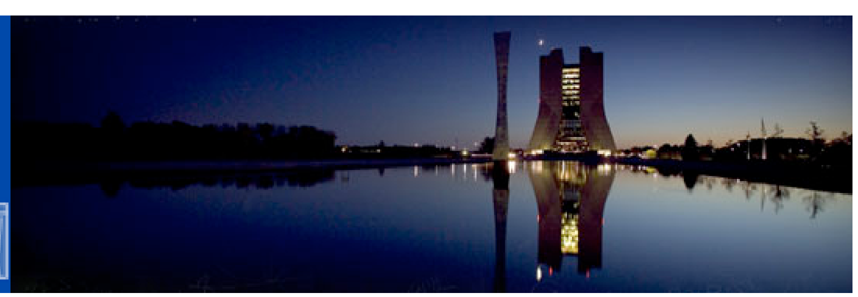
Quantum Chromodynamics

Lecture 1: All about color

Hadron Collider Physics Summer School 2010

John Campbell, Fermilab



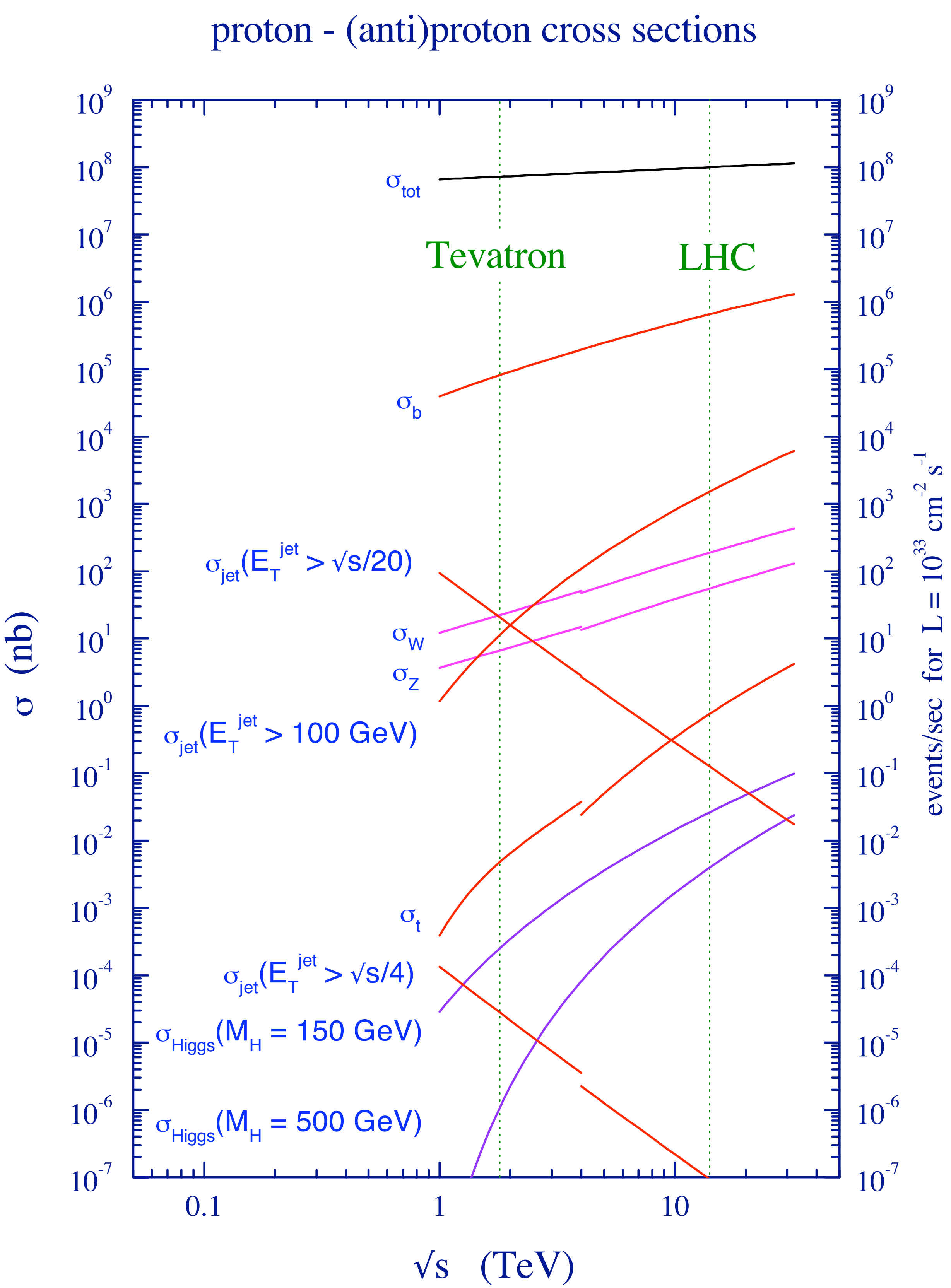
References and thanks

- Useful references for this short course are:
- QCD and Collider Physics
  R. K. Ellis, W. J. Stirling and B. R. Webber
  Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology
- Hard Interactions of Quarks and Gluons: a Primer for LHC Physics
  J. C., J. W. Huston and W. J. Stirling
  Rept. Prog. Phys. 70, 89 (2007) [hep-ph/0611148]
- Resource Letter: Quantum Chromodynamics
  A. S. Kronfeld and C. Quigg
  arXiv:1002.5032 [hep-ph] (for the American Journal of Physics)

- Thanks to R. K. Ellis and G. Zanderighi, for lecture notes from previous schools - upon which much of these lectures will be based.

QCD: why we care

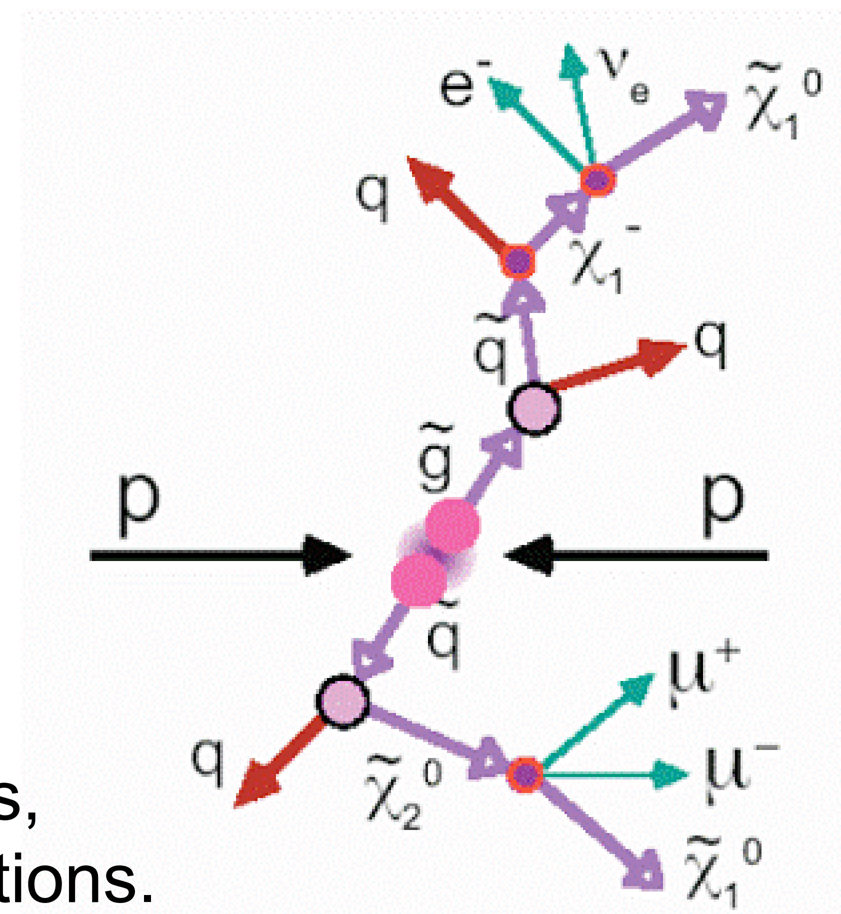
- It is no surprise that hadron colliders require an understanding of QCD.
- This plot demonstrates the extent to which we must have a good understanding,
    - cross sections for inclusive bottom production and final states with jets of hadrons are near the top.
    - Higgs boson cross sections are at the bottom.
- Discovering such New Physics requires a sophisticated, quantitative understanding of QCD.
- In these lectures, we will develop the tools necessary for such a task.



QCD: why we care even more

- If a Higgs-like signal is observed, to confirm its interpretation as the Higgs boson requires measurement of its couplings and quantum numbers.
  - need an accurate understanding of the production/decay mechanisms.

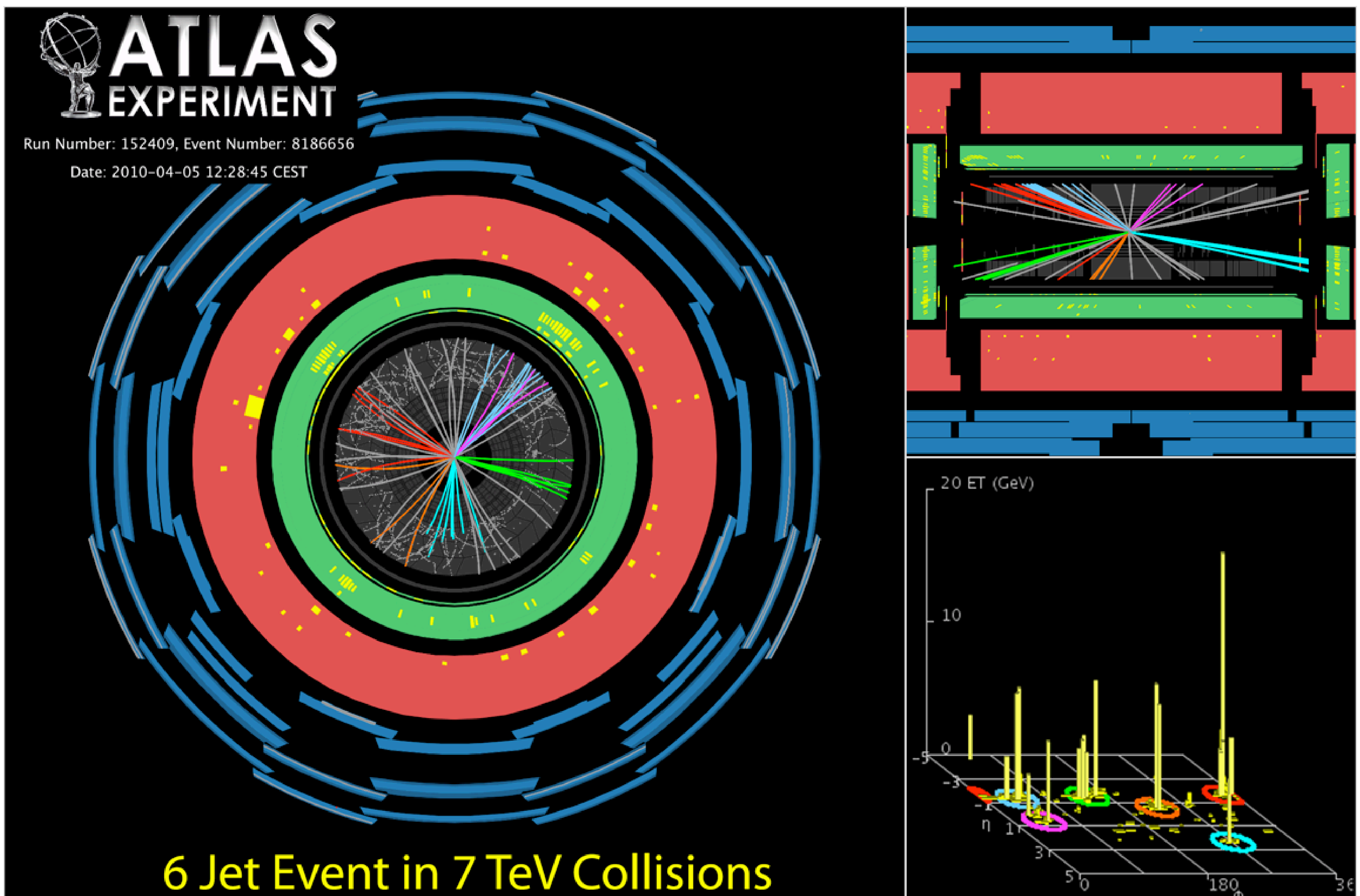
- Hopefully, we will see more than just a Higgs boson.
  - supersymmetry?
  - extra dimensions?
  - technicolor?



- All of these models of New Physics introduce new particles that will (most likely) decay as they traverse the detectors, into “old” colored particles → QCD interactions.

The challenge of QCD

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A + \sum_{\text{flavors}} \bar{q}_i \left(i \not{D} - m \right)_{ij} q_j$$



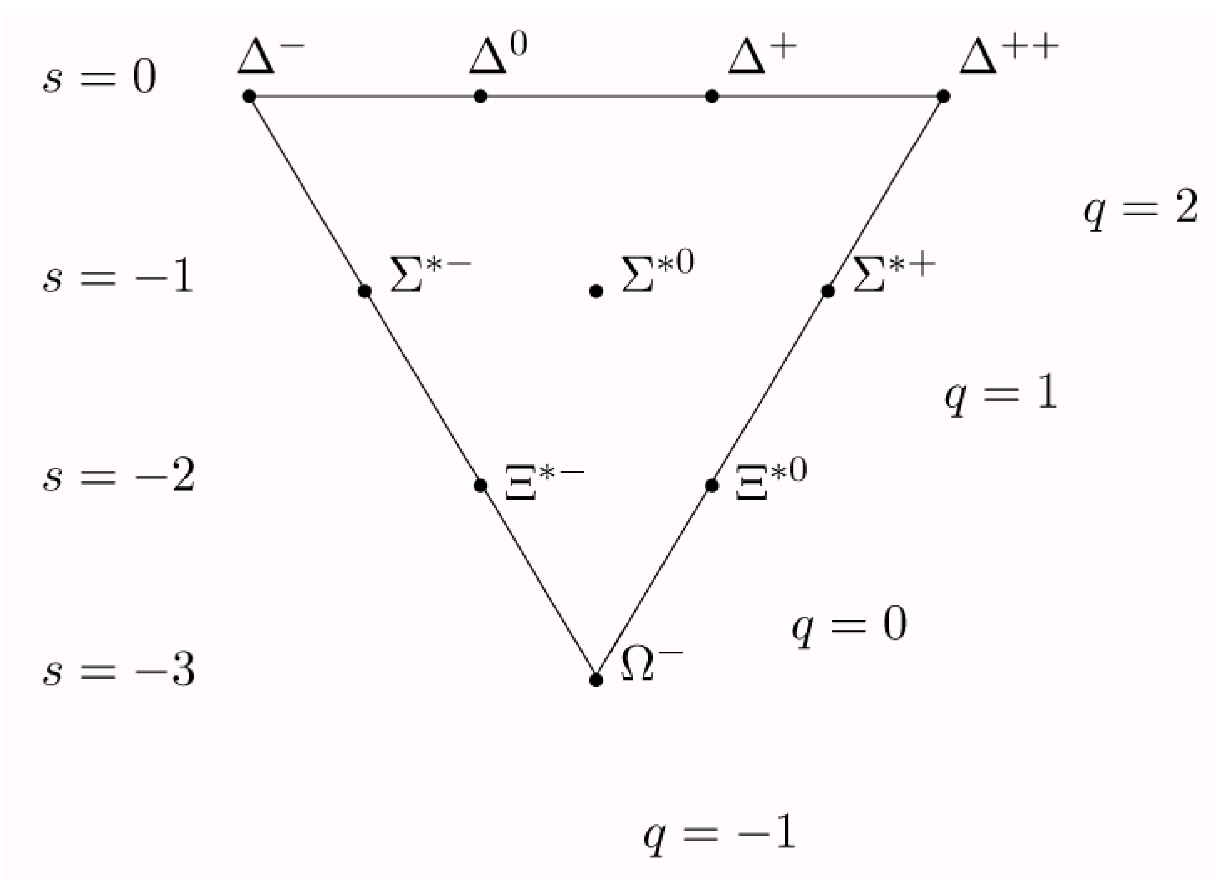
Tasks for today

- Understand why the Lagrangian looks like this:
  - why color and why SU(3)?
- Understand some features of this Lagrangian:
  - in practical terms, how does QCD differ from QED?
- Understand how to use this Lagrangian:
  - how can we use it to make predictions?

Quarks and color

- The quark model is a useful way of categorizing mesons (baryons) in terms of two (three) constituent quarks.

Q=+2/3	up m_u~4 MeV	charm m_c~1.5 GeV	top m_t~172 GeV
Q=-1/3	down m_d~7 MeV	strange m_s~135 MeV	bottom m_b~5 GeV

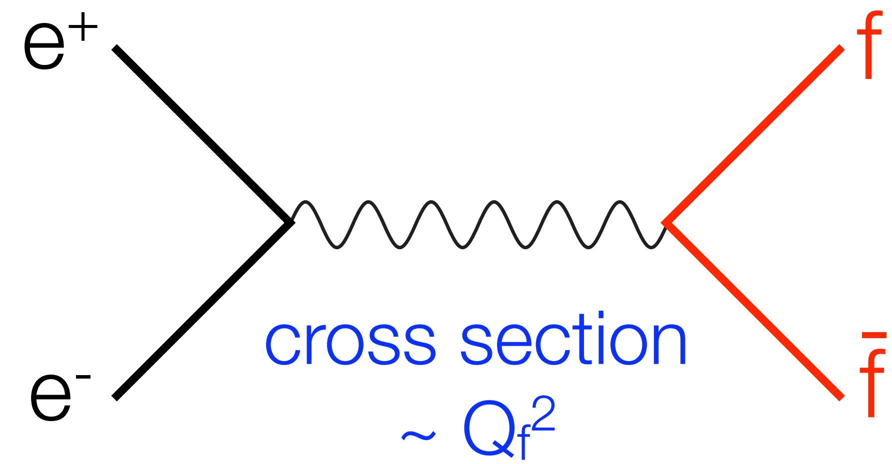


Baryon decuplet (S=3/2)

- Simple picture must be amended due to, for example, Δ^{++}=(u,u,u) in a symmetric spin state.
- The baryons should obey the Pauli principle: the overall wavefunction should be antisymmetric.
- In order to accommodate this, the antisymmetry should be carried by another quantum number: color.
- Observed particles are colorless.

Probing color

- Subsequent realization that color could be probed directly in e^+e^- collisions.
  - production of fermion pairs through a virtual photon sensitive to electric charge of fermion and the number of degrees of freedom allowed.



- Hence investigate quarks through "R-ratio":

$$R = \frac{\sigma\left(e^+e^- \rightarrow \text{hadrons}\right)}{\sigma\left(e^+e^- \rightarrow \mu^+\mu^-\right)} = N_c \sum_f Q_f^2$$

quark charge

assume N_c colors of quark

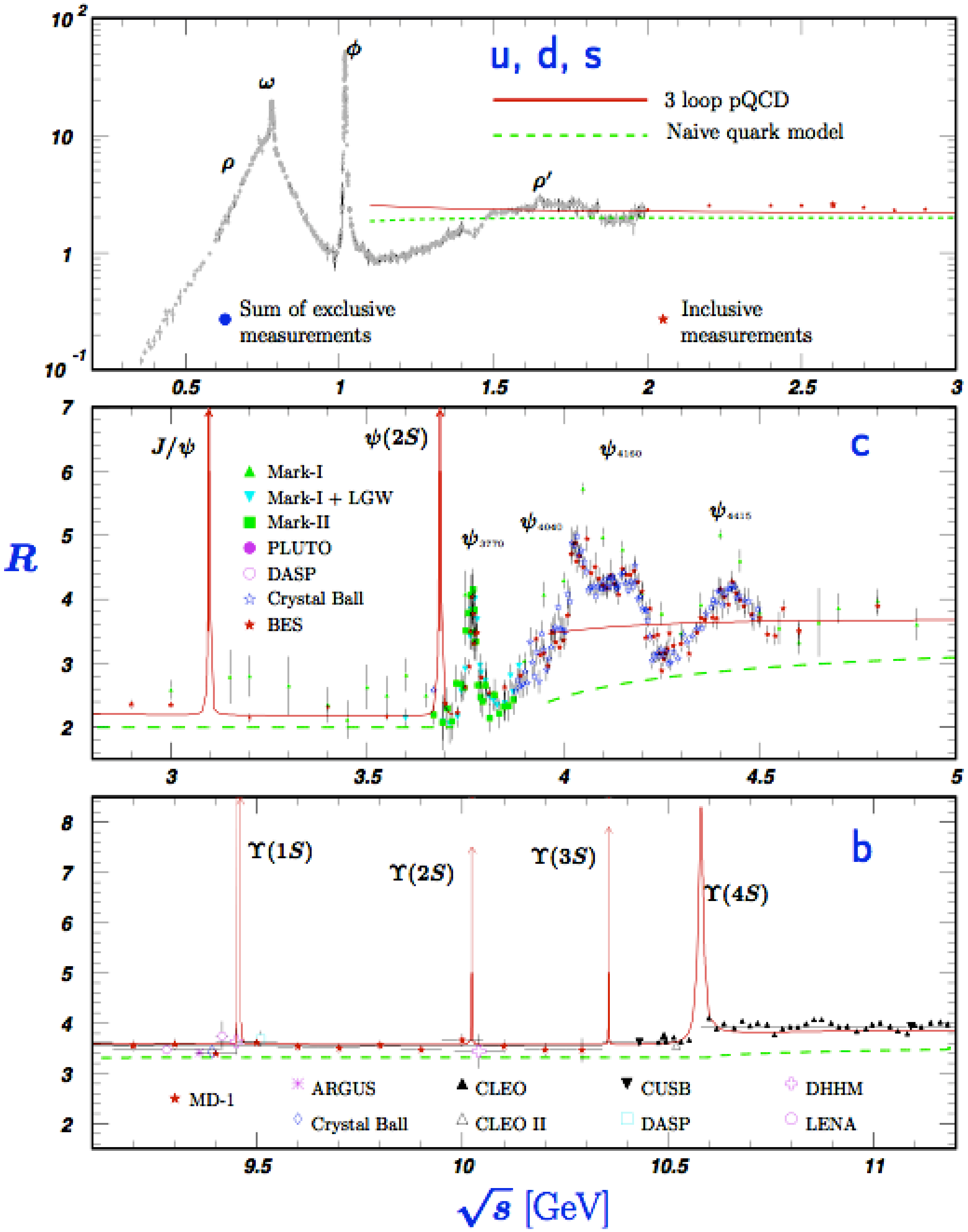
sum over active quarks

(this is at least the most basic expectation - corrections later)

- Each active quark is produced in N_c colors: must be above the kinematic threshold for each quark in the sum, i.e. $\sqrt{s} > 2m_q$.

Experimental measurements

Broad support for N_C=3



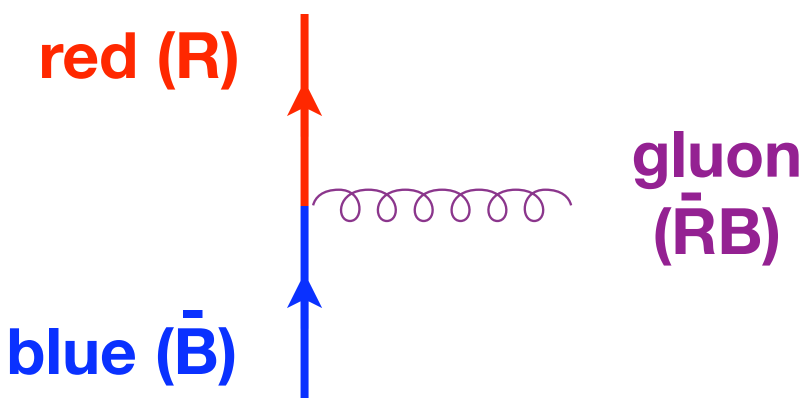
$$R_{u,d,s} = 3 \times \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2$$

$$R_{u,d,s,c} = R_{u,d,s} + 3 \times \left(\frac{2}{3}\right)^2 = \frac{10}{3}$$

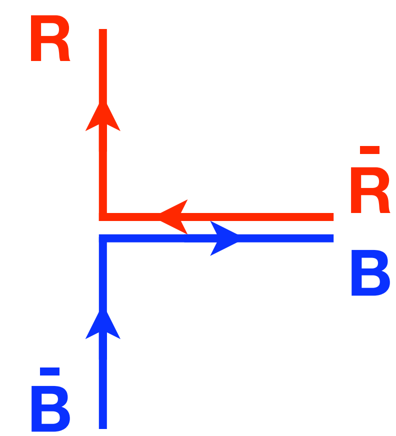
$$R_{u,d,s,c,b} = R_{u,d,s,c} + 3 \times \left(-\frac{1}{3}\right)^2 = \frac{11}{3}$$

QCD interactions

- In QCD, the color quantum number is mediated by the gluon, analogous to the photon in QED.
  - it will be responsible for changing quarks from one color to another; as such it must also carry a color charge (not neutral, as in QED).
- 1st try: mediating quark and anti-quark of 3 different colors → 3 x 3 = 9 gluons.



or as
"color flow"



- In fact we should take six such combinations, plus three mutually orthogonal combinations of same-color states.

$\bar{R}B$ $\bar{R}G$

$\bar{G}B$ $\bar{G}R$

$\bar{B}R$ $\bar{B}G$

$(\bar{R}R - \bar{B}B)/\sqrt{2}$

$(\bar{R}R + \bar{B}B - 2\,\bar{G}G)/\sqrt{6}$

$(\bar{R}R + \bar{B}B + \bar{G}G)/\sqrt{3}$

QCD interactions

- Since color is an internal degree of freedom, we expect invariance of the theory under rotations in this color space.
  - this requires that eight of our color combinations share the same coupling:

$\bar{R}B$ $\bar{R}G$ $\qquad (\bar{R}R - \bar{B}B)/\sqrt{2}$

$\bar{G}B$ $\bar{G}R$ $\qquad (\bar{R}R + \bar{B}B - 2\,\bar{G}G)/\sqrt{6}$

$\bar{B}R$ $\bar{B}G$

  - the remaining combination only transforms into itself - it is a color singlet:

$$(\bar{R}R + \bar{B}B + \bar{G}G)/\sqrt{3}$$

- Such a combination is not present in QCD: we are left with 8 gluons.
- The color charge of each gluon is represented by a matrix in color space.
  - the eight combinations result in eight matrices, T^A, with A=1,..8.
  - a conventional choice is to write these in terms of the Gell-Mann matrices, which are just an extension of Pauli Matrices:

$$T^A = \frac{1}{2}\lambda^A$$

Gell-Mann matrices

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- These matrices are Hermitian, $(\lambda^A)^\dagger = \lambda^A$, and traceless.
  - only two diagonal matrices: the color singlet would not have been traceless.
- They obey the two relations:

completely antisymmetric set of real constants, f^{ABC}

$$\mathrm{Tr}\,(\lambda^A \lambda^B) = 2\delta^{AB}\,, \qquad [\lambda^A, \lambda^B] = 2 i f^{ABC} \lambda^C$$

Color matrices

- Translating back to color matrices, we have:

$$\left[T^A, T^B\right] = i f^{ABC} T^C \;, \qquad \mathrm{Tr}\left(T^A T^B\right) = T_R \delta^{AB} \quad (\text{ with } T_R = 1/2)$$

- The first of these relations reflects that fact that:
  - the matrices T^A are the generators of the SU(3) group, A=1,...,8;
  - the antisymmetric set, f^{ABC}, contains the SU(3) structure constants.
- The second relation is just a normalization convention.
- The group structure is also characterized by two other relations:

$$\sum_{C,D} f^{ACD} f^{BCD} = C_A\, \delta^{AB} \qquad \text{with } C_A = N_c = 3$$

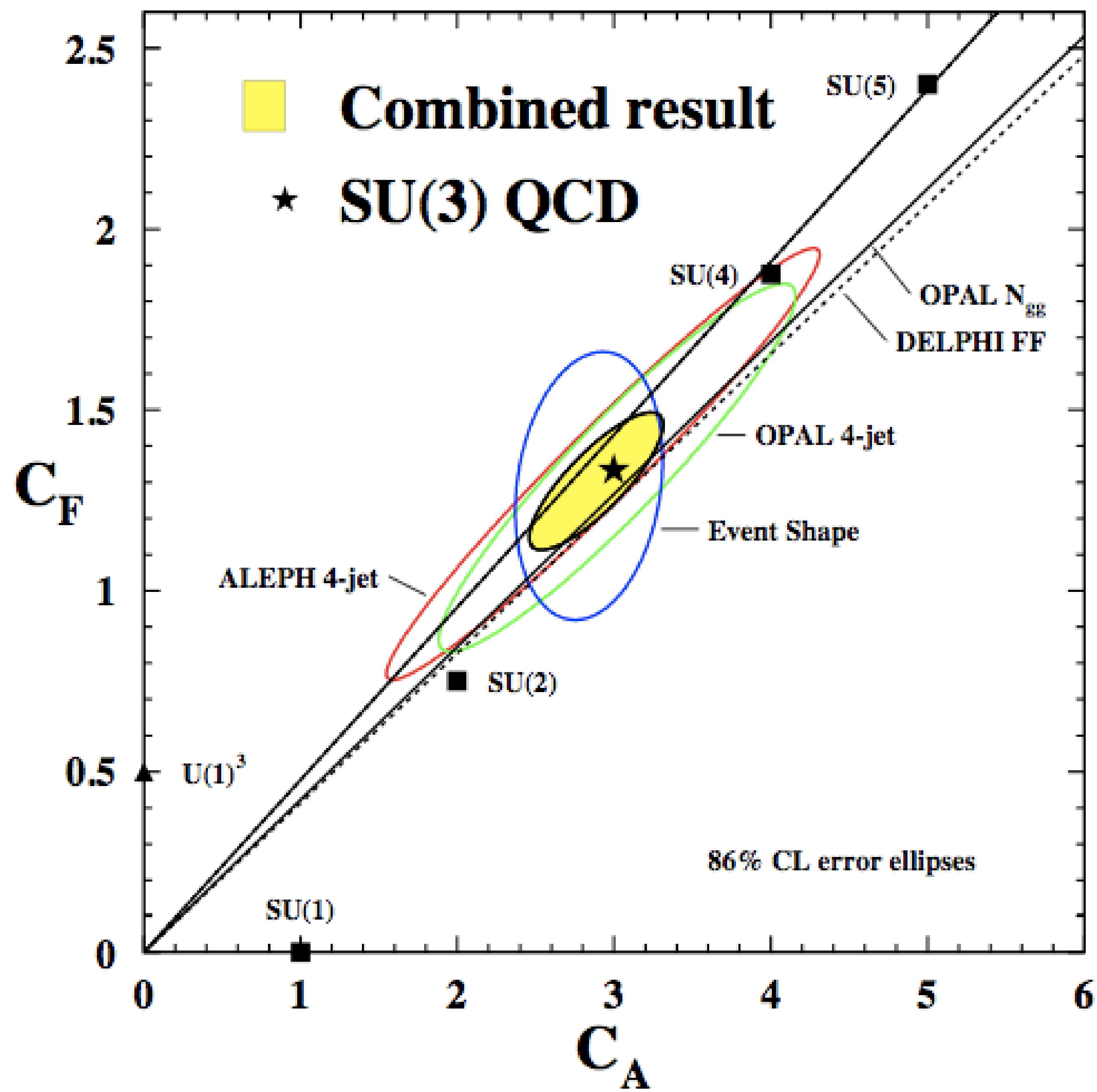
$$\sum_A T^A T^A = C_F\, \mathbf{1} \qquad \text{with } C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}$$

"Casimir"

3x3 identity matrix

Further support for SU(3)

- These color sums are exactly the quantities which will appear when we compute cross sections involving QCD.
- In particular, the cross section for 4-jet production in e^+e^- annihilation at LEP is sensitive to both C_A and C_F.
- At this point, no one expected that SU(3) was not the correct description.
- However, demonstrates that the group structure is an important phenomenological aspect - not just math!



The QCD Lagrangian

- The quantum field theory of QCD is then based on the Lagrangian:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^A_{\mu\nu} F_A^{\mu\nu} + \sum_{\text{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$

field strength tensor, gluon degrees of freedom

in the non-interacting case, the Dirac term for quark d.o.f.

- Color plays a crucial role in the Lagrangian:

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu - g_s f^{ABC} A^B_\mu A^C_\nu$$

A^A_μ : field for the spin-1 gluon (just like the photon in QED, but with an extra color label)

self-interaction term for gluon fields: called "non-Abelian" since it arises from the SU(3) structure

QCD gauge transformations

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^{A}_{\mu\nu} F^{\mu\nu}_{A} + \sum_{\mathrm{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$

- Color also appears in the definition of the covariant derivative:

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s (T^A A^A_\mu)_{ij}$$

which couples together quarks and gluons in the interacting theory.

- Such a definition ensures that the QCD Lagrangian remains invariant under local gauge transformations of the form,

$$q_i(x) \to q'_i(x) = \Omega_{ij}(x) q_j(x) \qquad \left(\Omega^\dagger_{ik}(x) \Omega_{kj}(x) = \delta_{ij} \right)$$

$$(D_\mu)_{ik}\, q_k(x) \to \left(D'_\mu \right)_{ik} q'_k(x) = \Omega_{ij}(x) \left(D_\mu \right)_{jk} q_k(x)$$

- Covariant means that it transforms in the same way as the quark field itself.
- Imposing these transformation laws ensures invariance of the second term.

QCD gauge transformations

- To apply the argument on the first term relies upon the specific form we have introduced for the covariant derivative.
- This is easiest to see by manipulating the field strength tensor into a new form,

$$T^A F^A_{\mu\nu} = \partial_\mu (T^A A^A_\nu) - \partial_\nu (T^A A^A_\mu) - g_s T^A f^{ABC} A^B_\mu A^C_\nu$$

(use comm. relation)

$$= \partial_\mu T^A A^A_\nu - (T^A A^A_\nu)\partial_\mu - \partial_\nu T^A A^A_\mu + (T^A A^A_\mu)\partial_\nu$$
$$+ i g_s \left[(T^B A^B_\mu)(T^C A^C_\nu) - (T^C A^C_\nu)(T^B A^B_\mu) \right]$$

(consider action on a field)

$$= \left[\partial_\mu + i g_s (T^B A^B_\mu)\right] \left[T^A A^A_\nu + \frac{1}{i g_s}\partial_\nu\right] - \left[\partial_\nu + i g_s (T^B A^B_\nu)\right] \left[T^A A^A_\mu + \frac{1}{i g_s}\partial_\mu\right]$$
$$= \frac{1}{i g_s} \left[D_\mu, D_\nu\right]$$

- Lastly, exploit the fact that the commutator transforms in the same way as the covariant derivative itself:

$$\left[D_\mu, D_\nu\right]_{ik} q_k(x) \to \Omega_{ij}(x) \left[D_\mu, D_\nu\right]_{jk} q_k(x)$$

QCD gauge transformations

- Putting it all together:

$$\left(T^A F^A_{\mu\nu}\right)_{ij} q_j(x) \to \left(T^A F'^A_{\mu\nu}\right)_{ij} q'_j(x)$$

$$\Omega_{ij}(x)\left(T^A F^A_{\mu\nu}\right)_{jk} q_k(x) = \left(T^A F'^A_{\mu\nu}\right)_{ij} \Omega_{jk}(x) q_k(x)$$

so that the field strength transforms as,

$$\left(T^A F^A_{\mu\nu}\right)_{ij} \to \Omega_{ik}(x)\left(T^A F^A_{\mu\nu}\right)_{k\ell} \Omega^{-1}_{\ell j}(x)$$

- The field strength is no longer gauge invariant as in QED, a reflection of the self-interacting nature of gluons.

- However the combination that appears in the Lagrangian is invariant, as required:

$$-\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu}_A = -\frac{1}{2} \mathrm{Tr}\left(T^A F^A_{\mu\nu}\, T^B F^{\mu\nu}_B\right)$$

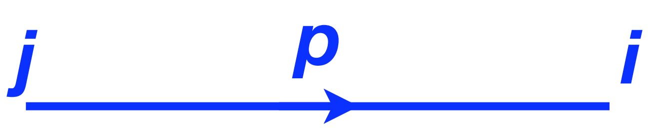
$$\to -\frac{1}{2} \mathrm{Tr}\left(\Omega\, T^A F^A_{\mu\nu}\, T^B F^{\mu\nu}_B\, \Omega^{-1}\right)$$

Using the QCD Lagrangian

- Armed with a Lagrangian that is invariant under gauge transformations, we can investigate many features of QCD.
- In these lectures, we're interested in perturbative QCD and cross sections computed from Feynman diagrams: convert Lagrangian into Feynman rules.
- Simplest place to start: free, or non-interacting Lagrangian ($g_s \to 0$).
- Prescription: make the replacement $\partial_\mu \to -ip_\mu$ (c.f. Fourier expansion) and then multiply by *i* to obtain inverse propagator.

$$\bar{q}_i \left(i\partial_\mu \gamma^\mu - m \right) \delta_{ij} q_j \to i q_i \left(p_\mu \gamma^\mu - m \right) \delta_{ij} q_j$$

quarks



$$\frac{i\left(\not{p} + m\right)}{p^2 - m^2}\,\delta_{ij}$$

trivial color factor

gluons $$-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \to \frac{i}{2} A_\mu \left(p^2 g^{\mu\nu} - p^\mu p^\nu\right) A_\nu$$

Cannot invert!

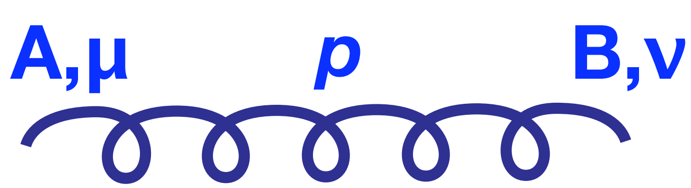
Gauge fixing

- The solution is to fix a gauge: add an additional term to the Lagrangian which depends upon an arbitrary gauge parameter λ.

$$\mathcal{L}_{\mathrm{gauge-fixing}} = -\frac{1}{2\lambda}\left(\partial^{\mu} A_{\mu}^{A}\right)^{2}$$

- This contributes an extra term: $\frac{i}{2\lambda} A_{\mu} p^{\mu} p^{\nu} A_{\nu}$ such that an inverse now exists.

gluons



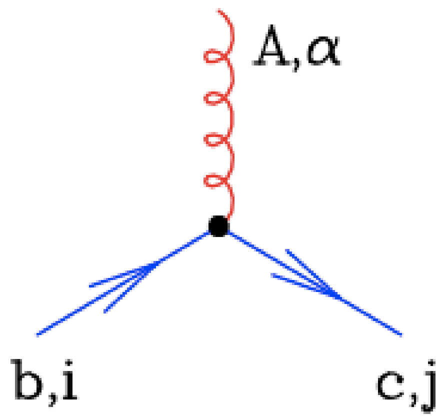
$$\frac{-i}{p^2}\left(g^{\mu\nu} - (1-\lambda)\frac{p^{\mu}p^{\nu}}{p^2}\right)\delta^{AB}$$

- Different gauges may be useful in different calculations, but ultimately must all give the same result.
  - a particularly simple choice is often the Feynman gauge, λ=1.
- Further complication: covariant gauge-fixing introduces unphysical d.o.f. that must be cancelled by ghost contributions - we will not discuss them here.

QCD interactions

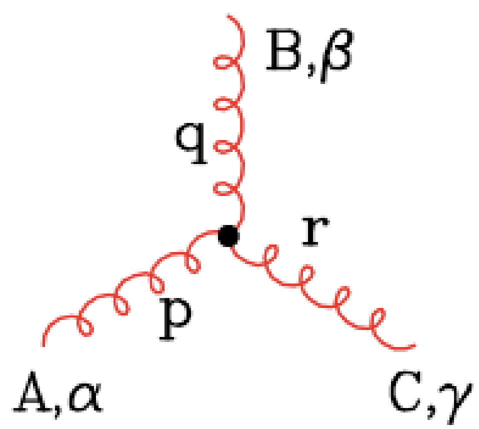
- Interactions between the quarks and gluons can be read off from the terms of order g_s and higher.

quark-gluon (from covariant derivative)



$$-ig\ (t^A)_{cb}\ (\gamma^\alpha)_{ji}$$

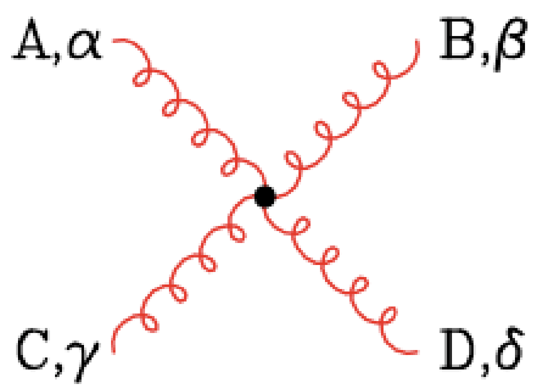
NB: sum over quark colors → trace over T strings



$$-g\ f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$

(all momenta incoming)

self interactions (from additional terms in the field strength)

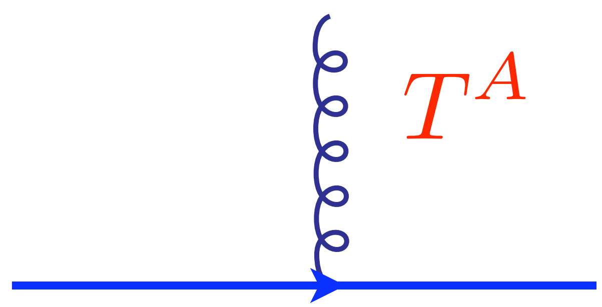


$$-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$

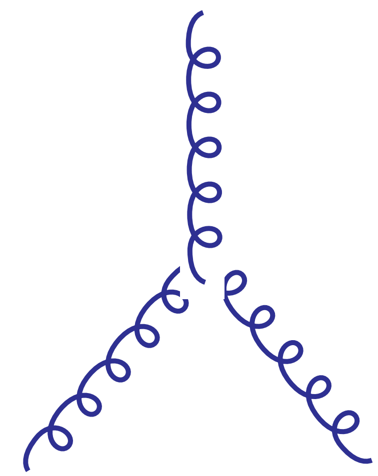
Quantum number management

- Since color is a completely separate degree of freedom, it is often useful to factorize out any dependence on color at an early stage of the calculation.
- Each Feynman diagram will be associated with a particular color factor, which it is often useful to calculate and account for separately.
- A pictorial way of doing this can be very useful.

from the
Feynman rules



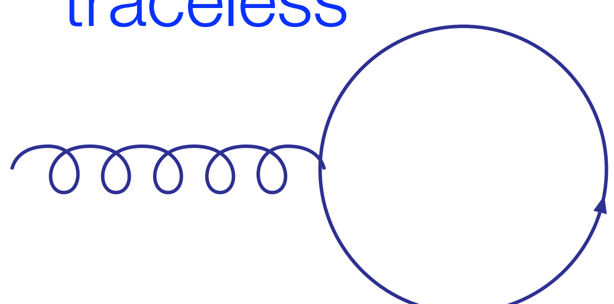
T^A



f^{ABC}

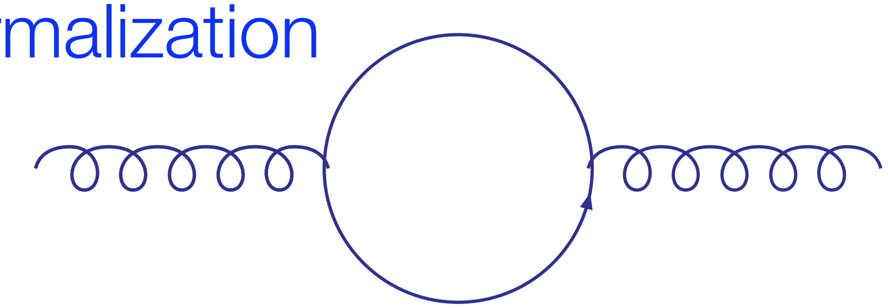
properties of the
color matrices

traceless



$Tr(T^A) = 0$

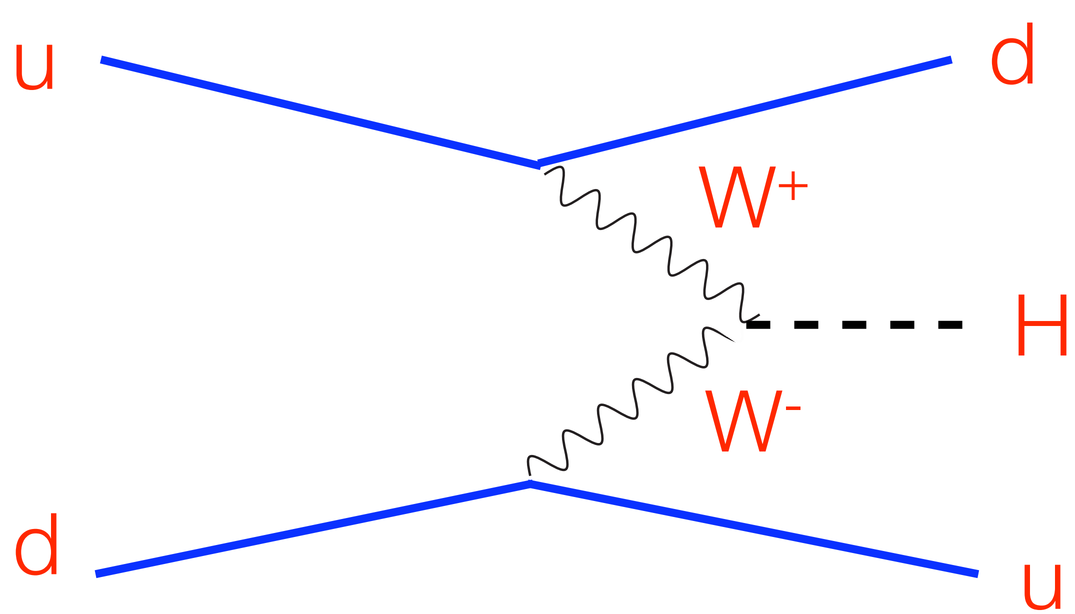
normalization



$= T_R$

Simple loop calculation

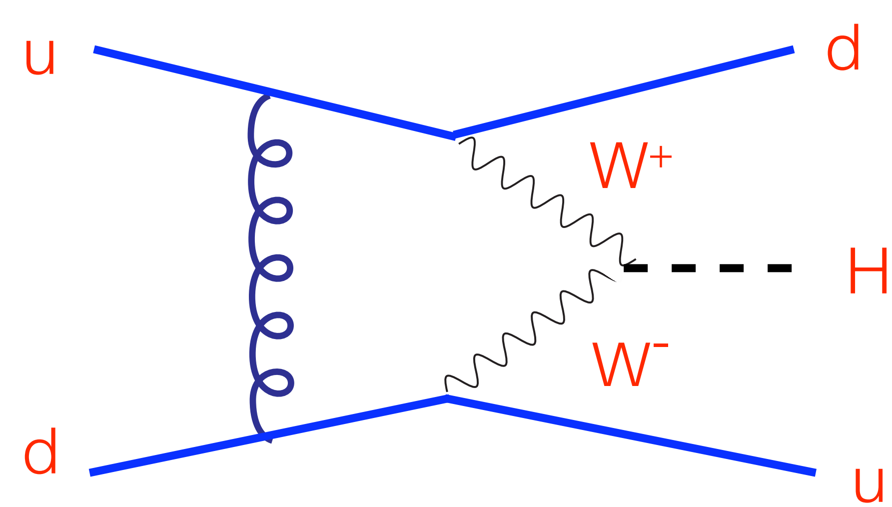
- Vector boson fusion is an important Higgs search channel at the LHC.



Basic idea: incoming quarks radiate W (or Z) bosons without changing direction much.

Higgs boson is produced in the central area of the detector relatively cleanly.

Simple picture corrected by gluon emission and absorption by the quarks:

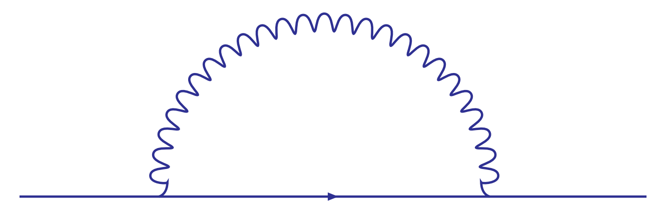


= 0 when interfered with diagram above!

Other color identities

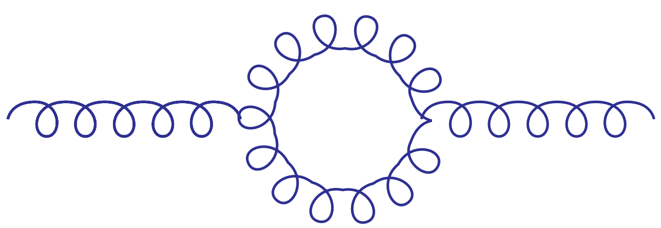
- Identities we have already seen:

$$\sum_A T^A T^A = C_F \, \mathbf{1}$$



$= C_F$

$$\sum_{C,D} f^{ACD} f^{BCD} = C_A \, \delta^{AB}$$

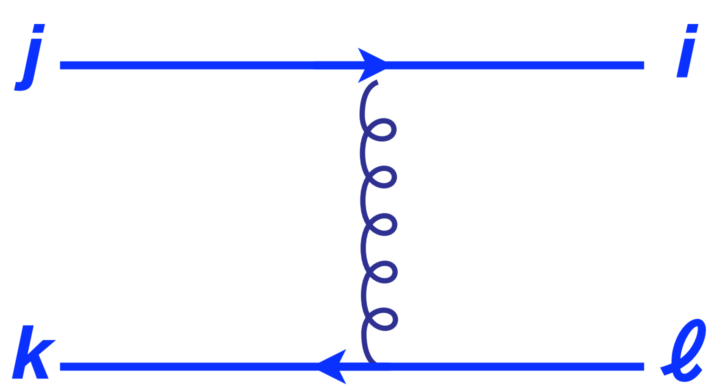


$= C_A$

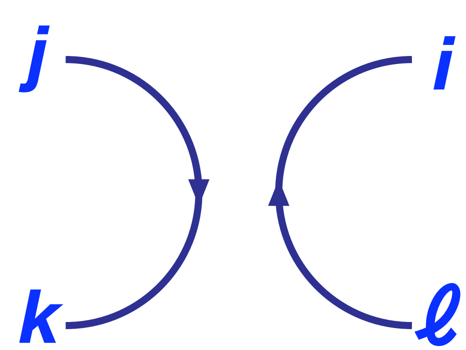
- A new relation, the Fierz identity:

$$\sum_A (T^A)_{ij} \, (T^A)_{k\ell} = \frac{1}{2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{k\ell} \right)$$

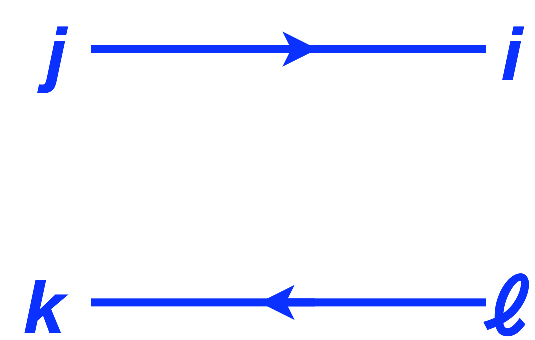
(note direction of arrows)



$= \frac{1}{2} \Bigg($



$- \frac{1}{N_c}$



$\Bigg)$

Color at work

H. Ita, Blackhat (June 2010)

W+4 jets on its way

W^{-}+4jets+X

Cuts: $\mu_R=\mu_f=\hat{H}_T/2$

$p_T^{jet} > 25$ GeV $|\eta^{jet}| < 3$

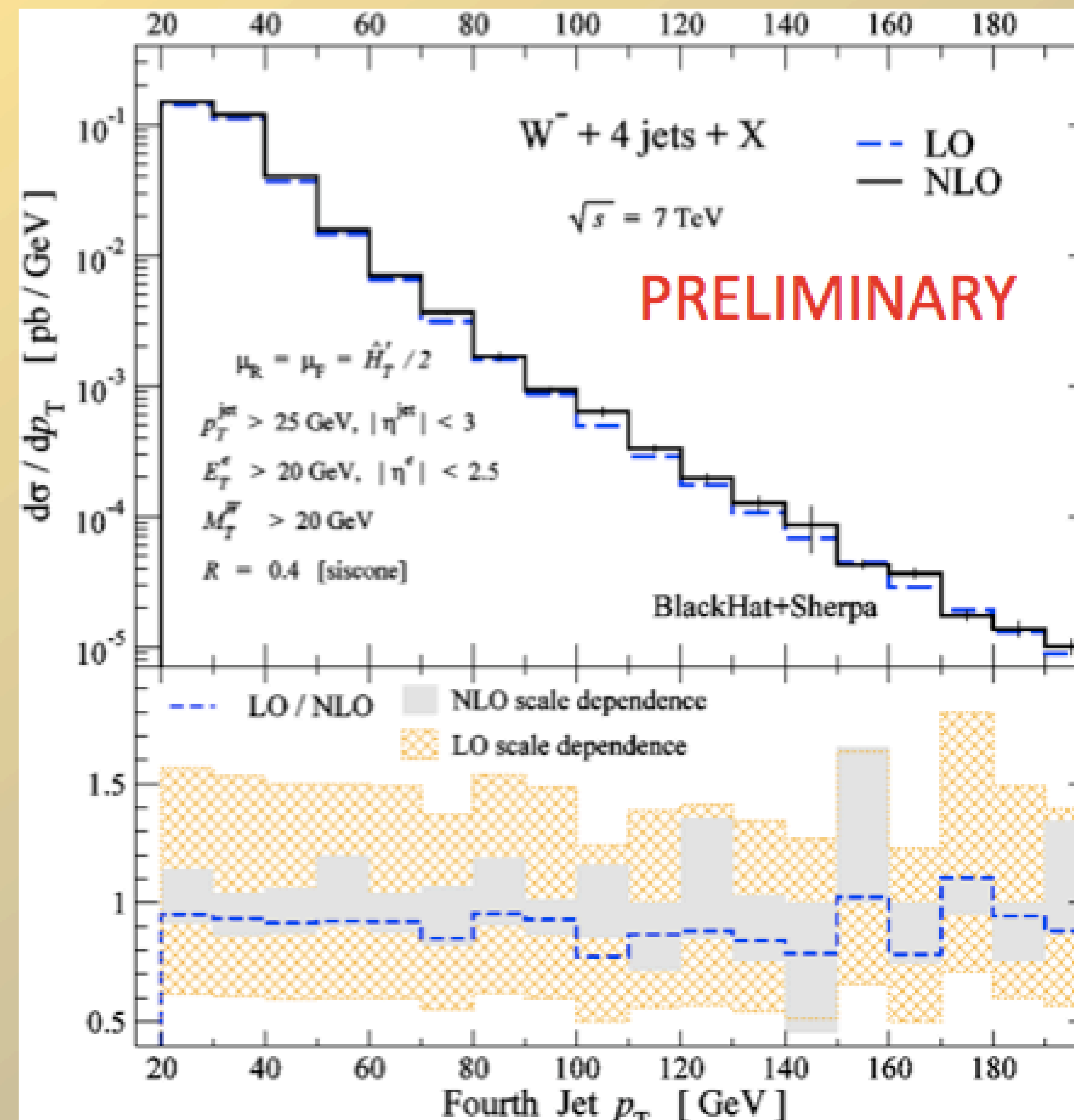
$E_T^{e} > 20$ GeV $|\eta_e| < 2.5$

$M_{TW} > 20$ GeV $R > 0.4$[siscone]

Leading color approximation:

- *8-point* virtual amplitudes
- Off-shell W
- in virtual keep up to n_f/N_c terms drop order $1/(N_c)^2$ terms and 6q real contribution

Under good control for physics!



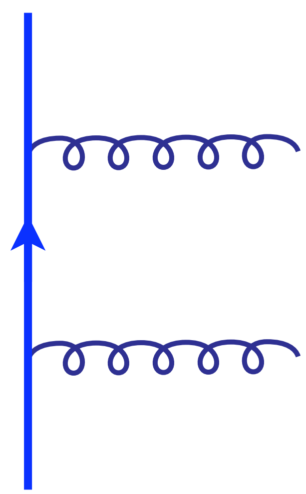
4^{th} jet p_T

How is approx. made?

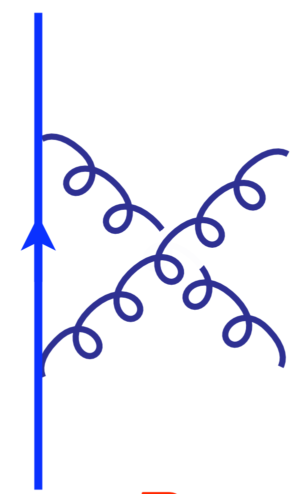
What is being dropped?

Simpler example

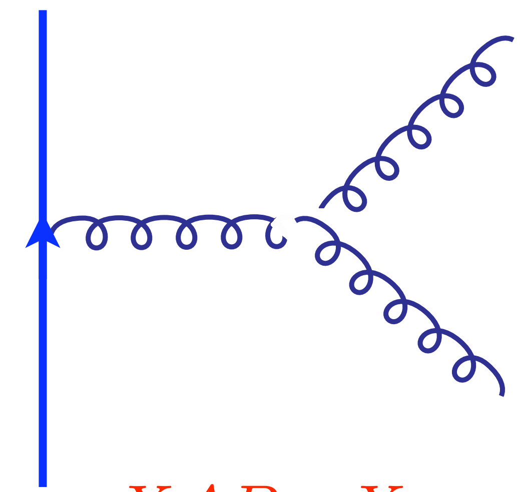
- quark+antiquark → W + 2 gluons is enough to see the main features.
  - in fact, we will drop the W in the pictures, since it is color-neutral.
- There are then three types of contribution, with the following color diagrams:



$\mathcal{C}_1 : T^A T^B$



$\mathcal{C}_2 : T^B T^A$

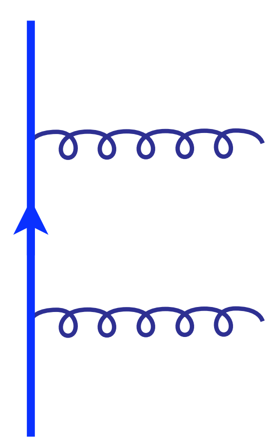


$\mathcal{C}_3 : f^{XAB} T^X$

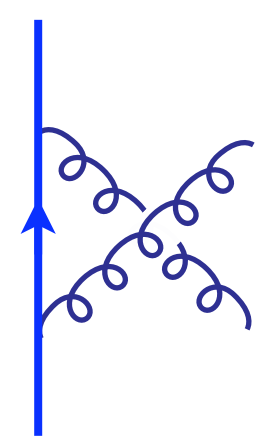
$= T^A T^B - T^B T^A$

- Hence we can already simplify our calculation to:

$\mathcal{C}_1 + \mathcal{C}_3 :$



$\mathcal{C}_2 - \mathcal{C}_3 :$

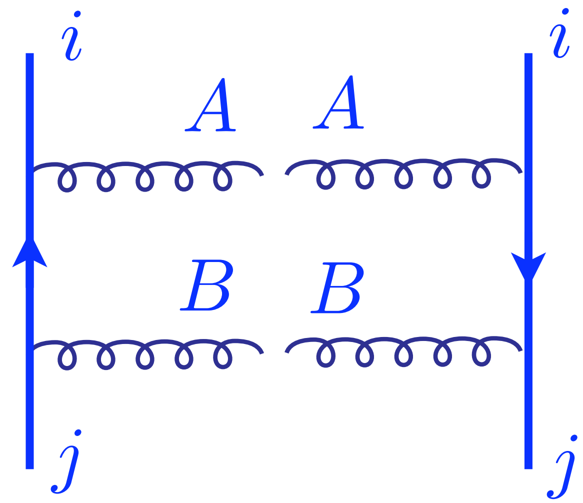


"color-ordered amplitudes"

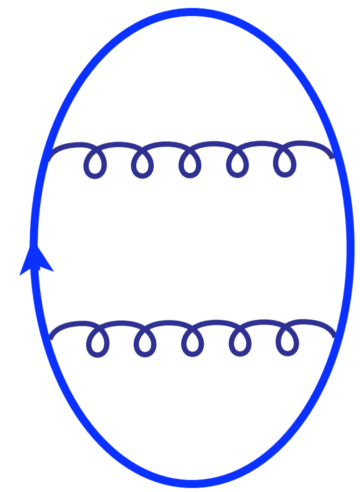
Color factors

- To compute the cross section we need the amplitude squared.

$|\mathcal{C}_1 + \mathcal{C}_3|^2$:

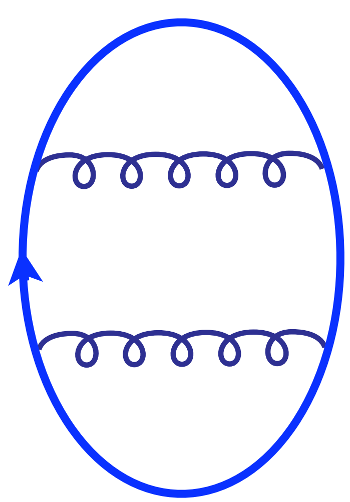


$=$

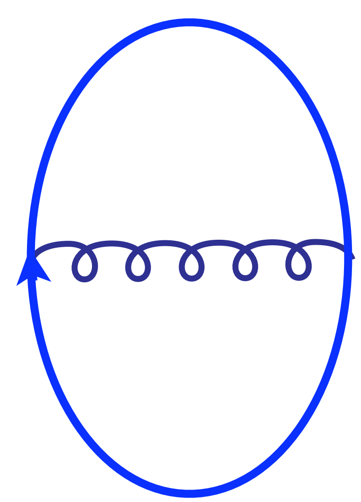


$\left[\text{recall: } (T^A_{ij})^* = T^A_{ji}\right]$

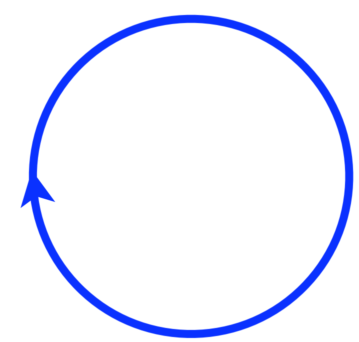
- Now we simplify using our pictorial rules:



$= C_F$



$= C_F^2$



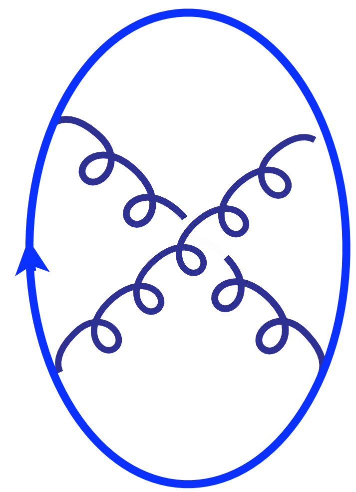
$= N_c C_F^2$

(same for $|\mathcal{C}_2 - \mathcal{C}_3|^2$)

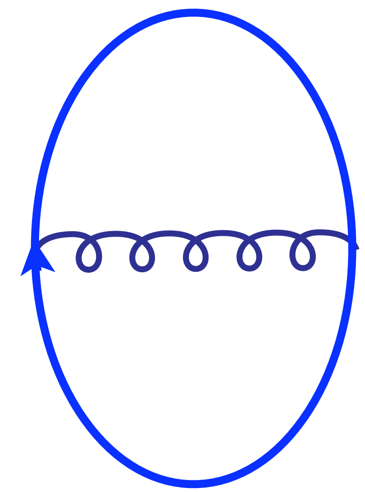
Color factors

- The interference term is a little more complicated (use Fierz).

$(\mathcal{C}_1 + \mathcal{C}_3)(\mathcal{C}_2 - \mathcal{C}_3)^* :$



$= -\dfrac{1}{2N_c}$



$= -\dfrac{C_F}{2}$

- Sum all contributions, keeping one overall factor of C_F but expanding other.

$$\frac{N_c^2 C_F}{2}\left(|\mathcal{C}_1 + \mathcal{C}_3|^2 + |\mathcal{C}_2 - \mathcal{C}_3|^2 - \frac{1}{N_c^2}|\mathcal{C}_1 + \mathcal{C}_2|^2 \right)$$

this is the leading-color contribution

sub-leading: does not contain any remnant of the triple-gluon diagrams (i.e. QED-like)

(color-ordered contributions)

Recap

- The role of color in the theory of QCD is experimentally measurable.
  - good evidence for $N_c=3$.
- The Lagrangian of QCD is based on the SU(3) gauge group.
  - QCD interactions can be represented by a relatively short list of Feynman rules, which can be read off from the Lagrangian.
  - color leads to self-interaction between gluons (triple- and 4-gluon) vertices.
  - more profound differences between QCD and QED we will discuss later.
- Accounting for color is performed using Gell-Mann matrices, whose properties can be used to write amplitudes in terms of color factors $C_F=4/3$ and $C_A=N_c=3$.
  - a pictorial method for computing color factors is a handy tool.