$

$$\begin{aligned} P_{ee} &= 1 - \sin^2 2\phi \sin^2 \frac{(\lambda_+ - \lambda_-)x}{2} \\ &+ s_{12}^2 \Delta_{21} \sin^2 2\phi \cos 2(\phi - \theta_{13}) \sin(\lambda_+ - \lambda_-)x \\ &- 2 s_{12}^2 \Delta_{21} \sin 2\phi \cos 2\phi \sin 2(\phi - \theta_{13}) \frac{1}{\frac{(\lambda_+ - \lambda_-)x}{2}} \sin^2 \frac{(\lambda_+ - \lambda_-)x}{2} \end{aligned}$$

By "magic" or redefining the quantities such that

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi \ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E}$$

Can you do it systematically ?

$\varepsilon = \Delta m^2_{21} / \Delta m^2_{31}$

Helio-perturbation theory

More precisely,
helio-to-terrestrial ratio
perturbation theory

What people usually do (expansion by ε)

$$\boxed{i\frac{d}{dx}\nu = H\nu}$$

$$H = \frac{1}{2E}\left\{U_{23}U_{13}U_{12}\begin{bmatrix}0 & 0 & 0\\ 0 & \Delta m^2_{21} & 0\\ 0 & 0 & \Delta m^2_{31}\end{bmatrix}U^\dagger_{12}U^\dagger_{13}U^\dagger_{23} + \begin{bmatrix}a(x) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right\}$$

Tilde basis

$$\tilde{H} = U^\dagger_{23}HU_{23} = \frac{1}{2E}\left\{U_{13}U_{12}\begin{bmatrix}0 & 0 & 0\\ 0 & \Delta m^2_{21} & 0\\ 0 & 0 & \Delta m^2_{31}\end{bmatrix}U^\dagger_{12}U^\dagger_{13} + \begin{bmatrix}a(x) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\right\}$$

$$\boxed{\tilde{H}(x) = \frac{\Delta m^2_{31}}{2E}\left\{\begin{bmatrix}\frac{a(x)}{\Delta m^2_{31}} + s^2_{13} & 0 & c_{13}s_{13}\\ 0 & 0 & 0\\ c_{13}s_{13} & 0 & c^2_{13}\end{bmatrix} + \frac{\Delta m^2_{21}}{\Delta m^2_{31}}\begin{bmatrix}s^2_{12}c^2_{13} & c_{12}s_{12}c_{13} & -s^2_{12}c_{13}s_{13}\\ c_{12}s_{12}c_{13} & c^2_{12} & -c_{12}s_{12}s_{13}\\ -s^2_{12}c_{13}s_{13} & -c_{12}s_{12}s_{13} & s^2_{12}s^2_{13}\end{bmatrix}\right\}}$$

Conventional way:
Zeroth order, 1st order

$$U = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{-i\delta}\end{bmatrix}U_{\rm PDG}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{i\delta}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0\\ 0 & c_{23} & s_{23}e^{i\delta}\\ 0 & -s_{23}e^{-i\delta} & c_{23}\end{bmatrix}\begin{bmatrix}c_{13} & 0 & s_{13}\\ 0 & 1 & 0\\ -s_{13} & 0 & c_{13}\end{bmatrix}\begin{bmatrix}c_{12} & s_{12} & 0\\ -s_{12} & c_{12} & 0\\ 0 & 0 & 1\end{bmatrix} \equiv U_{23}U_{13}U_{12}$$

What we do: Renormalized helio-perturbation theory

$\tilde{H}(x) = \tilde{H}_0(x) + \tilde{H}_1(x)$:

$\varepsilon = \Delta m^2_{21}/\Delta m^2_{\text{ren}}$

$$\tilde{H}_0(x) = \frac{\Delta m^2_{\text{ren}}}{2E}\left\{\begin{bmatrix} \frac{a(x)}{\Delta m^2_{\text{ren}}} + s^2_{13} & 0 & c_{13}s_{13} \\ 0 & 0 & 0 \\ c_{13}s_{13} & 0 & c^2_{13} \end{bmatrix} + \epsilon \begin{bmatrix} s^2_{12} & 0 & 0 \\ 0 & c^2_{12} & 0 \\ 0 & 0 & s^2_{12} \end{bmatrix}\right\}$$

$$\tilde{H}_1(x) = \epsilon c_{12}s_{12}\frac{\Delta m^2_{\text{ren}}}{2E}\begin{bmatrix} 0 & c_{13} & 0 \\ c_{13} & 0 & -s_{13} \\ 0 & -s_{13} & 0 \end{bmatrix}$$

$$\Delta m^2_{\text{ren}} \equiv \Delta m^2_{31} - s^2_{12}\Delta m^2_{21}$$

$= \Delta m^2_{ee}$

NOTAION ISSUE

[3] The authors respectfully disagree with each other on this point.

THAT IS ALL YOU NEED TO KNOW. JUST DO IT !

Diagonalization of tilde{H} → Hat basis

$$\hat{H}_0 = U_\phi^\dagger \tilde{H}_0 U_\phi, \qquad \hat{H}_1 = U_\phi^\dagger \tilde{H}_1 U_\phi$$

$$U_\phi = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$\cos 2\phi = \frac{\Delta m^2_{\rm ren}\cos 2\theta_{13} - a}{\lambda_+ - \lambda_-},$$

$$\sin 2\phi = \frac{\Delta m^2_{\rm ren}\sin 2\theta_{13}}{\lambda_+ - \lambda_-}.$$

$$\hat{H}_0 = \frac{1}{2E}\begin{bmatrix} \lambda_- & 0 & 0 \\ 0 & \lambda_0 & 0 \\ 0 & 0 & \lambda_+ \end{bmatrix}$$

$$\hat{H}_1 = U_\phi^\dagger \tilde{H}_1 U_\phi$$

$$= \epsilon c_{12} s_{12} \frac{\Delta m^2_{\rm ren}}{2E}\begin{bmatrix} 0 & \cos(\phi-\theta_{13}) & 0 \\ \cos(\phi-\theta_{13}) & 0 & \sin(\phi-\theta_{13}) \\ 0 & \sin(\phi-\theta_{13}) & 0 \end{bmatrix}$$

$$\lambda_- = \frac{1}{2}\left[(\Delta m^2_{\rm ren} + a) - {\rm sign}(\Delta m^2_{\rm ren})\sqrt{(\Delta m^2_{\rm ren} - a)^2 + 4 s_{13}^2 a \Delta m^2_{\rm ren}}\right] + \epsilon \Delta m^2_{\rm ren} s_{12}^2,$$

$$\lambda_0 = c_{12}^2\, \epsilon\, \Delta m^2_{\rm ren},$$

$$\lambda_+ = \frac{1}{2}\left[(\Delta m^2_{\rm ren} + a) + {\rm sign}(\Delta m^2_{\rm ren})\sqrt{(\Delta m^2_{\rm ren} - a)^2 + 4 s_{13}^2 a \Delta m^2_{\rm ren}}\right] + \epsilon \Delta m^2_{\rm ren} s_{12}^2.$$

There are two ways to compute P: V-matrix & S-matrix methods

Agree nontrivially !

S-matrix method: Doing perturbation theory with hat basis

$$S(L) = U_{23}\tilde{S}(L)U_{23}^\dagger = U_{23}U_\phi\hat{S}(L)U_\phi^\dagger U_{23}^\dagger$$

$$\tilde{S}(L) = T\exp\left[-i\int_0^L dx\tilde{H}(x)\right]$$

Hat-S matrix is computed as

$$\Omega(L) = e^{i\hat{H}_0 L}\hat{S}(L).$$

$$i\frac{d}{dx}\Omega(x) = \check{H}_1\Omega(x)$$

$$\check{H}_1 \equiv e^{i\hat{H}_0 x}\hat{H}_1 e^{-i\hat{H}_0 x}$$

$$\Omega(L) = 1 + (-i)\int_0^L dx\check{H}_1(x) + \mathcal{O}(\epsilon^2).$$

Then, finally

$$S(L) = U_{23}U_\phi e^{-i\hat{H}_0 L}\Omega(L)U_\phi^\dagger U_{23}^\dagger$$

V matrix method

$$P(\nu_\beta \to \nu_\alpha) = \delta_{\alpha\beta} - 4\sum_{j>i} \mathrm{Re}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}]\sin^2\frac{(\lambda_{mj}-\lambda_{mi})x}{4E} - 2\sum_{j>i}\mathrm{Im}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}]\sin\frac{(\lambda_{mj}-\lambda_{mi})x}{2E}$$

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = V\begin{pmatrix}\nu_-\\ \nu_0\\ \nu_+\end{pmatrix}$$

mass eigenstate in matter to O(ε)

Zeroth order=basis of perturbation theory

$$\nu_\alpha = (U_{23}U_\phi)_{\alpha i}\hat{\nu}_i^{(0)}$$

1st order correction

$$\nu_{mi} = \hat{\nu}_i^{(0)} + \hat{\nu}_i^{(1)}: \qquad \hat{\nu}_i^{(1)} = \sum_{j\neq i}\frac{(\hat{H}_1)_{ji}}{\lambda_i-\lambda_j}\hat{\nu}_j^{(0)}$$

$$\nu_{m-} = \hat{\nu}_-^{(0)} + \epsilon\Delta m^2_{\mathrm{ren}}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0}\hat{\nu}_0^{(0)},$$

$$\nu_{m0} = \hat{\nu}_0^{(0)} - \epsilon\Delta m^2_{\mathrm{ren}}\left[\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0}\hat{\nu}_-^{(0)} + \frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0}\hat{\nu}_+^{(0)}\right]$$

$$\nu_{m+} = \hat{\nu}_+^{(0)} + \epsilon\Delta m^2_{\mathrm{ren}}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0}\hat{\nu}_0^{(0)}.$$

Inverting ν_{mi} = [...] $\nu(\text{hat})_l$ to obtain

$$\begin{pmatrix} \hat{\nu}_-^{(0)} \\ \hat{\nu}_0^{(0)} \\ \hat{\nu}_+^{(0)} \end{pmatrix} = V^{(1)} \begin{pmatrix} \nu_{m-} \\ \nu_{m0} \\ \nu_{m+} \end{pmatrix}$$

$$V^{(1)} = \begin{bmatrix} 1 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 0 \\ \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 1 & \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} \\ 0 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} & 1 \end{bmatrix}$$

Then, inserting this to $\nu_\alpha = (U_{23}U_\phi)_{\alpha i}\hat{\nu}_i^{(0)}$

$$V = U_{23}U_\phi \begin{pmatrix} 1 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 0 \\ \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 1 & \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} \\ 0 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} & 1 \end{pmatrix}$$

V matrix result

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_- \\ \nu_0 \\ \nu_+ \end{pmatrix}$$

Mass eigenstate in matter to O(ε)

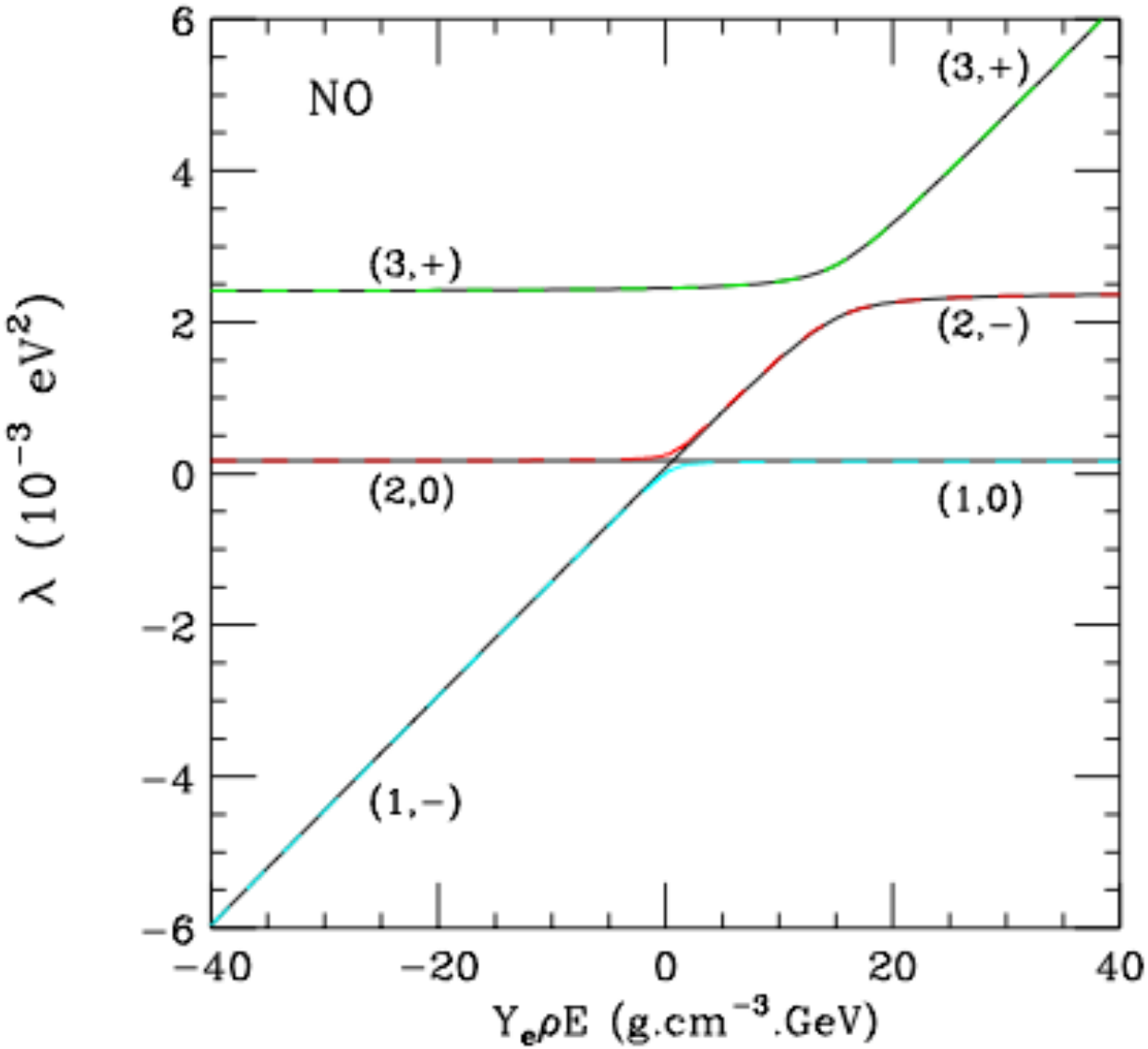
$$V = U_{23}U_\phi \begin{pmatrix} 1 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 0 \\ \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0} & 1 & \epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} \\ 0 & -\epsilon\Delta m^2_{\rm ren}\frac{c_{12}s_{12}s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0} & 1 \end{pmatrix}$$

$$= \begin{bmatrix} c_\phi & 0 & s_\phi \\ -s_\phi s_{23}e^{i\delta} & c_{23} & c_\phi s_{23}e^{i\delta} \\ -s_\phi c_{23} & -s_{23}e^{-i\delta} & c_\phi c_{23} \end{bmatrix}$$

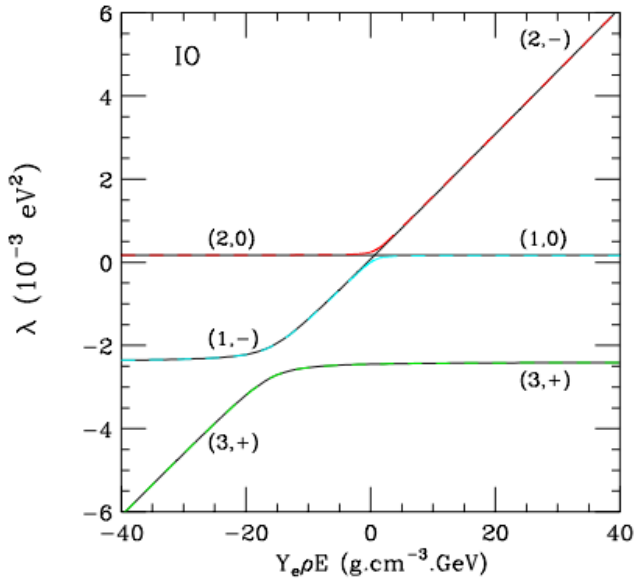
This zero is important ! → No ν_e in "0" state

$$+\epsilon c_{12}s_{12}\Delta m^2_{\rm ren}\left\{\frac{c_{(\phi-\theta_{13})}}{\lambda_- - \lambda_0}U_{23}U_\phi\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{s_{(\phi-\theta_{13})}}{\lambda_+ - \lambda_0}U_{23}U_\phi\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\right\}$$

3 ν level crossing diagram



Black solid: ours (-0+)
Colored dashed: = exact (123)



Then, use the general formula

$$P(\nu_\beta \to \nu_\alpha) = \left| \sum_i V_{\alpha i} V^*_{\beta i}\, e^{-i\frac{\lambda_i L}{2E}} \right|^2$$

$$= \delta_{\alpha\beta} - 4\sum_{j>i} \mathrm{Re}[V_{\alpha i} V^*_{\beta i} V^*_{\alpha j} V_{\beta j}] \sin^2 \frac{(\lambda_j - \lambda_i)L}{4E} - 2\sum_{j>i} \mathrm{Im}[V_{\alpha i} V^*_{\beta i} V^*_{\alpha j} V_{\beta j}] \sin \frac{(\lambda_j - \lambda_i)L}{2E}$$

$$P(\nu_e \to \nu_e) = 1 - 4|V_{e+}|^2|V_{e-}|^2 \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E}$$

$$- 4|V_{e+}|^2|V_{e0}|^2 \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E}$$

$$- 4|V_{e0}|^2|V_{e-}|^2 \sin^2 \frac{(\lambda_0 - \lambda_-)L}{4E}$$

O(ε^2)

$$\boxed{P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi\ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E}}$$

$$\sin^2 2\phi = \left(\frac{\Delta m^2_{\mathrm{ren}}}{\lambda_+ - \lambda_-}\right)^2 \sin^2 2\theta_{13}$$

P_{ee} and $P_{e\mu}$ to order ε

P_{ee}

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\phi \ \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E}$$

$$\sin^2 2\phi = \left(\frac{\Delta m^2_{\rm ren}}{\lambda_+ - \lambda_-}\right)^2 \sin^2 2\theta_{13}$$

$$\sqrt{(\Delta m^2_{\rm ren} - a)^2 + 4 s^2_{13} a \Delta m^2_{\rm ren}}$$

$P_{e\mu}$

$$J_r \equiv c_{12} s_{12} c_{23} s_{23} c^2_{13} s_{13}.$$

$$\begin{aligned} & P(\nu_e \to \nu_\mu) \\ &= \left[s^2_{23} \sin^2 2\theta_{13} + 4\epsilon J_r \cos\delta \left\{ \frac{(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a)}{(\lambda_+ - \lambda_0)} \right\} \right] \left(\frac{\Delta m^2_{\rm ren}}{\lambda_+ - \lambda_-}\right)^2 \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\ &+ 8\epsilon J_r \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \cos\left(\delta - \frac{(\lambda_+ - \lambda_0)L}{4E}\right) \end{aligned}$$

~ as simple as Cervera et al formula NPB 2000

$P_{e\mu}$

M.Freund PRD01

$$P_0 = \sin^2\theta_{23}\frac{\sin^2 2\theta_{13}}{\hat{C}^2}\sin^2(\hat{\Delta}\hat{C}), \tag{36a}$$

$$P_{\sin\delta} = \frac{1}{2}\alpha\frac{\sin\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$
$$\times\{\cos(\hat{C}\hat{\Delta}) - \cos((1+\hat{A})\hat{\Delta})\}, \tag{36b}$$

$$P_{\cos\delta} = \frac{1}{2}\alpha\frac{\cos\delta\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}\sin 2\theta_{23}}{\hat{A}\hat{C}\cos\theta_{13}^2}\sin(\hat{C}\hat{\Delta})$$
$$\times\{\sin((1+\hat{A})\hat{\Delta}) \mp \sin(\hat{C}\hat{\Delta})\}, \tag{36c}$$

$$P_1 = -\alpha\frac{1-\hat{A}\cos 2\theta_{13}}{\hat{C}^3}\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\hat{\Delta}$$
$$\times\sin(2\hat{\Delta}\hat{C}) + \alpha\frac{2\hat{A}(-\hat{A}+\cos 2\theta_{13})}{\hat{C}^4}$$
$$\times\sin^2\theta_{12}\sin^2 2\theta_{13}\sin^2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36d}$$

$$P_2 = \alpha\frac{\mp 1 + \hat{C} \pm \hat{A}\cos 2\theta_{13}}{2\hat{C}^2\hat{A}\cos^2\theta_{13}}\cos\theta_{13}\sin 2\theta_{12}\sin 2\theta_{13}$$
$$\times\sin 2\theta_{23}\sin^2(\hat{\Delta}\hat{C}), \tag{36e}$$

$$P_3 = \alpha^2\frac{2\hat{C}\cos^2\theta_{23}\sin^2 2\theta_{12}}{\hat{A}^2\cos^2\theta_{13}(\mp\hat{A}+\hat{C}\pm\cos 2\theta_{13})}$$
$$\times\sin^2\left(\frac{1}{2}(1+\hat{A}\mp\hat{C})\hat{\Delta}\right). \tag{36f}$$

$P_{\mu\mu}$ to order ε

$P_{\mu\mu}$

$$
\begin{aligned}
& P(\nu_\mu \to \nu_\mu) \\
&= 1 - \left[s_{23}^4 \sin^2 2\phi + 8\epsilon J_r \cos\delta \; s_{23}^2 \; \frac{(\Delta m^2_{\rm ren})^2 \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a) \right\}}{(\lambda_+ - \lambda_-)^2 (\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&- \left[\sin^2 2\theta_{23} c_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \; \cos 2\theta_{23} \; \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)} \right] \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E} \\
&- \left[\sin^2 2\theta_{23} s_\phi^2 - 4\epsilon \left(J_r \cos\delta / c_{13}^2 \right) \; \cos 2\theta_{23} \; \frac{\Delta m^2_{\rm ren} \left\{ (\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} + a) \right\}}{(\lambda_+ - \lambda_-)(\lambda_- - \lambda_0)} \right] \sin^2 \frac{(\lambda_- - \lambda_0)L}{4E} \\
&- 16\epsilon J_r \cos\delta \; s_{23}^2 \; \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \cos \frac{(\lambda_+ - \lambda_0)L}{4E}.
\end{aligned}
$$

$P_{\mu\tau}$ to order ε

$P_{\mu\tau}$

$$
\begin{aligned}
& P(\nu_\mu \to \nu_\tau) \\
&= -\left[c_{23}^2 s_{23}^2 \sin^2 2\phi + 4\epsilon J_r \cos\delta \cos 2\theta_{23} \frac{(\Delta m^2_{\rm ren})^2 \left\{(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} - a)\right\}}{(\lambda_+ - \lambda_-)^2(\lambda_+ - \lambda_0)}\right] \sin^2 \frac{(\lambda_+ - \lambda_-)L}{4E} \\
&+ \left[\sin^2 2\theta_{23} c_\phi^2 - 4\epsilon J_r \cos\delta \left(\frac{\cos 2\theta_{23}}{c_{13}^2}\right) \frac{\Delta m^2_{\rm ren} \left\{(\lambda_+ - \lambda_-) - (\Delta m^2_{\rm ren} + a)\right\}}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)}\right] \sin^2 \frac{(\lambda_+ - \lambda_0)L}{4E} \\
&+ \left[\sin^2 2\theta_{23} s_\phi^2 - 4\epsilon J_r \cos\delta \left(\frac{\cos 2\theta_{23}}{c_{13}^2}\right) \frac{\Delta m^2_{\rm ren} \left\{(\lambda_+ - \lambda_-) + (\Delta m^2_{\rm ren} + a)\right\}}{(\lambda_+ - \lambda_-)(\lambda_- - \lambda_0)}\right] \sin^2 \frac{(\lambda_- - \lambda_0)L}{4E} \\
&- 8\epsilon J_r \frac{(\Delta m^2_{\rm ren})^3}{(\lambda_+ - \lambda_-)(\lambda_+ - \lambda_0)(\lambda_- - \lambda_0)} \sin \frac{(\lambda_+ - \lambda_-)L}{4E} \sin \frac{(\lambda_- - \lambda_0)L}{4E} \\
&\times \left[(c_{23}^2 - s_{23}^2) \cos\delta \cos \frac{(\lambda_+ - \lambda_0)L}{4E} - \sin\delta \sin \frac{(\lambda_+ - \lambda_0)L}{4E}\right]. \qquad (3.9)
\end{aligned}
$$

$P_{\mu\tau}$ by E. K. Akhmedov, T. Ohlsson et al. (JHEP04)

$P_{\mu\tau}$

$$P^{(0)}_{\mu\tau} = \frac{1}{2}\sin^2 2\theta_{23}\left[\left(1-\frac{\cos 2\theta_{13}-A}{C_{13}}\right)\sin^2\frac{1}{2}(1+A-C_{13})\Delta+\left(1+\frac{\cos 2\theta_{13}-A}{C_{13}}\right)\times\right.$$
$$\left.\times\sin^2\frac{1}{2}(1+A+C_{13})\Delta-\frac{1}{2}\frac{\sin^2 2\theta_{13}}{C_{13}^2}\sin^2 C_{13}\Delta\right]$$

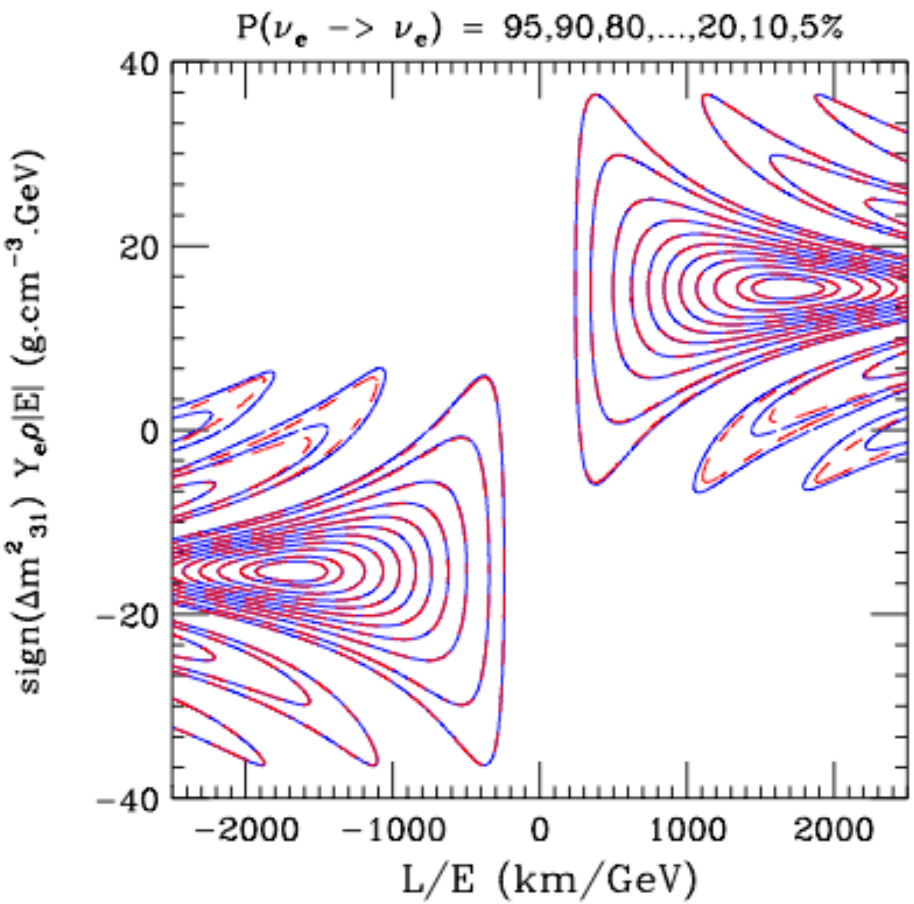
$$P^{(1)}_{\mu\tau} = -\frac{1}{2}\sin^2 2\theta_{23}\,\Delta\left\{2\left[c_{12}^2-s_{12}^2 s_{13}^2\frac{1}{C_{13}^2}(1+2s_{13}^2A+A^2)\right]\cos C_{13}\Delta\sin(1+A)\Delta+\right.$$
$$+2[c_{12}^2c_{13}^2-c_{12}^2s_{13}^2+s_{12}^2s_{13}^2+(s_{12}^2s_{13}^2-c_{12}^2)A]\times$$
$$\times\frac{\sin C_{13}\Delta}{C_{13}}\cos(1+A)\Delta+s_{12}^2\frac{\sin^2 2\theta_{13}}{C_{13}^2}\frac{\sin C_{13}\Delta}{C_{13}}\times$$
$$\times\left[\frac{A}{\Delta}\sin(1+A)\Delta+\frac{A}{\Delta}\frac{\cos 2\theta_{13}-A}{C_{13}}\sin C_{13}\Delta-\right.$$
$$\left.\left.-(1-A\cos 2\theta_{13})\cos C_{13}\Delta\right]\right\}+$$
$$+\frac{s_{13}\sin 2\theta_{12}\sin 2\theta_{23}}{2Ac_{13}^2}\left\{2c_{13}^2\sin\delta_{\rm CP}\frac{\sin C_{13}\Delta}{C_{13}}[\cos C_{13}\Delta-\cos(1+A)\Delta]-\right.$$
$$-\cos 2\theta_{23}\cos\delta_{\rm CP}(1+A)[\cos C_{13}\Delta-\cos(1+A)\Delta]^2+$$
$$+\cos 2\theta_{23}\cos\delta_{\rm CP}[\sin(1+A)\Delta+\frac{\cos 2\theta_{13}-A}{C_{13}}\sin C_{13}\Delta]\times$$
$$\left.\times\left[(1+2s_{13}^2A+A^2)\frac{\sin C_{13}\Delta}{C_{13}}-(1+A)\sin(1+A)\Delta\right]\right\}. \quad (4.11)$$

$$C_{13}\equiv\sqrt{\sin^2 2\theta_{13}+(A-\cos 2\theta_{13})^2}\,.$$

How accurate is the formula?

How accurate is the formula?: equi-probability contour

P_{ee}



$P_{e\mu}$

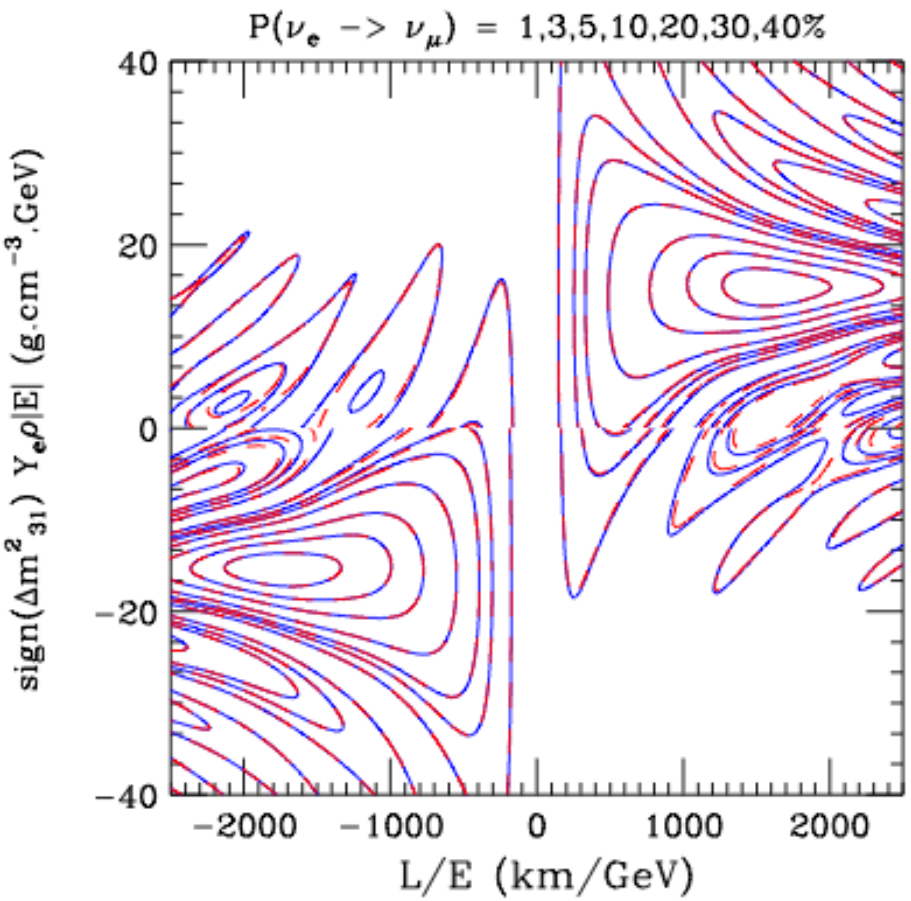
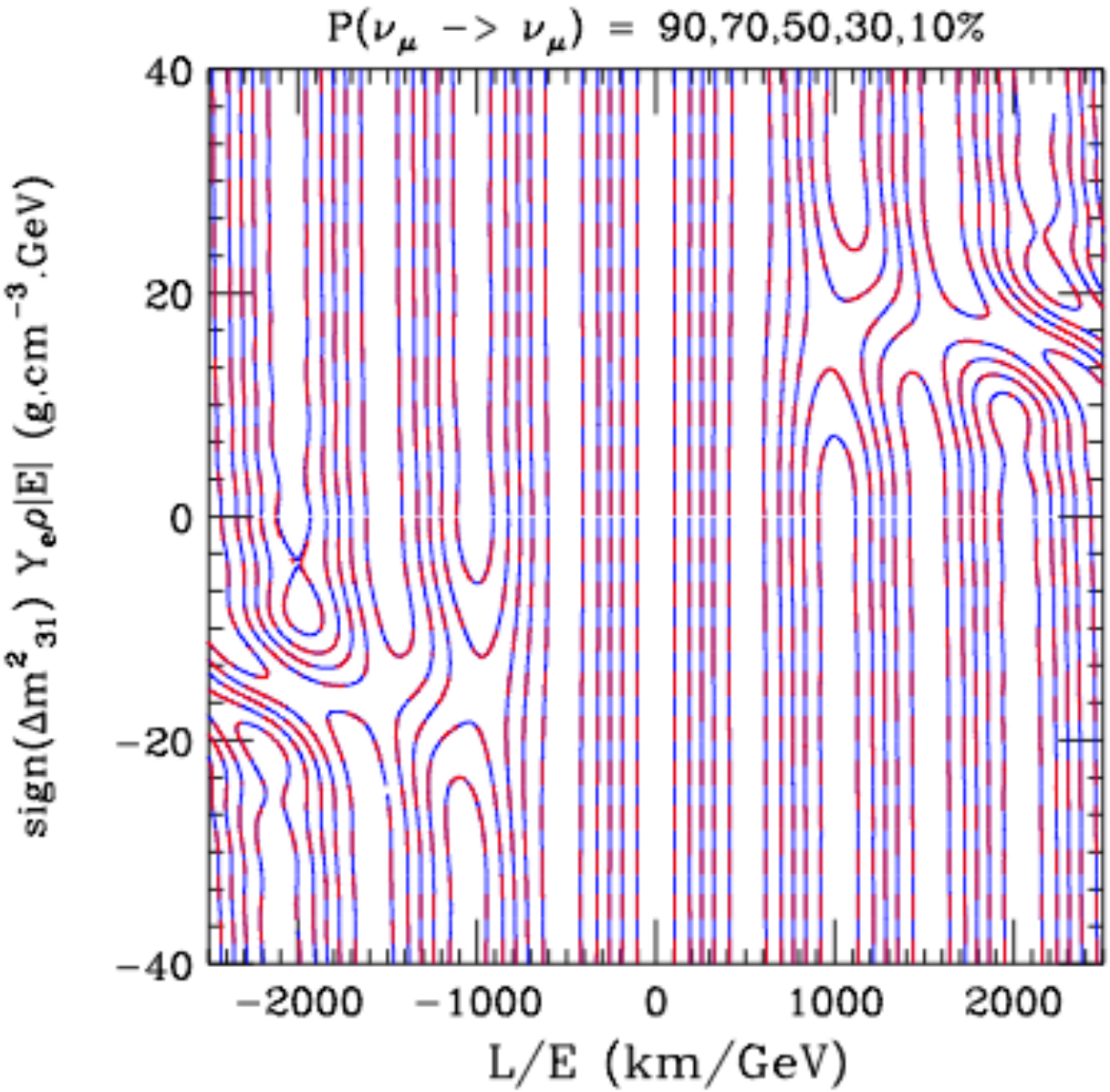


Figure 1. The iso-probability contours for the exact (solid blue) and approximate (dashed red)

How accurate is the formula? #2

$P_{\mu\mu}$

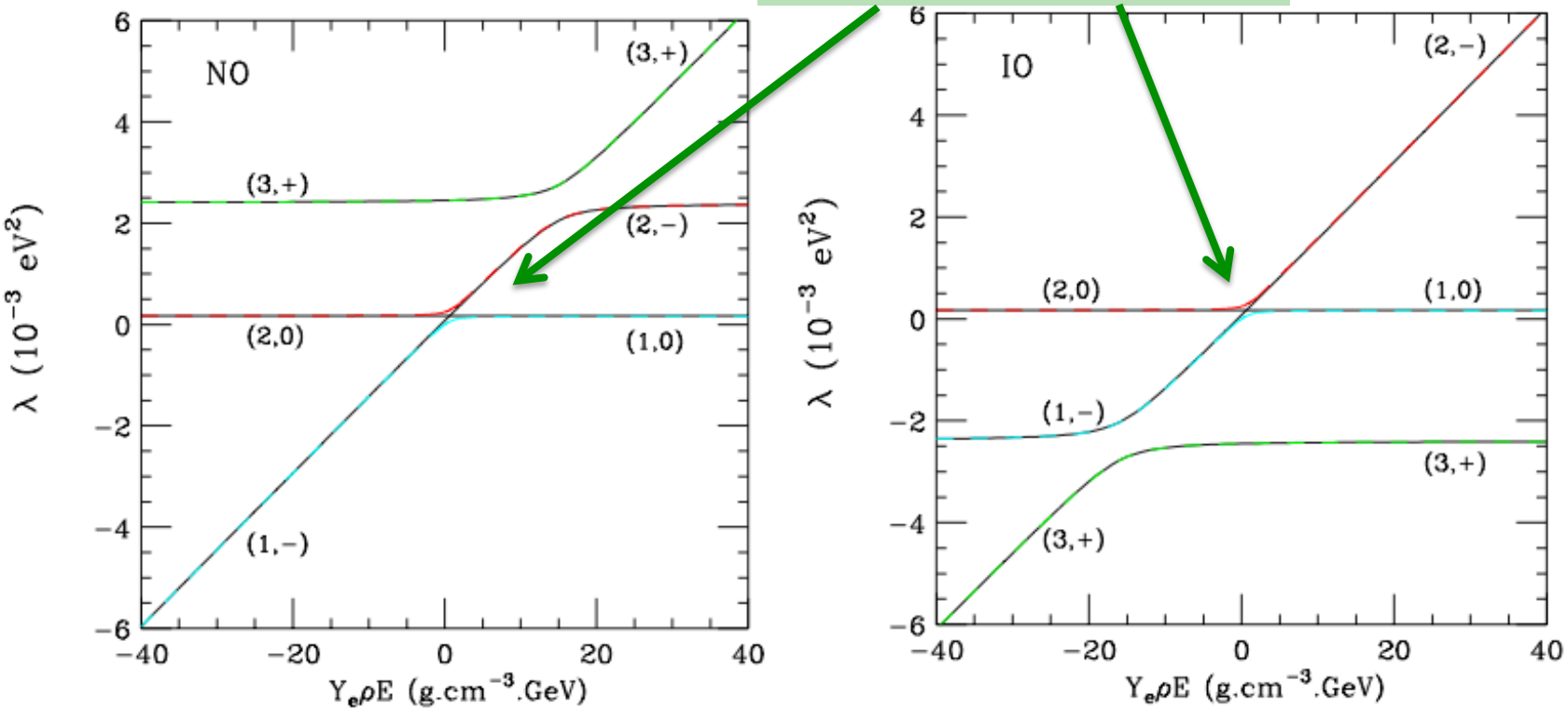


Not without problem

The problem is common for all the perturbative framework in the market

Not without problem: ν_- and ν_0 level crossing

ν_- and ν_0 level cross !



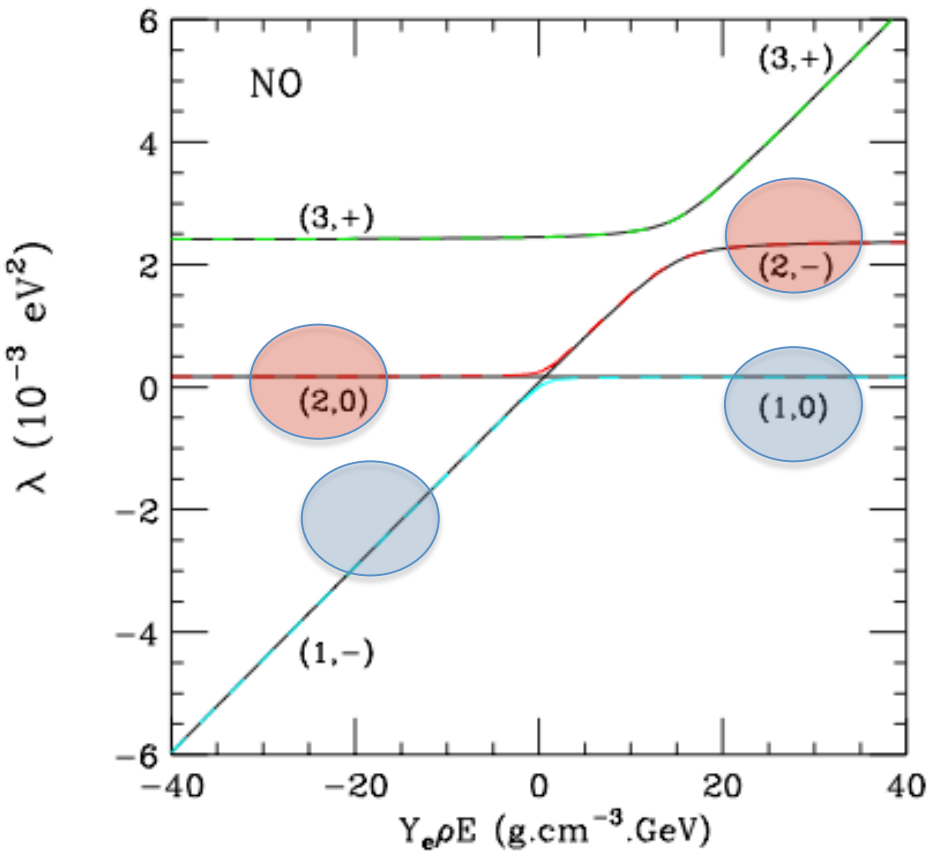
$$\lambda_- = \frac{1}{2}\left[(\Delta m^2_{\rm ren} + a) - {\rm sign}(\Delta m^2_{\rm ren})\sqrt{(\Delta m^2_{\rm ren} - a)^2 + 4 s^2_{13} a \Delta m^2_{\rm ren}}\right] + \epsilon \Delta m^2_{\rm ren} s^2_{12},$$

$$\lambda_0 = c^2_{12}\, \epsilon\, \Delta m^2_{\rm ren},$$

$$\lambda_+ = \frac{1}{2}\left[(\Delta m^2_{\rm ren} + a) + {\rm sign}(\Delta m^2_{\rm ren})\sqrt{(\Delta m^2_{\rm ren} - a)^2 + 4 s^2_{13} a \Delta m^2_{\rm ren}}\right] + \epsilon \Delta m^2_{\rm ren} s^2_{12}.$$

Flavor content of ν_- and ν_0

- Despite the failure of treating the solar resonance, the flavor content of the states (which participate the solar resonance) are properly reproduced off resonance



What is the charm of the formulas?

Harmony between perturbation theory and general expression

- Perturbation theory is good: relative simplicity of the elements of expressions
- The general expression

$$P(\nu_\beta \to \nu_\alpha) = \delta_{\alpha\beta}$$
$$- 4\sum_{j>i} \mathrm{Re}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}] \sin^2 \frac{(\lambda_{mj}-\lambda_{mi})x}{4E} - 2\sum_{j>i} \mathrm{Im}[V_{\alpha i}V^*_{\beta i}V^*_{\alpha j}V_{\beta j}] \sin \frac{(\lambda_{mj}-\lambda_{mi})x}{2E}$$

- Is good, because “structure revealing”
- Oscillation in matter ~ as simple as oscillation in vacuum

Some insights, problems, .. (personal opinion)

- So far P theorists (incl. myself) were lazy,, I mean, they produced lengthy formulas, but didn't try to make them simpler by combining terms
- We can calculate higher order
- The real issue is whether the expression remains simple and compact → renormalization to order ε^2 ?
- I suspect YES

$$P(\nu_\beta \to \nu_\alpha) = \delta_{\alpha\beta} - 4\sum_{j>i} \mathrm{Re}[V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* V_{\beta j}] \sin^2 \frac{(\lambda_{mj} - \lambda_{mi})x}{4E} - 2\sum_{j>i} \mathrm{Im}[V_{\alpha i} V_{\beta i}^* V_{\alpha j}^* V_{\beta j}] \sin \frac{(\lambda_{mj} - \lambda_{mi})x}{2E}$$

- To all order P has simple general form
- Order ε OK, so probably OK to order ε^N (my guess)

Back up slides