GENEVA Monte Carlo: status and new developments

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LoopFest XIII

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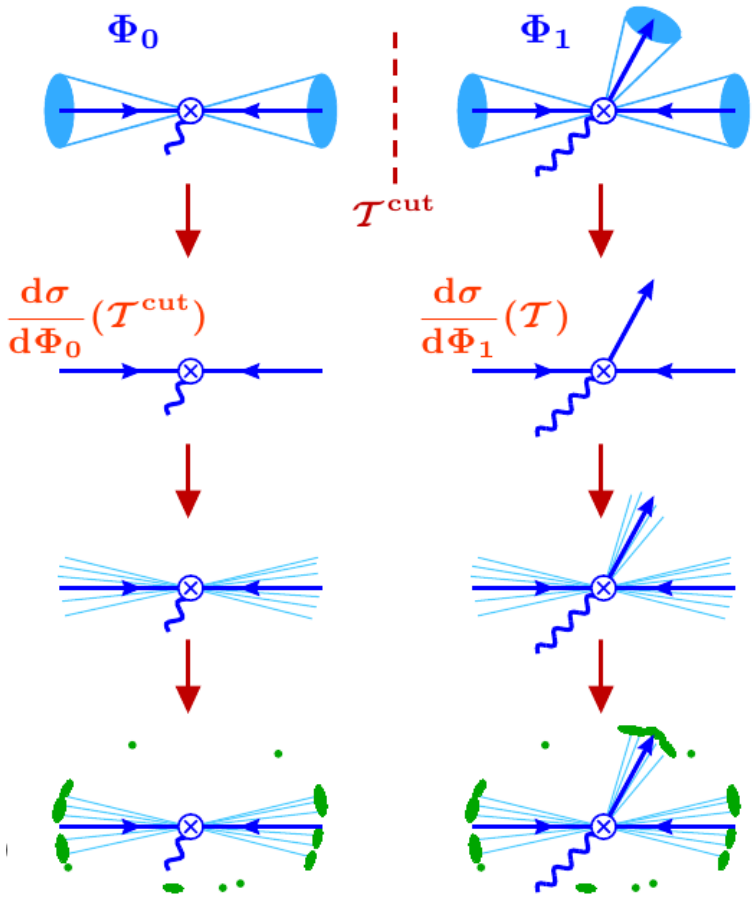
arXiv:1211.7049 SA, C. Bauer, C. Berggren, A. Hornig, F. Tackmann, C. Vermilion, J. Walsh, S. Zuberi
arXiv:1311.0286 SA, C. Bauer, C. Berggren, F. Tackmann, J. Walsh, S. Zuberi

GENEVA

Combines 3 key ingredients in a single framework:

1. Fully Exclusive NLO Calculations
   - NLO_N, NLO_{N+1}, ...
2. Higher-order Resummation (using SCET, but not limited to it)
   - $\text{LL}_{\mathcal{O}}$, $\text{NLL}_{\mathcal{O}}$, $\text{NLL'}_{\mathcal{O}}$, $\text{NNLL}_{\mathcal{O}}$ ...
3. Parton Shower and Hadronization
   - Pythia8, Herwig++, ...

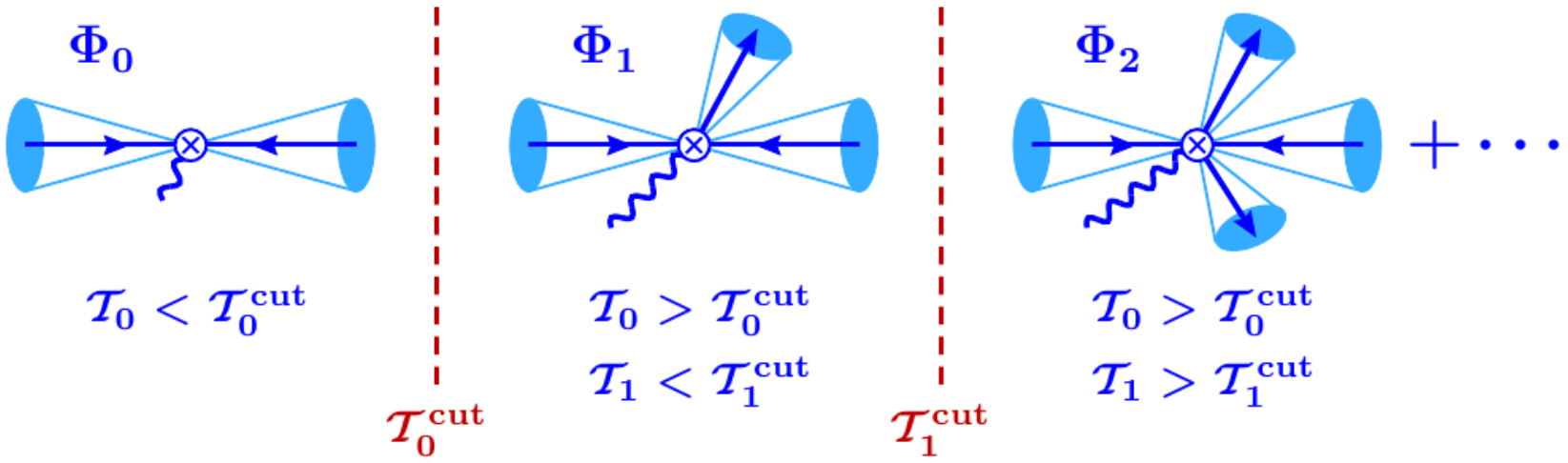


GENEVA guiding principle

Give a coherent description at the **Next-to-Lowest perturbative accuracy** in both fixed-order perturbation theory and logarithmic resummation, including **event-by-event theoretical uncertainties**, and combine it with parton shower and hadronization.

Recipe for an IR-safe definitions of events beyond LO

- Only generate "physical events", i.e. events to which one can assign an IR-finite cross section $\mathrm{d}\sigma^{\mathrm{MC}}$.
- Introduce a resolution parameter $\mathcal{T}_N$, $\mathcal{T}_N \to 0$ in the IR region. Emissions below $\mathcal{T}_N^{\mathrm{cut}}$ are unresolved (i.e. integrated over) and the kinematic considered is the one of the event before the emission.
- An M-parton event is thus really defined as an N-jet event, $N \leq M$, fully differential in Φ_N (standard "jet-algo" not needed)
  - Price to pay: corrections in powers of $\mathcal{T}_N^{\mathrm{cut}}$ due to PS projection.
  - Advantage: vanish for IR-safe observables as $\mathcal{T}_N^{\mathrm{cut}} \to 0$
- Iterating the procedure, the phase space is sliced into jet-bins



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Inclusive N-jet bin

$$\frac{\mathrm{d}\sigma^{\mathsf{MC}}_{\geq N}}{\mathrm{d}\Phi_N}$$

$\mathcal{T}_0^{\mathrm{cut}}$ $\quad \mathcal{T}_1 < \mathcal{T}_1$ $\quad \mathcal{T}_1^{\mathrm{cut}}$ $\quad \mathcal{T}_1 > \mathcal{T}_1$

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Exclusive N-jet bin

$$\frac{\mathrm{d}\sigma_N^{\mathsf{MC}}}{\mathrm{d}\Phi_N}(\mathcal{T}_N^{\mathrm{cut}})$$

Inclusive $(N+1)$-jet bin

$$\frac{\mathrm{d}\sigma_{\geq N+1}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}})$$

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Excl. $(N+1)$-jet

$$\frac{\mathrm{d}\sigma_{N+1}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}}; \mathcal{T}_{N+1}^{\mathrm{cut}})$$

Inclusive $(N+2)$-jet bin

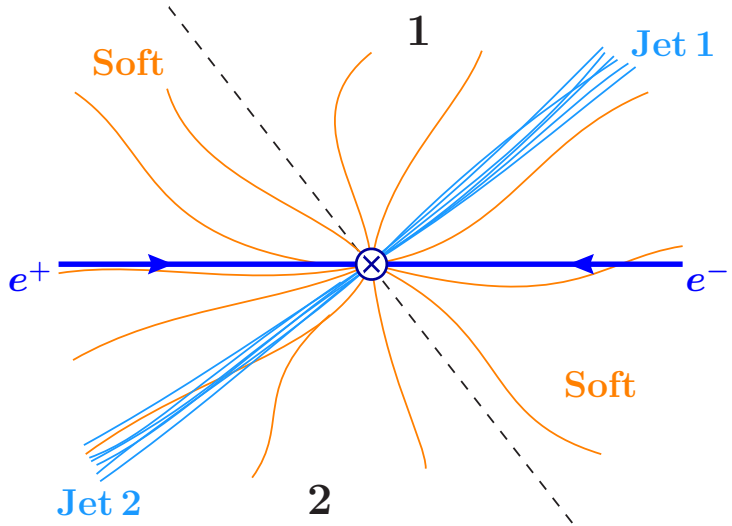
$$\frac{\mathrm{d}\sigma_{\geq N+2}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+2}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}}, \mathcal{T}_{N+1} > \mathcal{T}_{N+1}^{\mathrm{cut}})$$

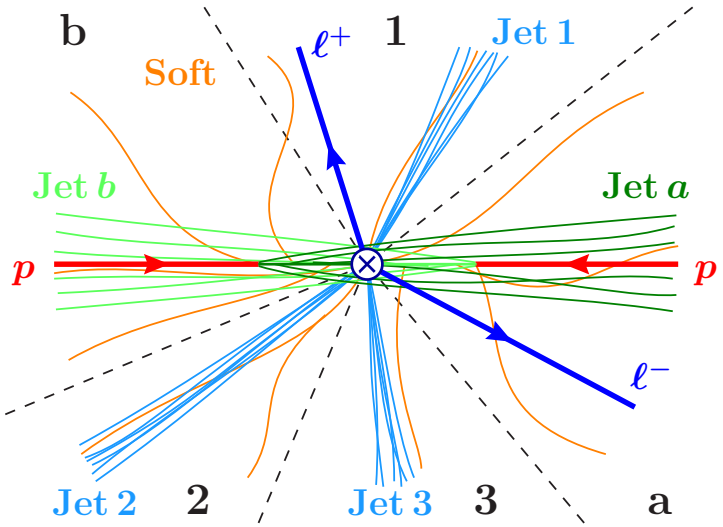
$\mathcal{T}_0^{\mathrm{cut}}$ $\mathcal{T}_1^{\mathrm{cut}}$

Jet-resolution variable

- Use N-jettiness as resolution parameter. Global physical observable with straightforward definitions for hadronic colliders, in terms of beams $q_{a,b}$ and jet-directions q_j

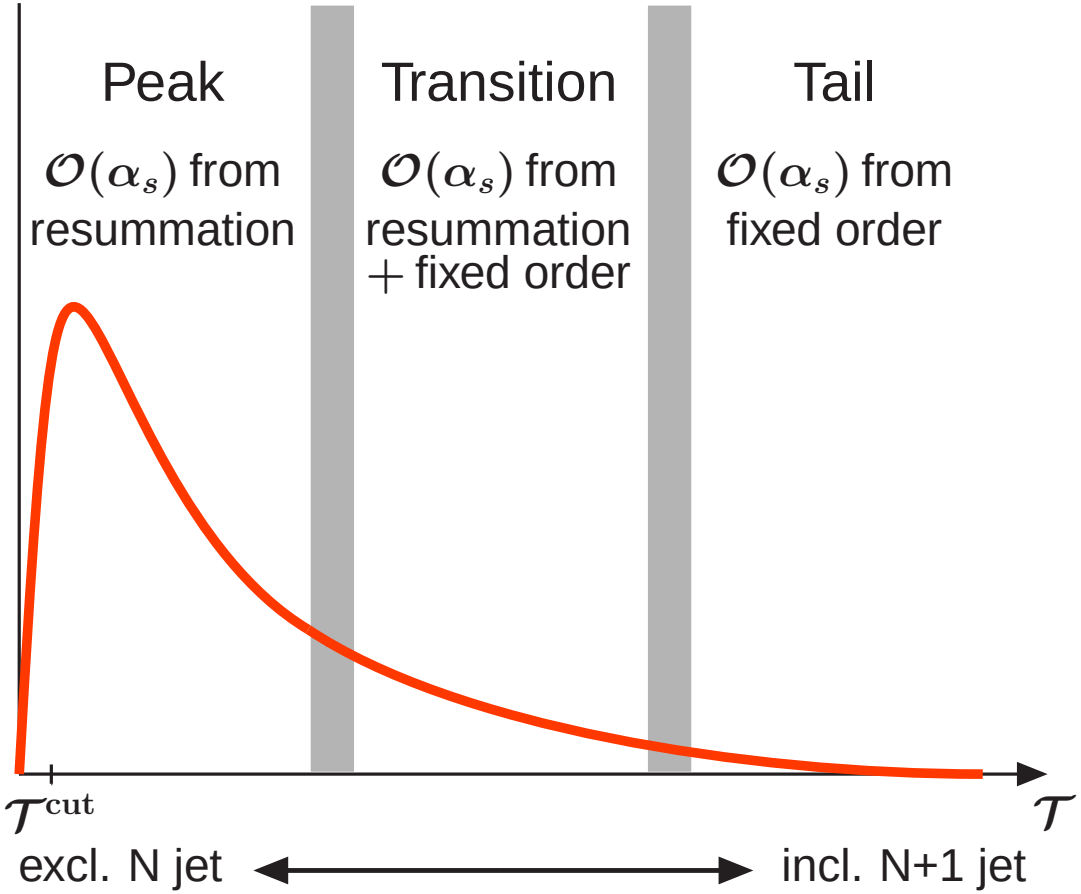
$$\mathcal{T}_N = \frac{2}{Q}\sum_k \min\{q_1\cdot p_k,\,\ldots,\,q_N\cdot p_k\} \;\Rightarrow\; \mathcal{T}_N = \frac{2}{Q}\sum_k \min\{q_a\cdot p_k,\,q_b\cdot p_k,\,q_1\cdot p_k,\,\ldots,\,q_N\cdot p_k\}$$





- N-jettiness has good factorization properties, IR safe and resummable at all orders. Resummation known at NNLL for any N in SCET [Stewart et al. 1004.2489, 1102.4344]
- $\mathcal{T}_N \to 0$ for N pencil-like jets, $\mathcal{T}_N \gg 0$ spherical limit.
- Framework not limited by resolution parameter choice: other options currently being investigated.

Perturbative accuracy



- Fixed NLO$_{N+1}$ calculations are insufficient outside FO region
- Lowest accuracy across the whole spectrum in MEPS: CKKW, MLM
- Standard NLO+PS (POWHEG, MC@NLO) improve total rate, not spectrum

Perturbative accuracy

(Notation: $\tau = \mathcal{T}/Q$, $L = \ln\tau$, $L_{\rm cut} = \ln\tau^{\rm cut}$)

$$
\begin{array}{rcccccccccl}
 & & \mathrm{LL}_\sigma & & \mathrm{NLL}_\sigma & & \mathrm{NLL}'_\sigma & & \mathrm{NNLL}_\sigma & & \\
\dfrac{\sigma(\tau^{\rm cut})}{\sigma_B} & = & 1 & & & & & & & & \mathrm{LO_N} \\
 & + \alpha_s\Big[& \frac{c_{11}}{2}L_{\rm cut}^2 & + & c_{10}L_{\rm cut} & + & c_{1,-1} & + & & F_1(\tau^{\rm cut})\Big] & \mathrm{NLO_N} \\
 & + \alpha_s^2\Big[& \vdots & + & \vdots & + & \vdots & + & \vdots & & \\
\dfrac{1}{\sigma_B}\dfrac{d\sigma}{d\tau} & = \alpha_s/\tau\Big[& c_{11}L & + & c_{10} & + & & & & \tau f_1(\tau)\Big] & \mathrm{LO_{N+1}} \\
 & + \alpha_s^2/\tau\Big[& c_{23}L^3 & + & c_{22}L^2 & + & c_{21}L & + & c_{20} & + \tau f_2(\tau)\Big] & \mathrm{NLO_{N+1}} \\
 & + \alpha_s^3/\tau\Big[& \vdots & + & \vdots & + & \vdots & + & \vdots & &
\end{array}
$$

- Lowest pert. accuracy everywhere requires $\mathrm{NLL}_{\mathcal{T}}$ + LO_{N+1}
  - NLL because $\alpha_{\rm s}(\alpha_{\rm s}^n L^{2n}) \approx \alpha_{\rm s}$ in the peak.
- Next-to-Lowest pert. accuracy everywhere requires $\mathrm{NNLL}_{\mathcal{T}}$ + NLO_{N+1}

GENEVA Master Formula

arXiv:1211.7049

$$\sigma_{\geq N} = \int d\Phi_N \, \frac{d\sigma}{d\Phi_N}(\mathcal{T}^{\rm cut}) + \int d\Phi_{N+1} \, \frac{d\sigma}{d\Phi_{N+1}}(\mathcal{T}) \, \theta(\mathcal{T} > \mathcal{T}^{\rm cut})$$

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- Cumulant: $\mathcal{T}$ integral over exclusive N-jets bin up to $\mathcal{T}^{\rm cut}$

$$\frac{d\sigma}{d\Phi_N}(\mathcal{T}^{\rm cut}) = \frac{d\sigma^{\rm resum}}{d\Phi_N}(\mathcal{T}^{\rm cut}) + \left[\frac{d\sigma^{\rm FO}}{d\Phi_N}(\mathcal{T}^{\rm cut}) - \frac{d\sigma^{\rm resum}}{d\Phi_N}(\mathcal{T}^{\rm cut})\Big|_{\rm FO}\right]$$

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- Spectrum: $\mathcal{T}$ distribution of inclusive $N+1$-jets sample above $\mathcal{T}^{\rm cut}$

$$\frac{d\sigma}{d\Phi_{N+1}}(\mathcal{T}) = \frac{d\sigma^{\rm FO}}{d\Phi_{N+1}}(\mathcal{T}) \left[\frac{d\sigma^{\rm resum}}{d\Phi_N d\mathcal{T}} \Big/ \frac{d\sigma^{\rm resum}}{d\Phi_N \, d\mathcal{T}} \Big|_{\rm FO} \right]$$

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- Correctly reproduces the expected limits for $\mathcal{T} \to 0$ and $\mathcal{T} \sim Q$.
- Enforcing that leftover dependence on $\mathcal{T}^{\rm cut}$ drops at the order we are working: spectrum is derivative of the cumulant

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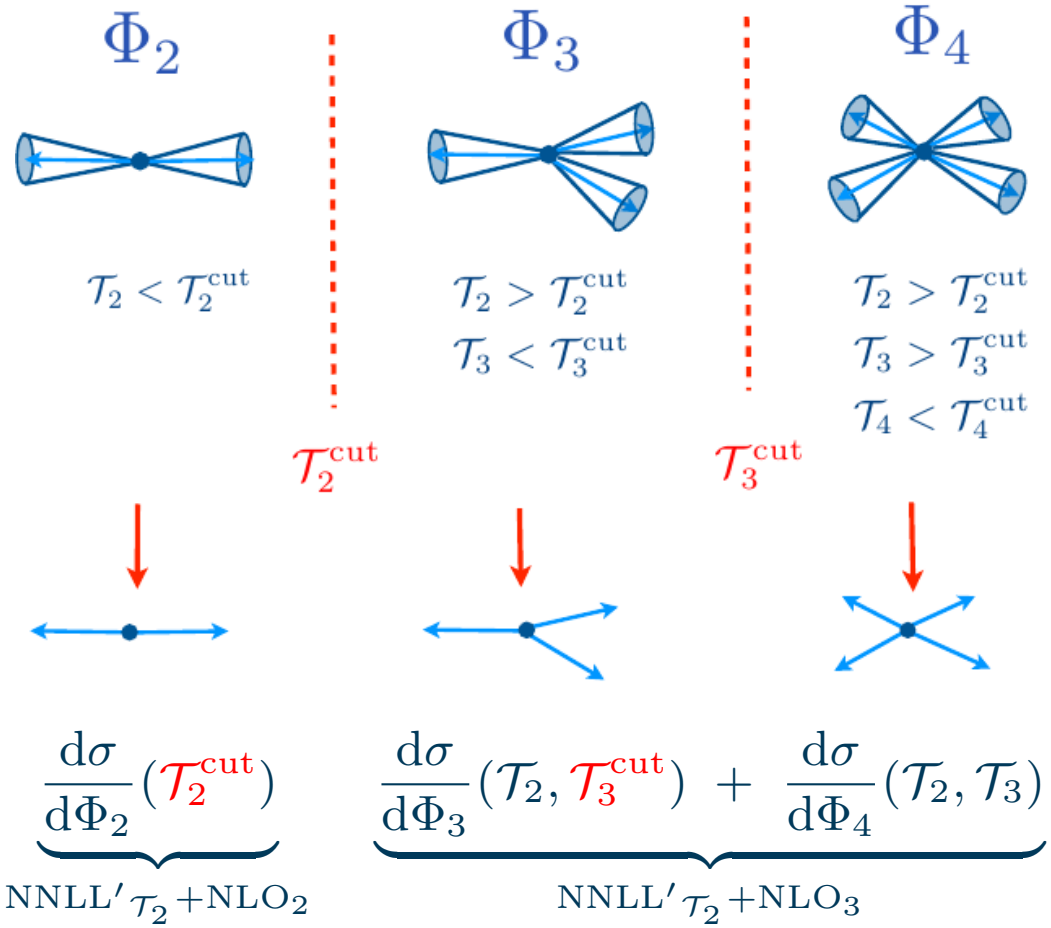
- MonteCarlo's perspective: increases SMC resummation while including multiple NLO.
- Resummation's perspective: takes the resummation of $\mathcal{T}$ and produces fully differential results.

First application: $e^+e^- \to$ jets



- Use $2-$jettiness and $3-$jettiness as resolution parameters

$$\mathcal{T}_2 = E_{\rm cm}\left(1 - \max_{\hat{n}} \frac{\sum_k |\hat{n}\cdot\vec{p}_k|}{\sum_k |\vec{p}_k|}\right) = E_{\rm cm}(1-T)$$

✓ Simpler process to test our construction.

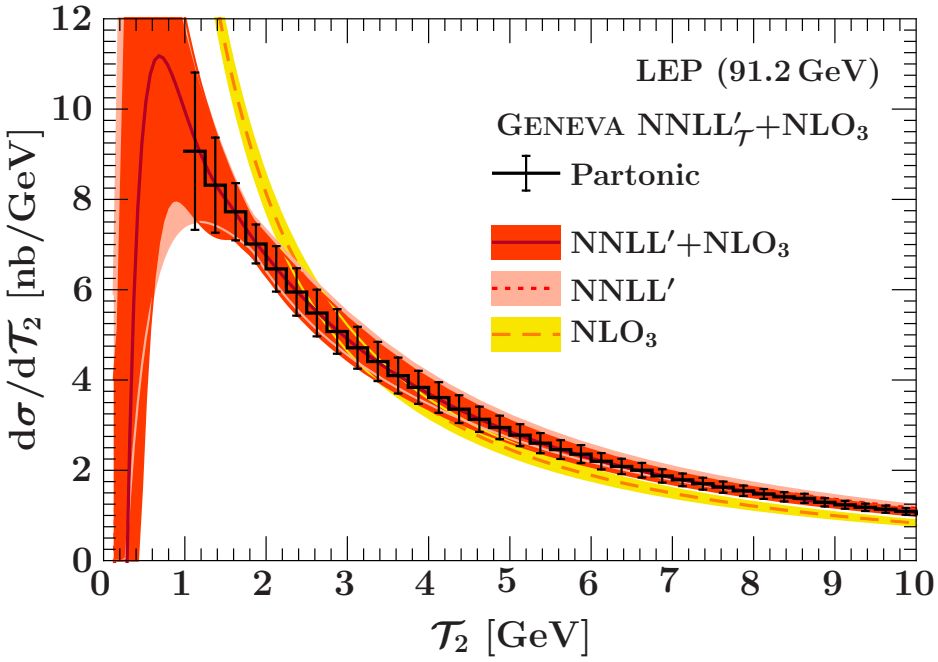
✓ Thrust spectrum known to N^{3}LL'$_{\mathcal{T}}$+ NNLO$_3$.

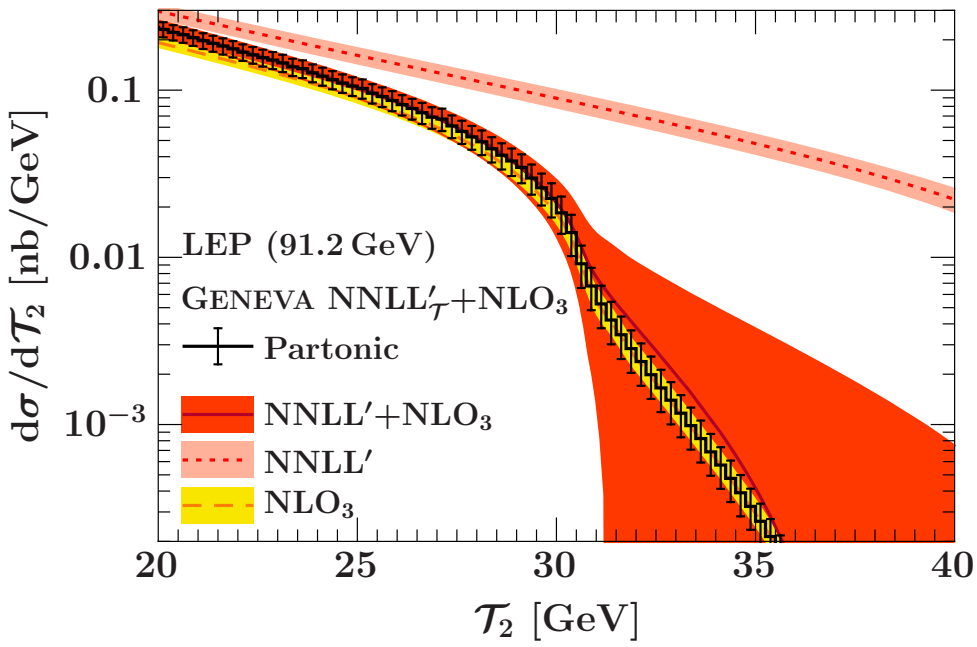
✓ Several 2-jet shapes known to NNLL$_{\mathcal{O}}$+NNLO$_3$.

✓ LEP data available for validation.

Validation of $\mathcal{T}_2$ resummation

- GENEVA precisely reproduces full NNLL'+NLO3 analytic result : simply getting out what we put in!





- Error bars are always theory uncertainties, obtained via scale variations. Statistical uncertainties negligible and not shown.
- Resummation unc. obtained via quadrature sum of single scale variations (μ_S, μ_J) inside profile scale bands plus direct sum of FO uncertainties (μ_H).
- Theoretical uncertainties agree across most of the spectrum, differences after kinematic 3-body endpoint consequence of different matching procedure (multiplicative vs. additive).

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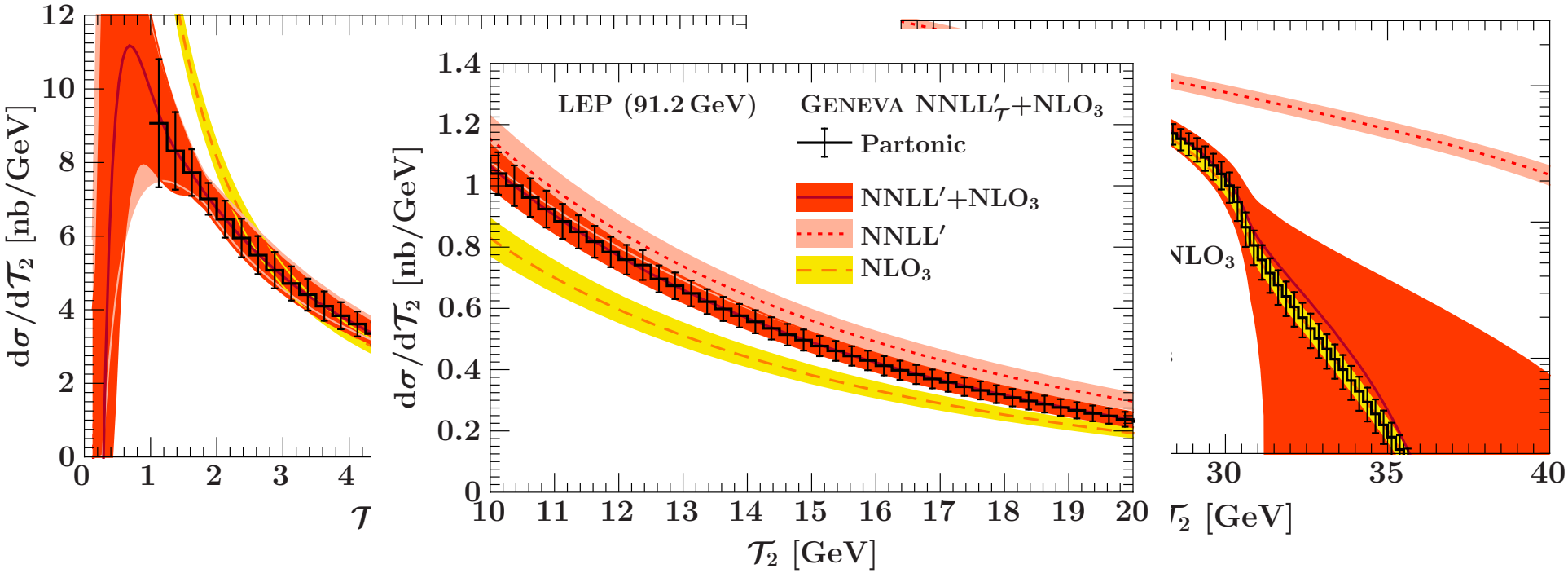

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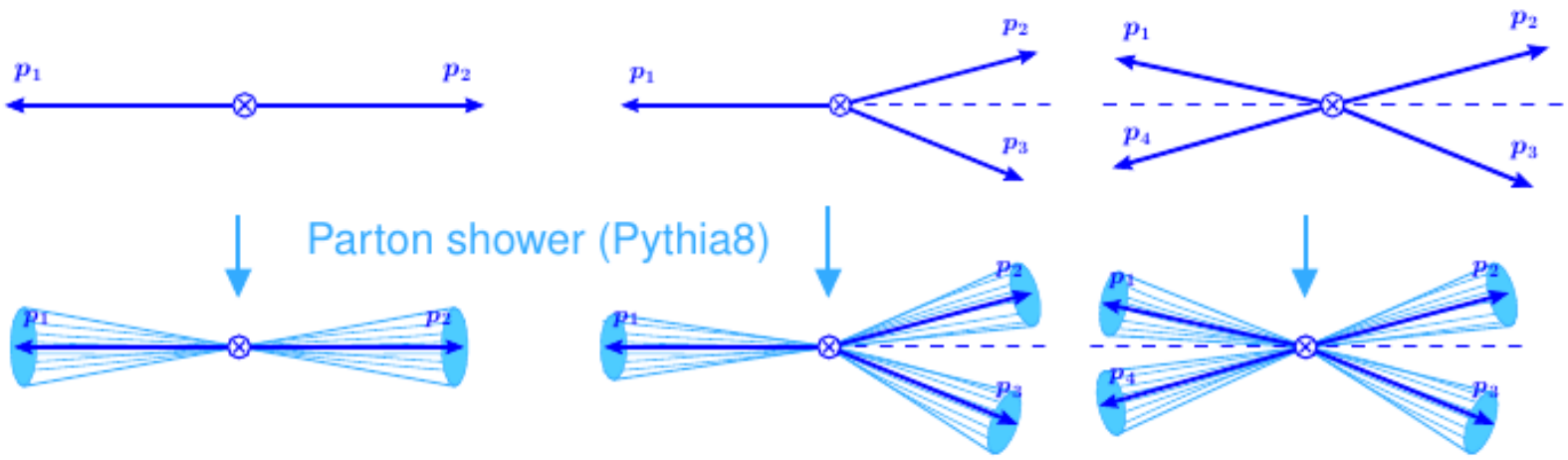
Interface with the parton shower

- The final-state shower must not be allowed to spoil NNLL'$_{\mathcal{T}}$ accuracy of GENEVA, but only used to fill out jets.



- $\mathcal{T}_2$ spectrum for 3 and 4-parton events constrained by higher-order resummation. Only allow small variations $\Delta\mathcal{T}_2 < \mathcal{T}_2^{\rm cut}(1+\epsilon)$.
- 2-parton events must remain in 2-jets bin, up to small corrections
- Similarly for $\mathcal{T}_3(\Phi_4)$ spectrum and 3-jets bin. Proxy for $\mathcal{T}$-ordered PS.
- Shower unconstrained in the far tail at the moment, since only LO$_4$ there.

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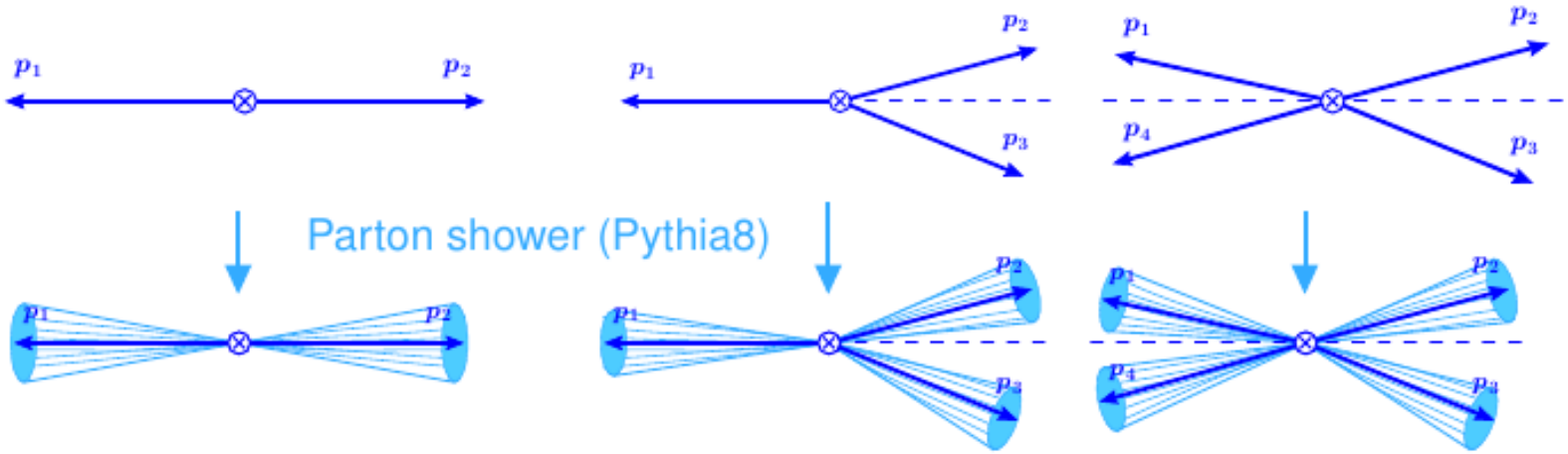

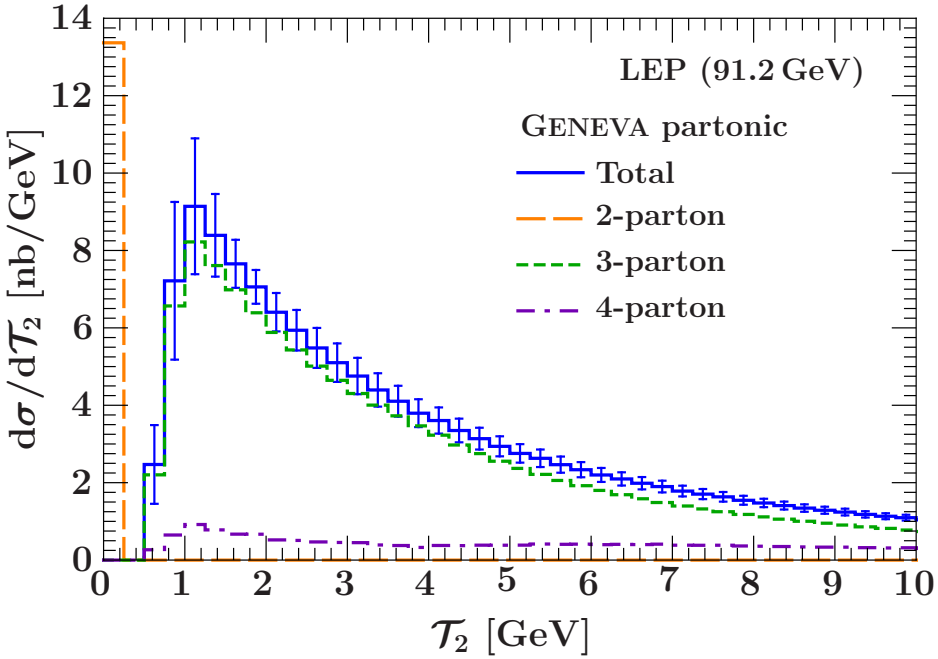

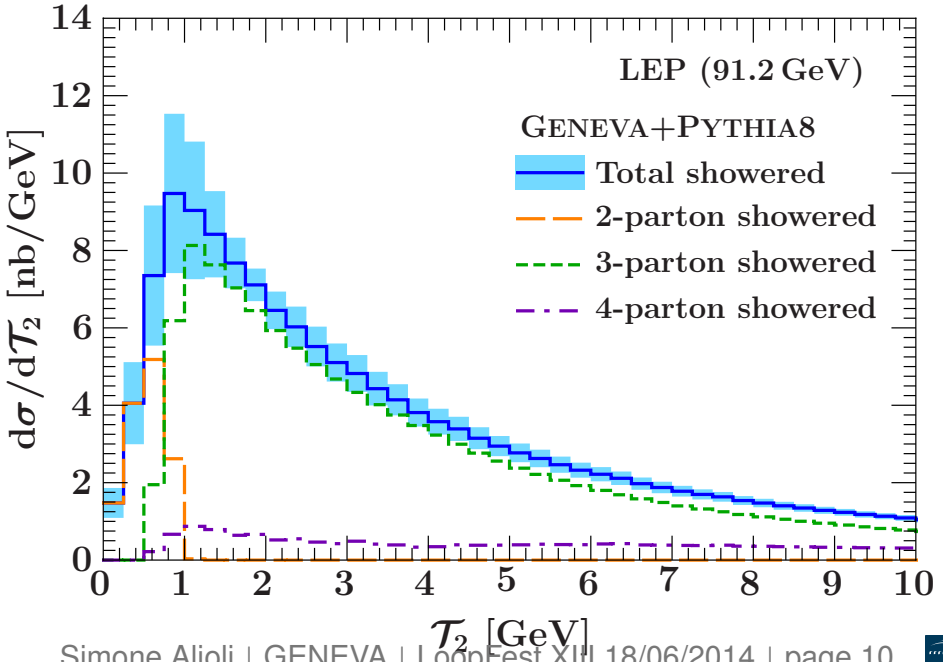

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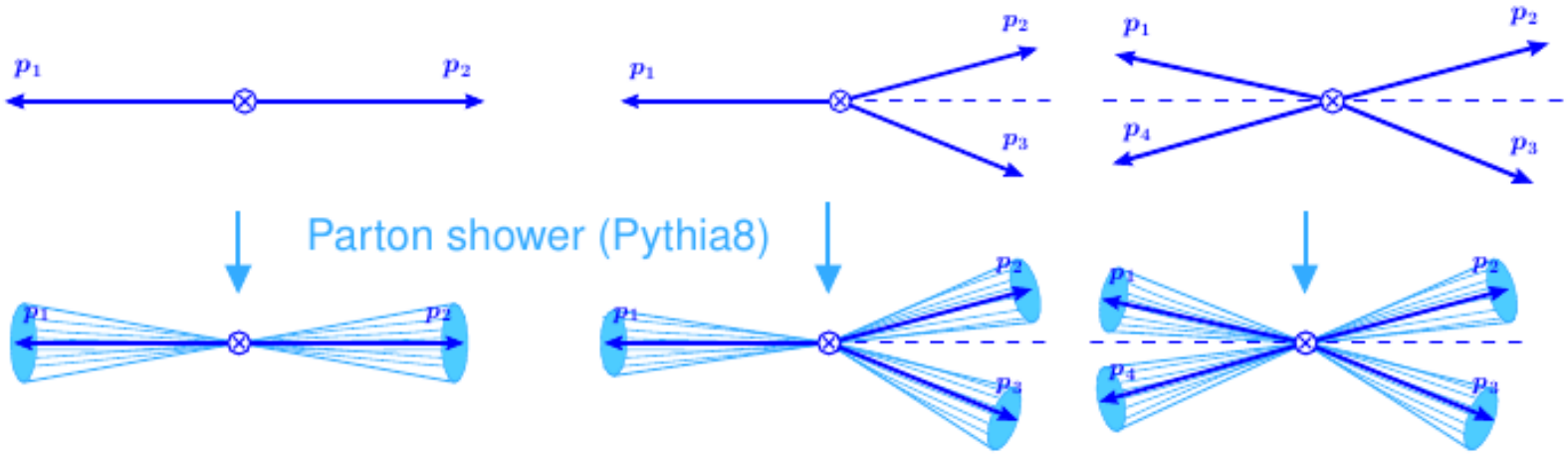
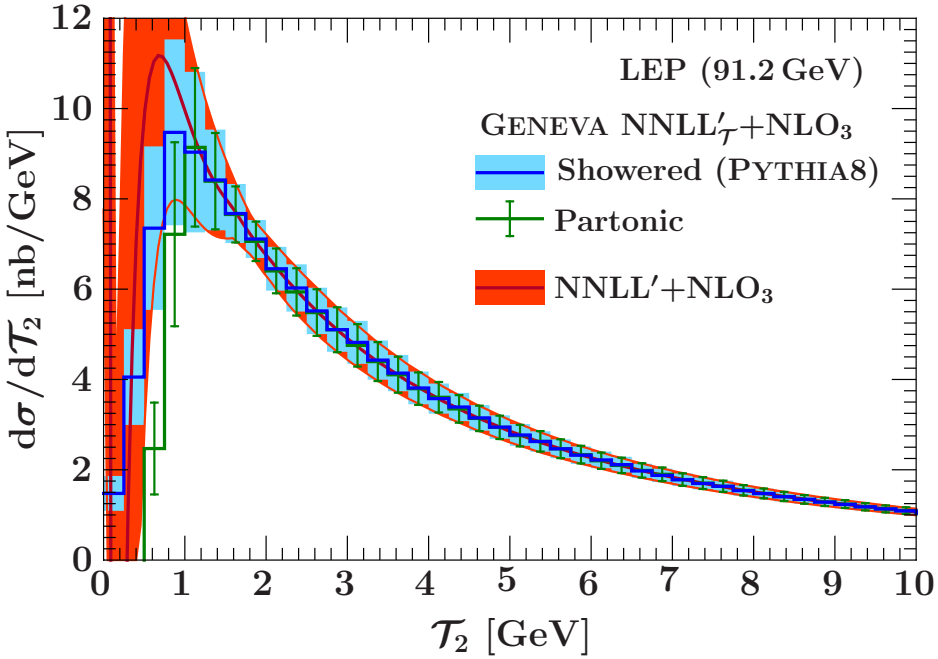
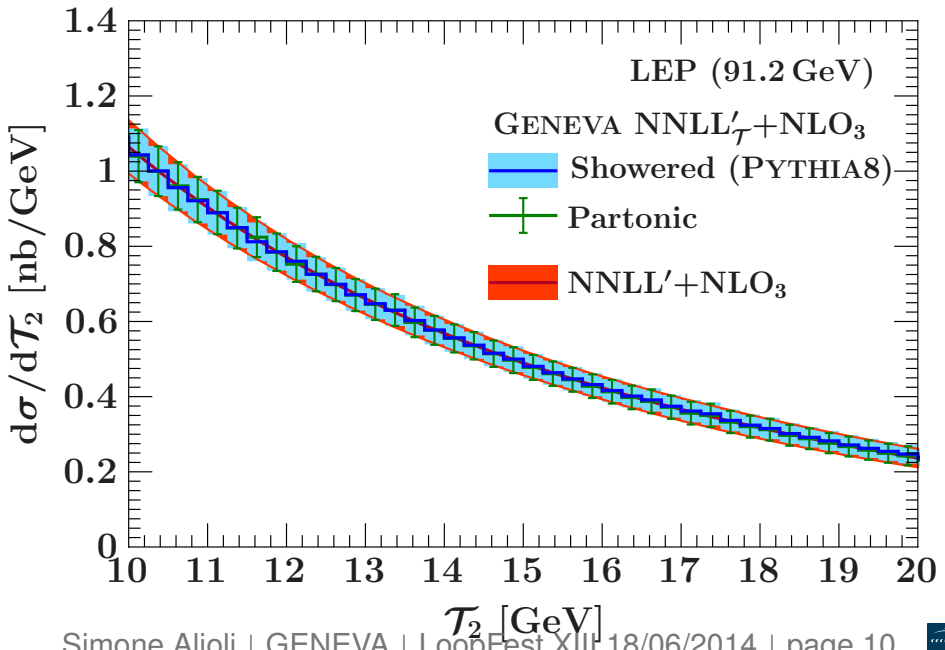
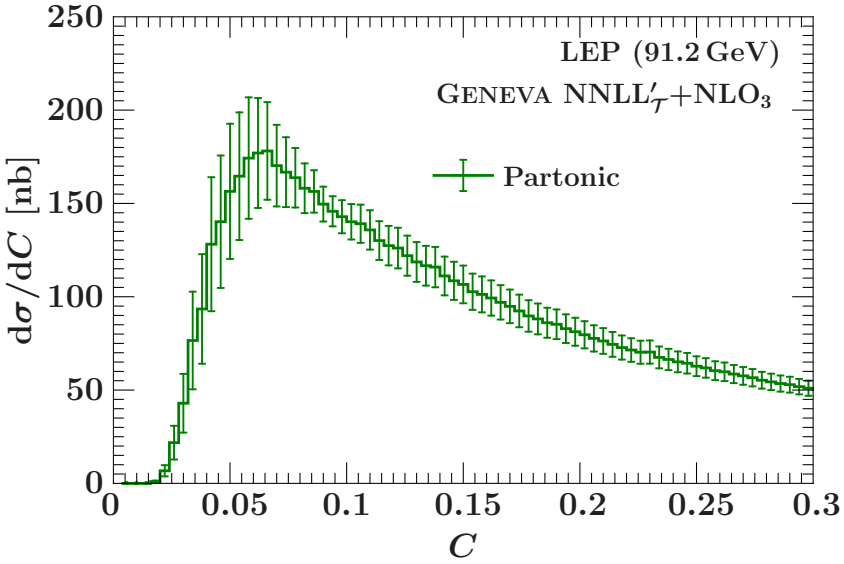


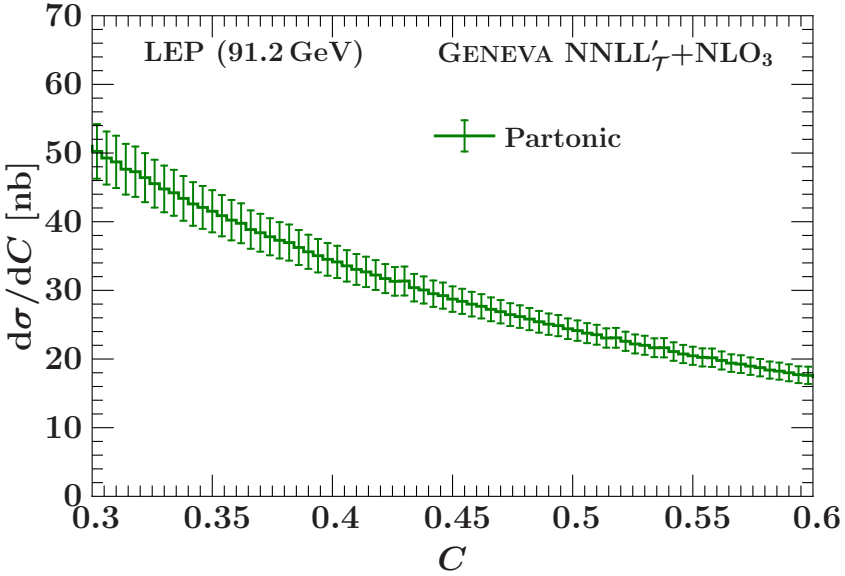




Accuracy of observables different from resolution parameter

- After showering we are formally limited by shower resummation for generic observables $\mathcal{O} \neq \mathcal{T}_2$. Naively, only (N)LL is expected.
- What is the perturbative accuracy we obtain for other $\mathcal{O}$?
- C-parameter – perturbative structure very similar to $\mathcal{T}_2$





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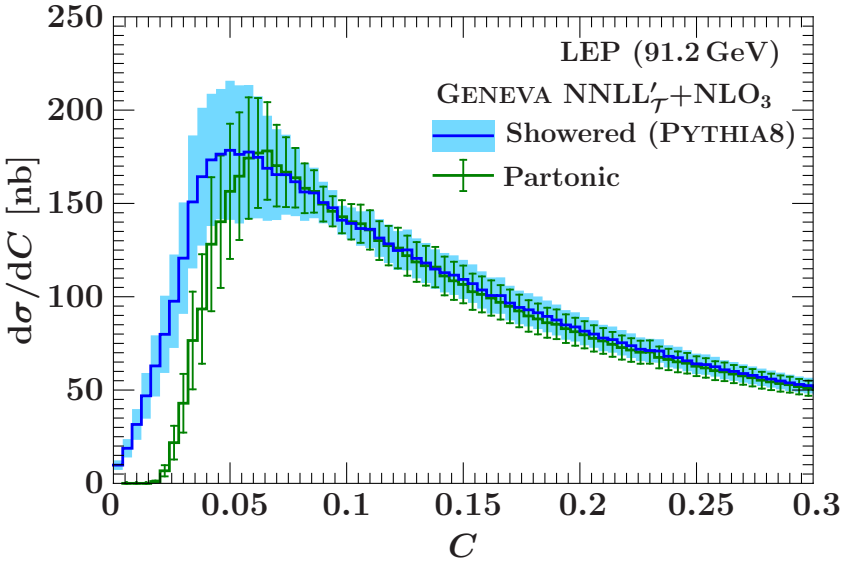

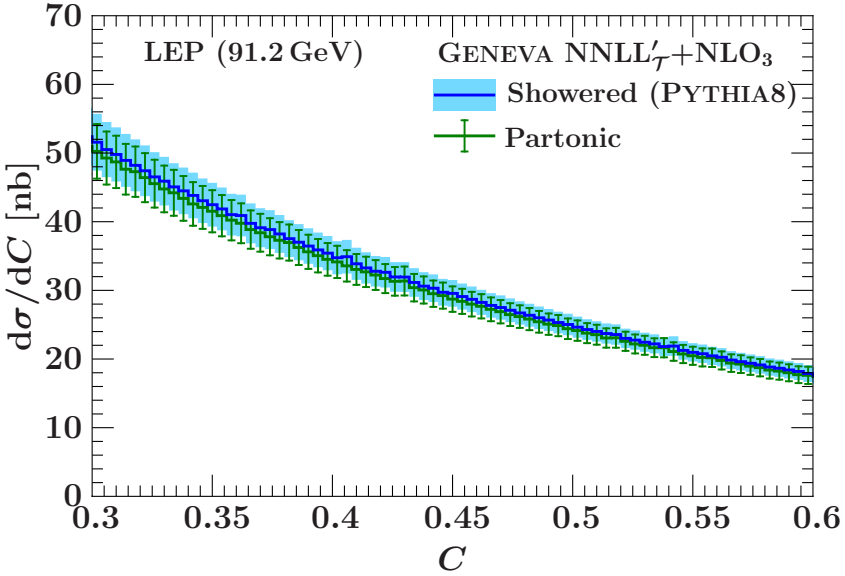

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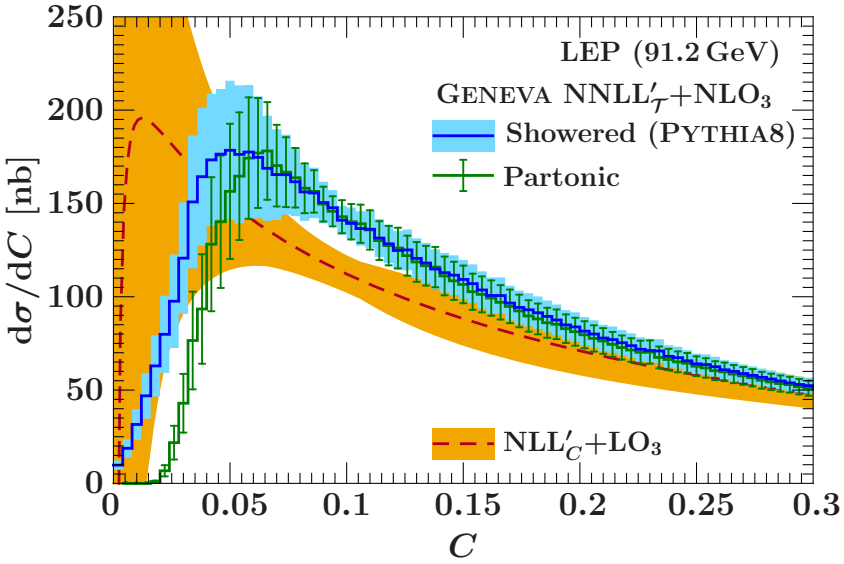

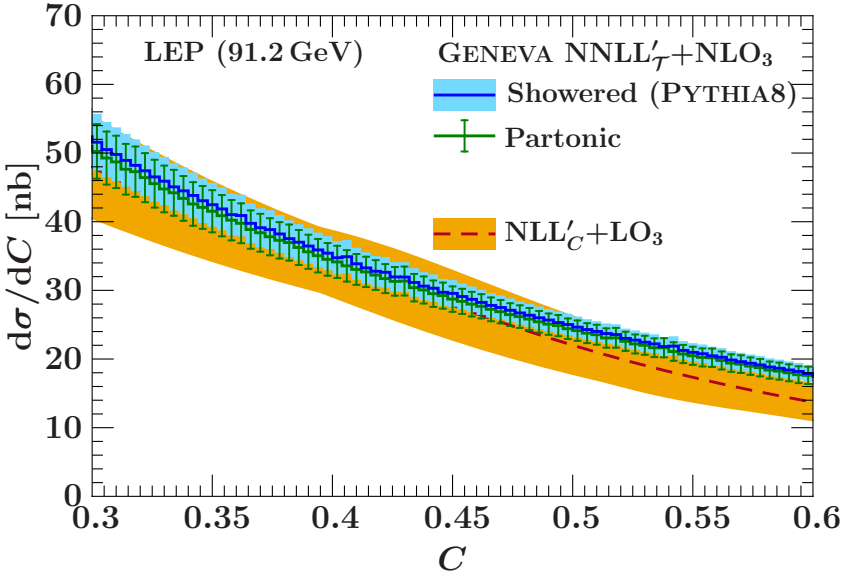

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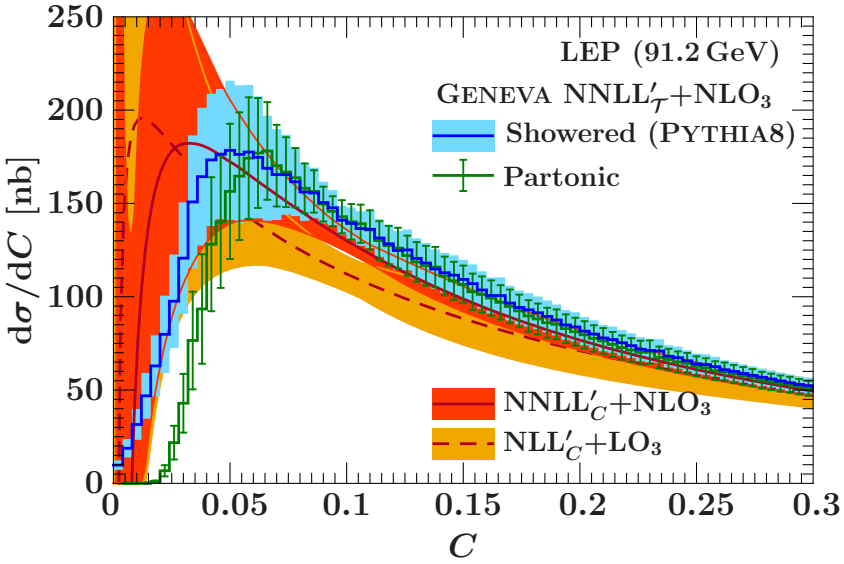
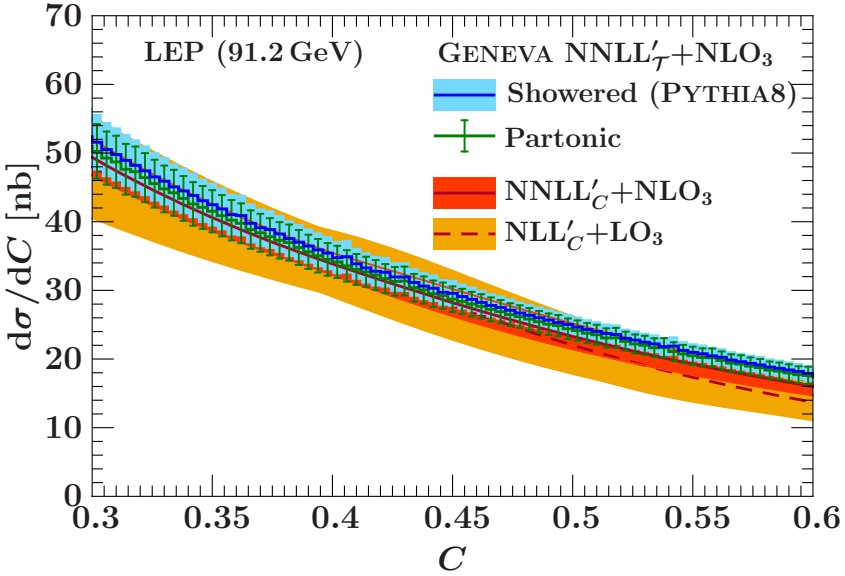
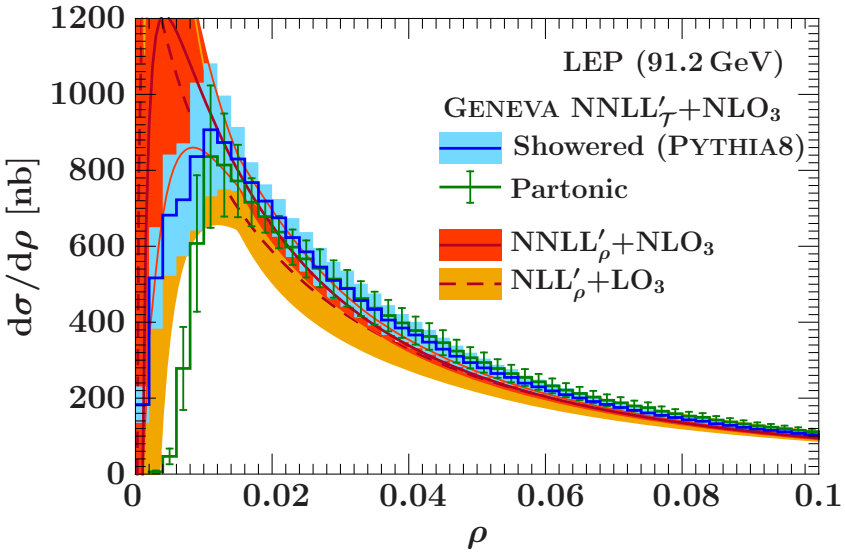
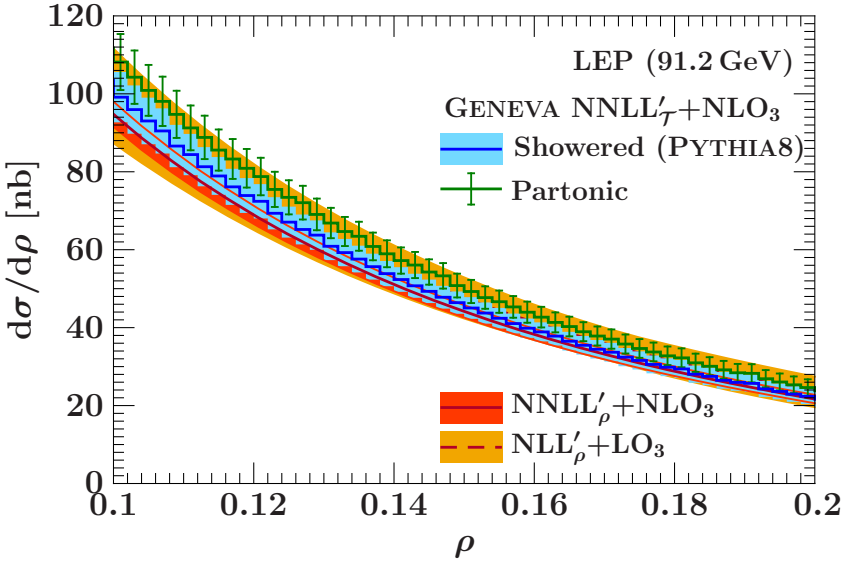




Accuracy of observables different from resolution parameter

- After showering we are formally limited by shower resummation for generic observables $\mathcal{O} \neq \mathcal{T}_2$. Naively, only (N)LL is expected.
- What is the perturbative accuracy we obtain for other $\mathcal{O}$?
- Heavy jet mass – perturbative structure partially related to $\mathcal{T}_2$





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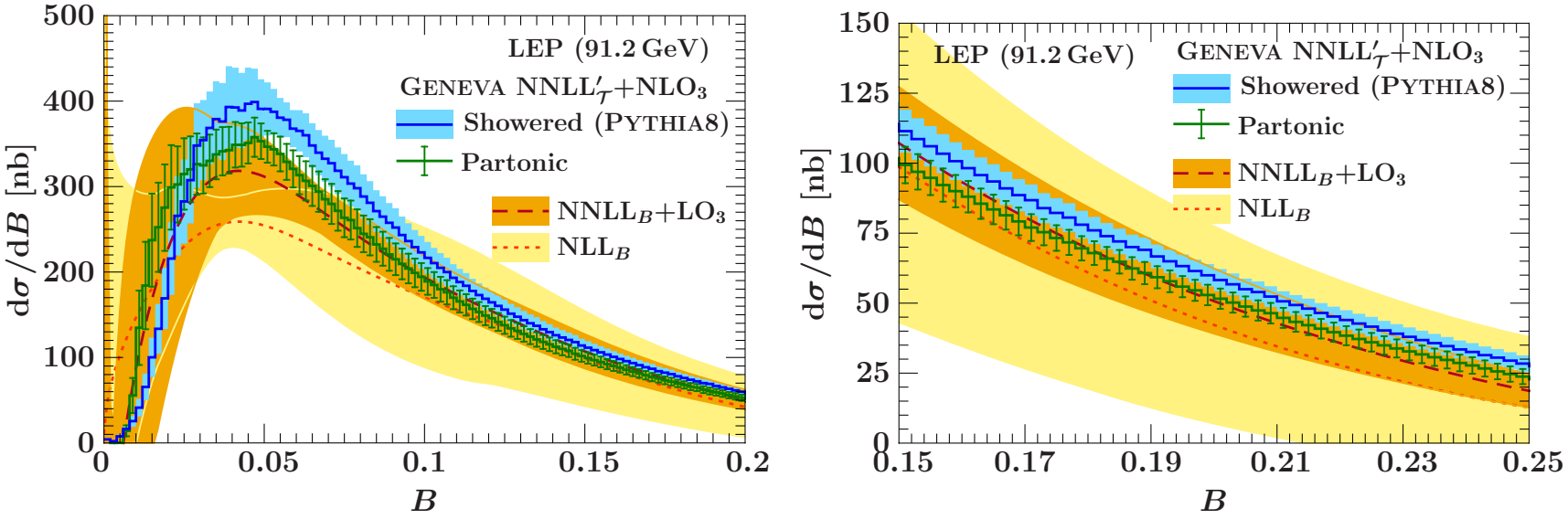
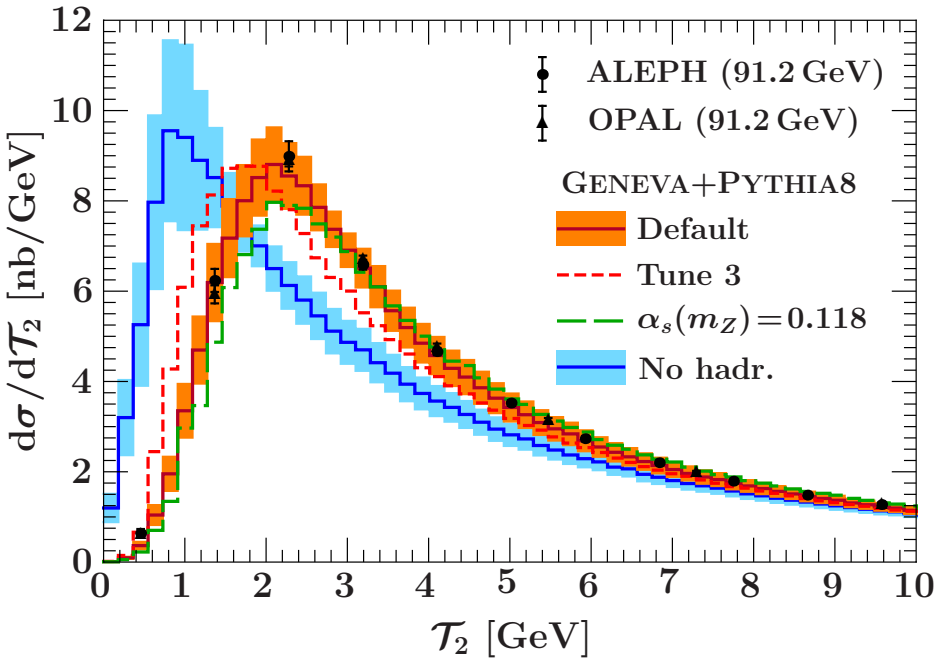
- Jet Broadening – perturbative structure completely different from $\mathcal{T}_2$

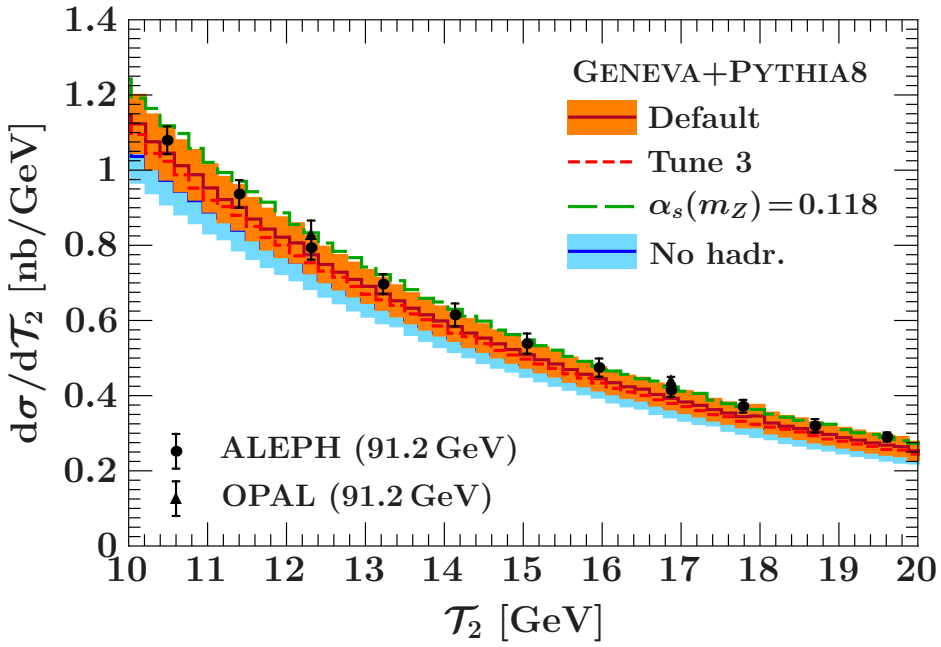


- Good agreement in central values and scale uncertainties envelopes at NNLL, also for observables with a very different resummation structure.
- NNLL resummation allows to push $\mathcal{T}_2^{\rm cut}$ to very small values, effectively replacing the shower evolution.
- Ultimately, we rely on Pythia8 hadronization model for non-pert. physics.

Hadronization and comparison with LEP data.

- Two-jettiness $= E_{\rm cm}(1-T)$





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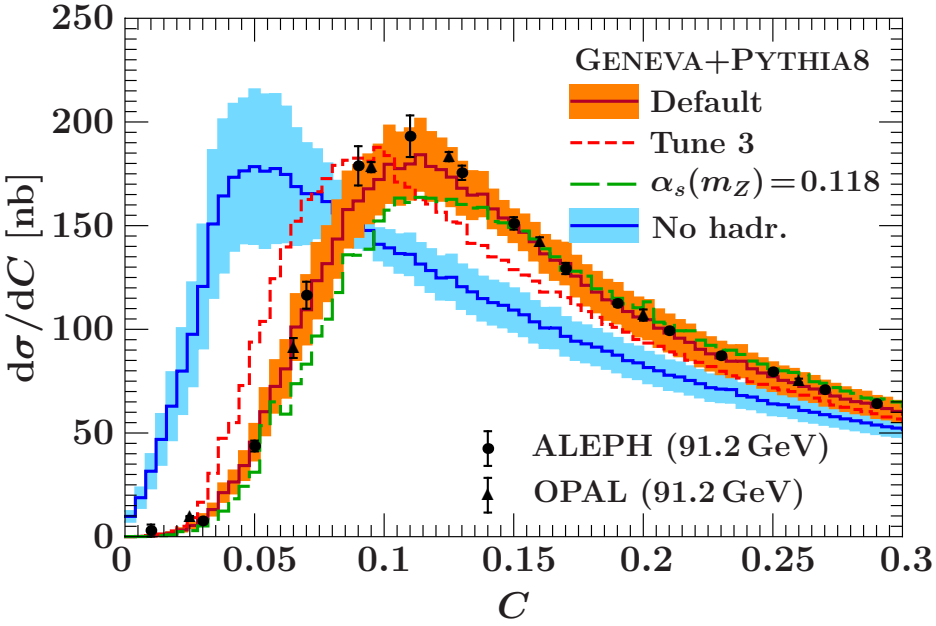

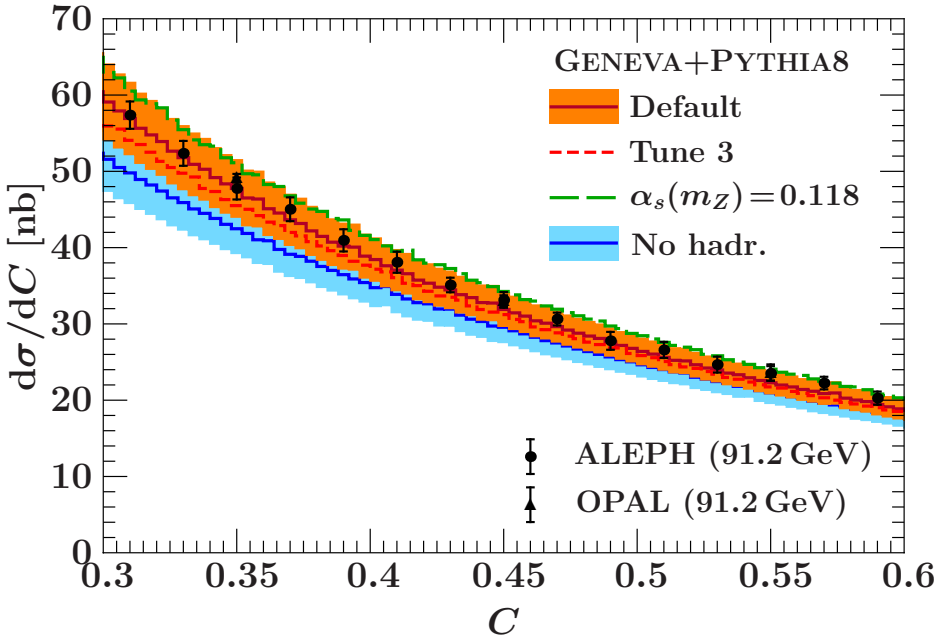

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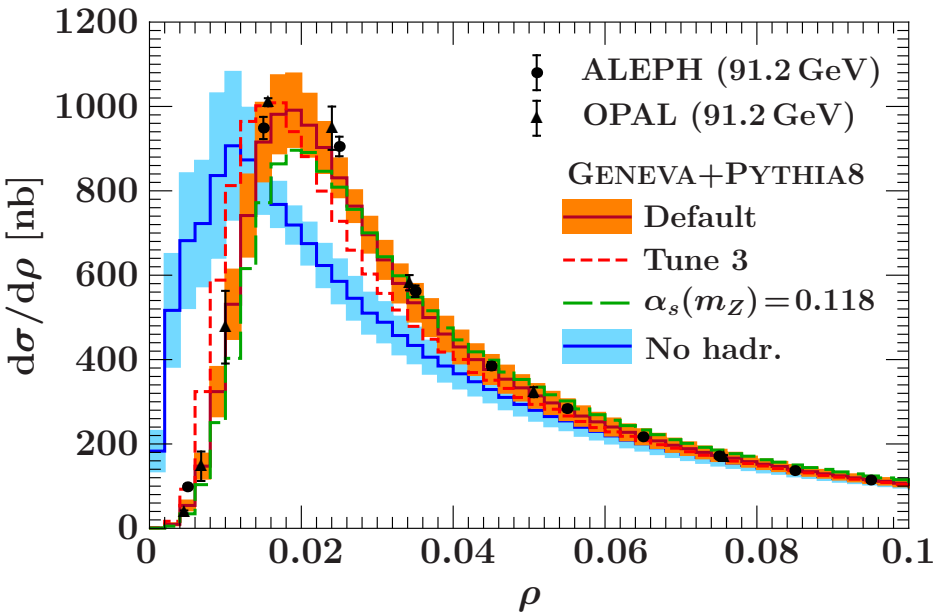

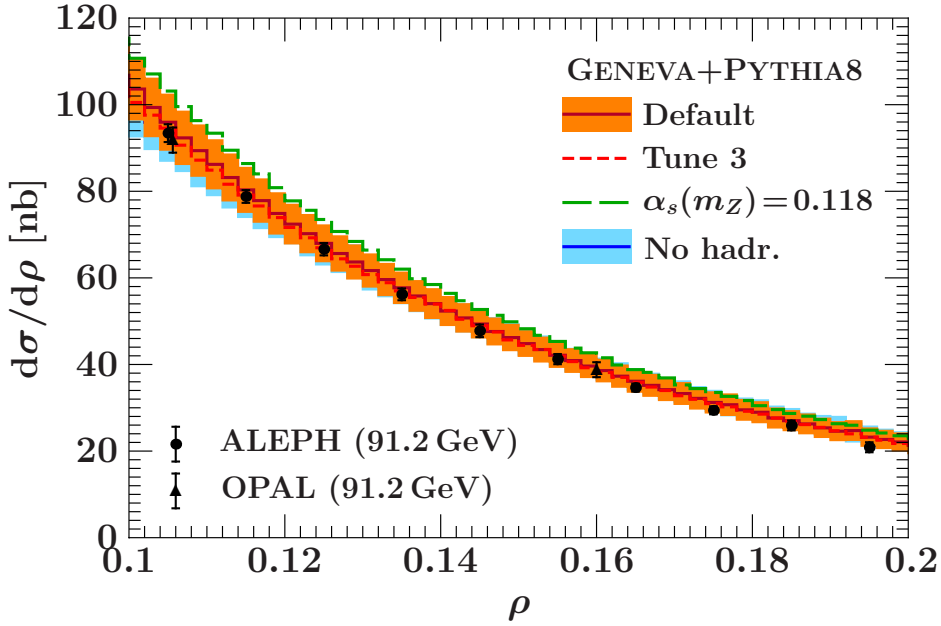

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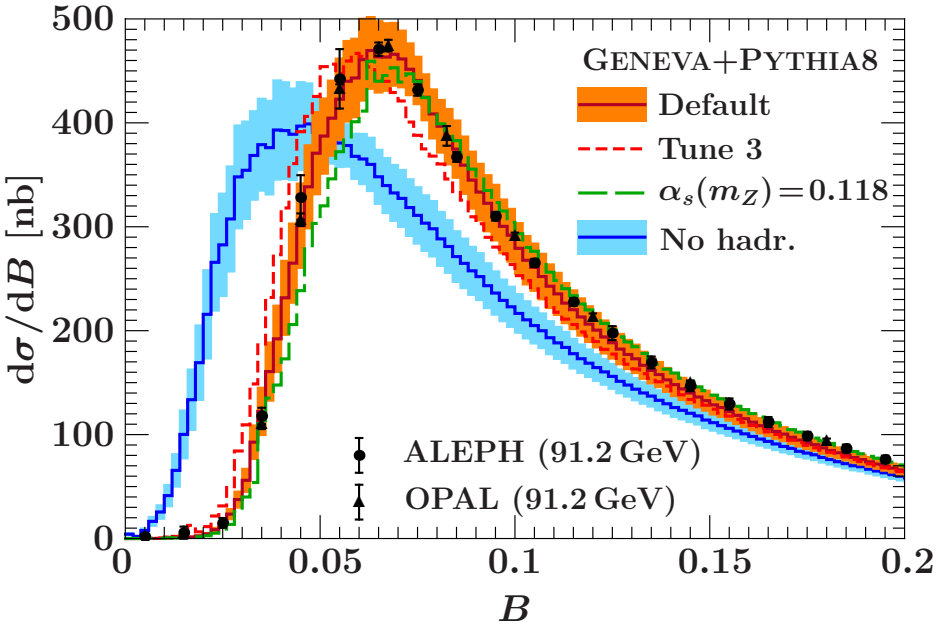

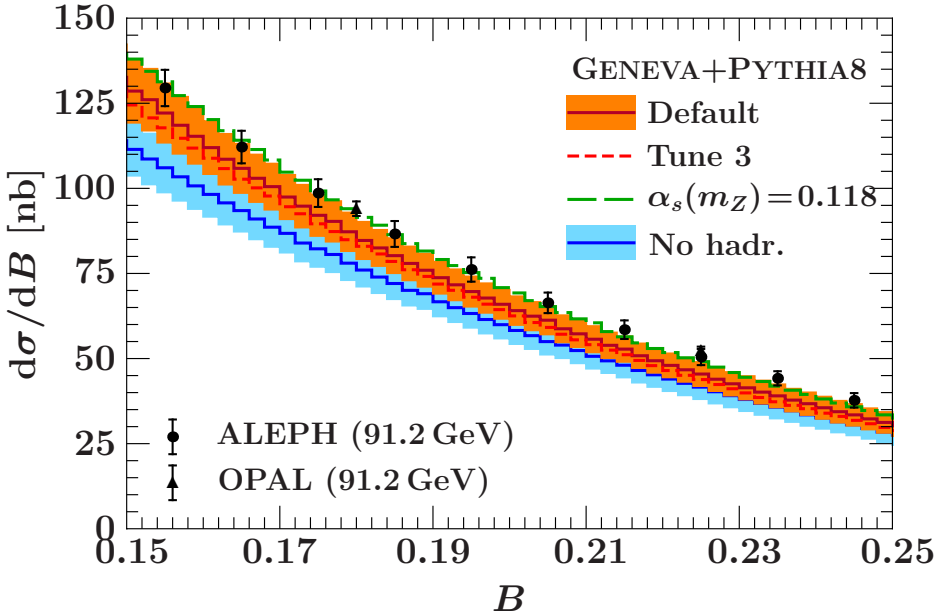

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Hadronic collisions: $pp \to (Z/\gamma^* \to \ell^+\ell^-)$ + jets

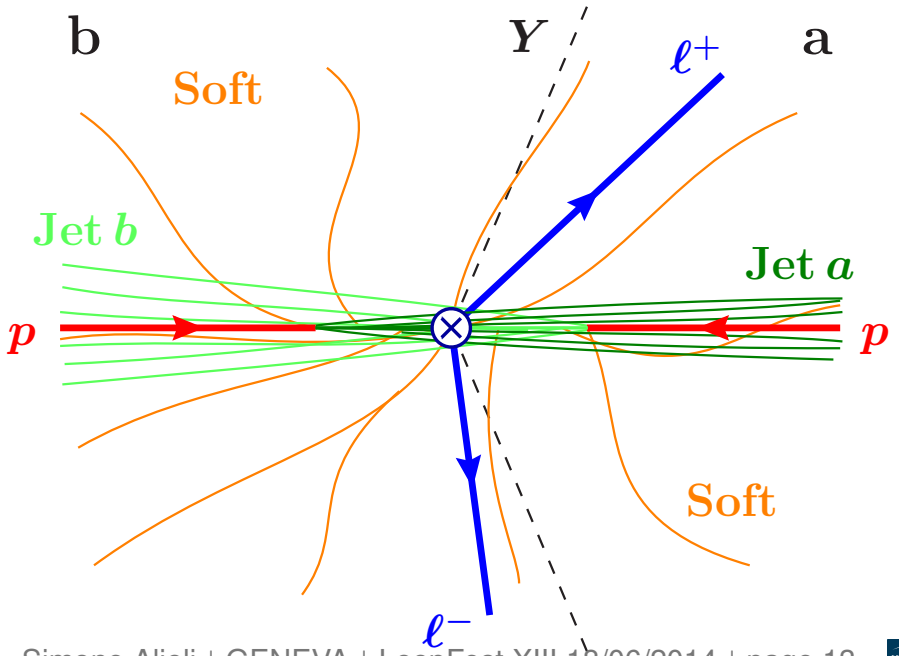
- To extend GENEVA approach to Drell-Yan we need :
  - A factorizable and resummable resolution parameter: Beam Thrust $\mathcal{T}_0 = \sum_i p_{T,i} e^{-|\eta_i - y_V|}$. Resummation known to NNLL'

$$\frac{\mathrm{d}\sigma^{\mathrm{s}}}{\mathrm{d}x_a \mathrm{d}x_b \, \mathrm{d}\mathcal{T}_B} = \sigma_B \cdot H(\mu_H) \otimes U_H(\mu_H, \mu) \cdot B(x_a, \mu_{B_a}) \otimes U_B(\mu_{B_a}, \mu) \otimes B(x_b, \mu_{B_b}) \otimes U_B(\mu_{B_b}, \mu) \otimes S(\mathcal{T}_B, \mu_S) \otimes U_S(\mu_S, \mu)$$

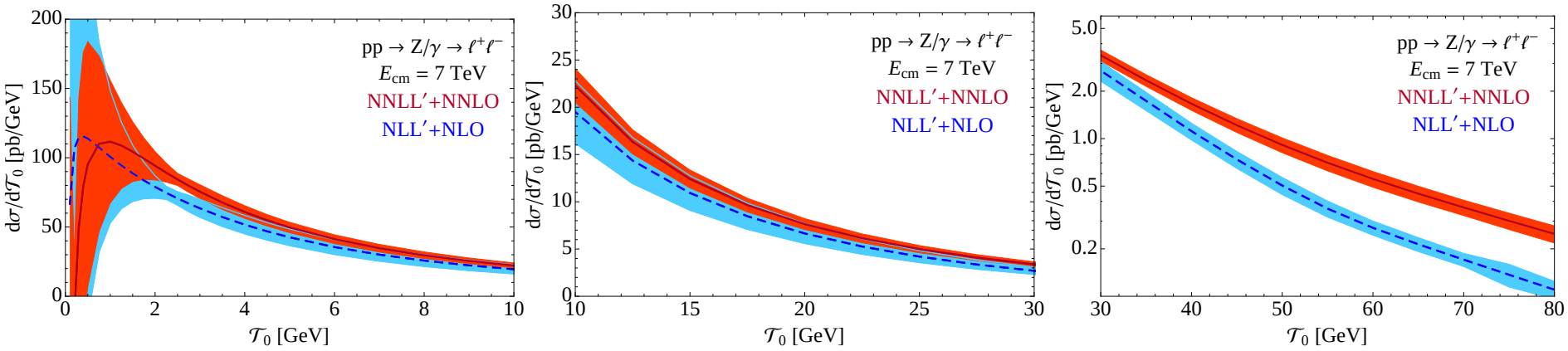
  - Factorization on terms of Soft S, Beam B and Hard H functions. Evolution kernels U are obtained by RGE running at NNLL in SCET.
  - Two-loop Beam functions needed for NNLL' in hadronic collisions recently became available

[Gaunt et al. 1405.1044, 1401.5478]

- Complications in going from e^+e^- to hadron collisions:
  - Beam Functions contain convolutions with PDFs
  - Preserving Beam Thrust and full V kinematics in $V + 1 \to V + 2$
  - Efficient Pythia8 showering without changing $\mathcal{T}_0$ for ISR
  - Proper framework to deal with MPI
  - Re-tuning of GENEVA+PYTHIA ?



Validation of $pp \to (Z/\gamma^* \to \ell^+\ell^-)$ + jets



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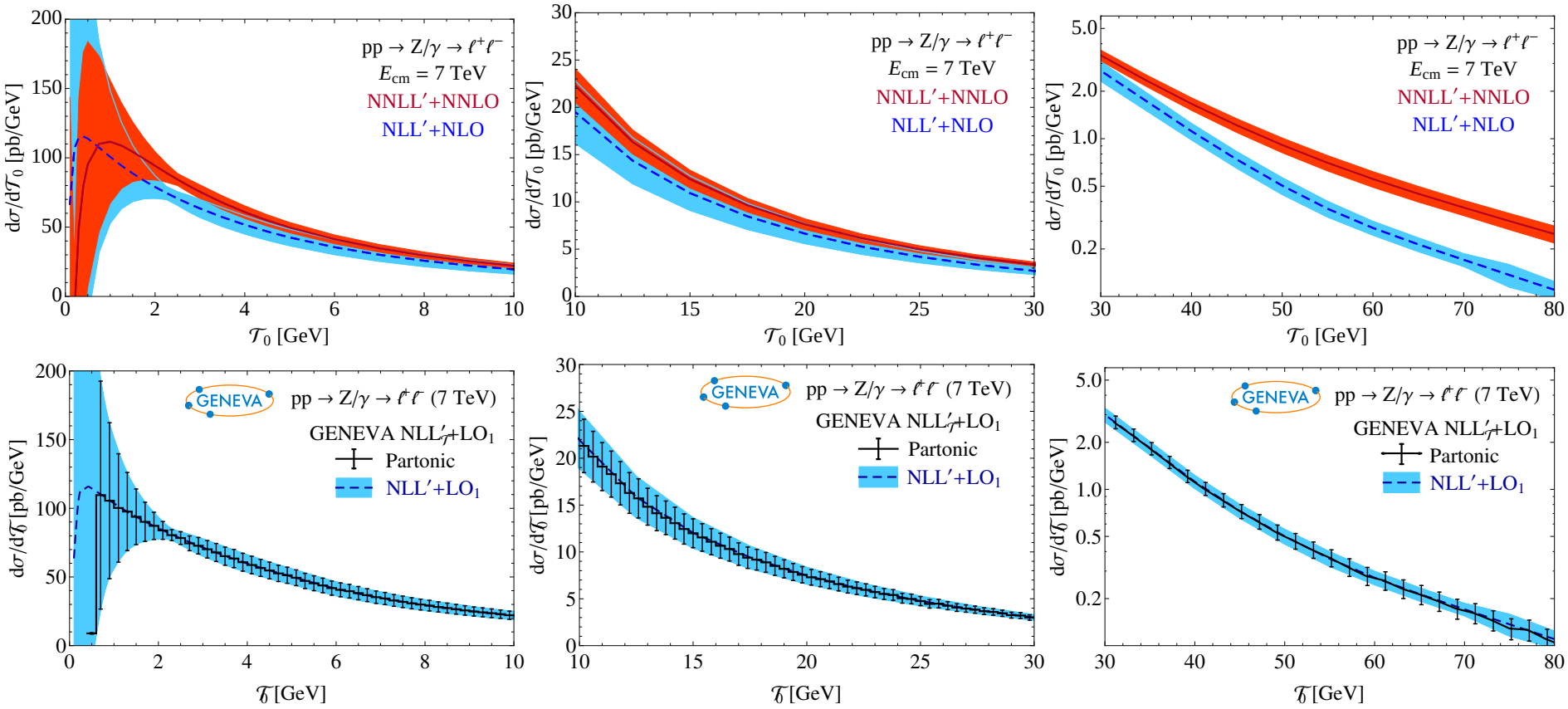

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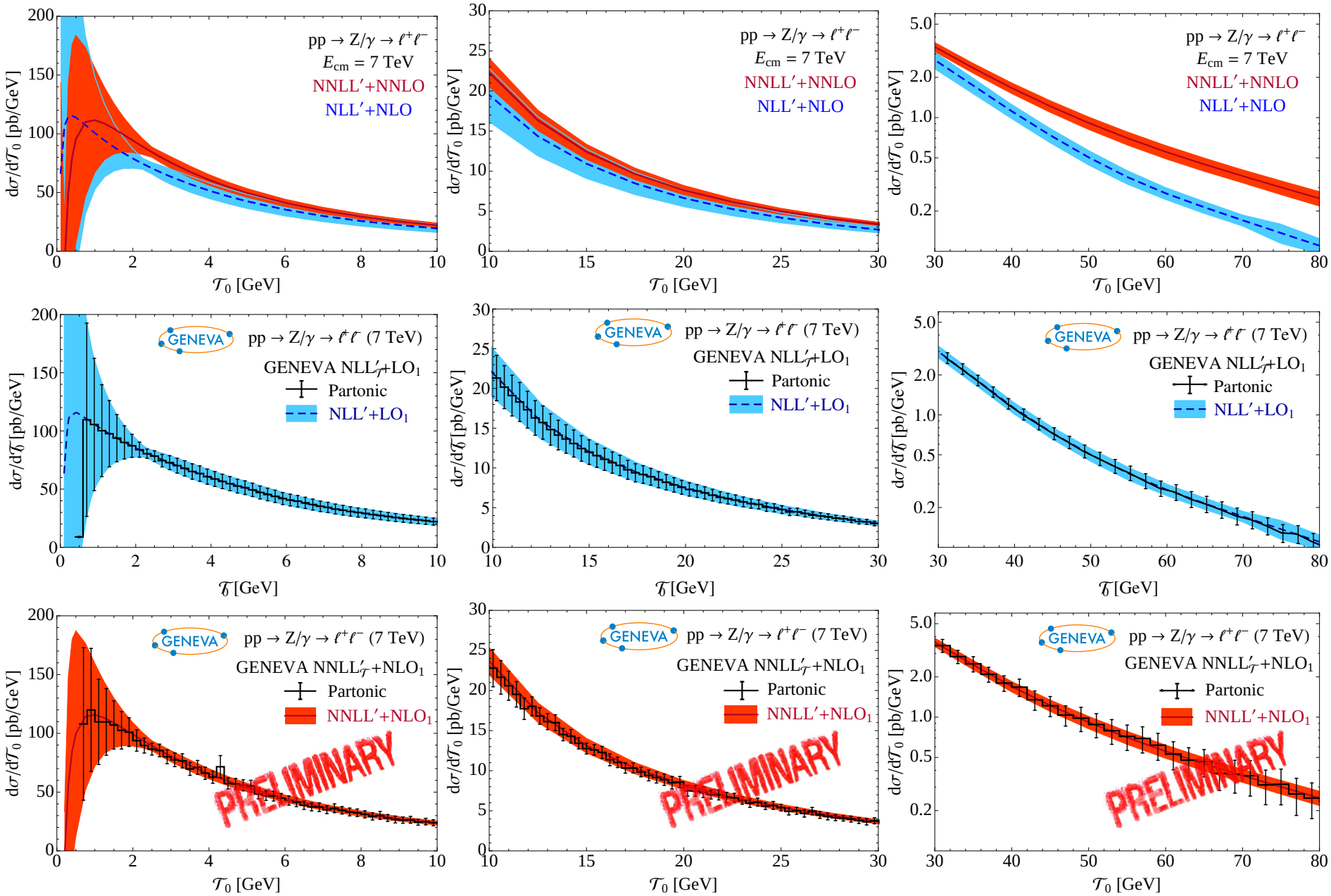
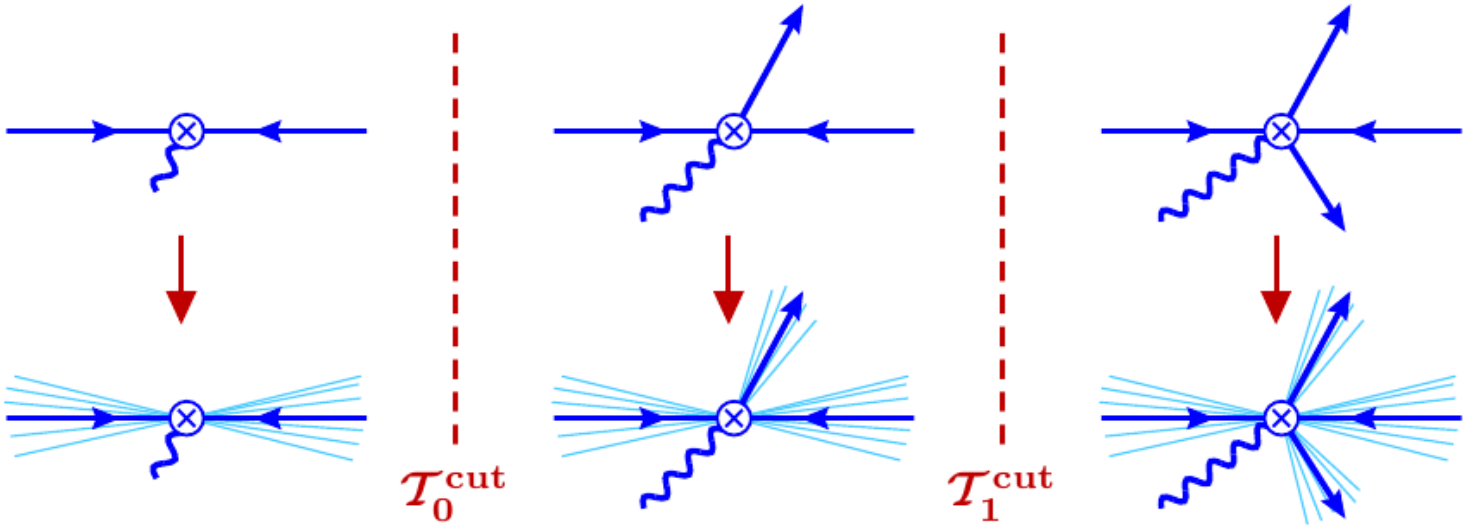


Showering



- Similarly to e^+e^-, both initial- and final-state showering should not spoil $\mathcal{T}_0$ resummation.
- Internal shower machinery untouched. Running shower repeatedly until event accepted. Only small corrections $\frac{|\mathcal{T}_0^{GE+PY} - \mathcal{T}_0^{GE}|}{\mathcal{T}_0^{GE}} < \lambda$ allowed.
- Higher starting scale makes ISR less efficient than FSR. Investigating better veto strategies.

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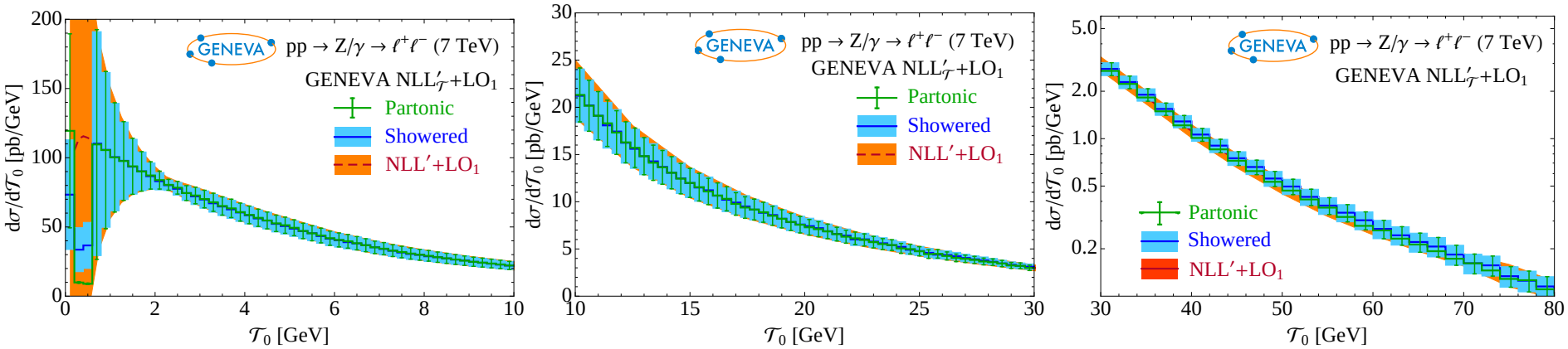

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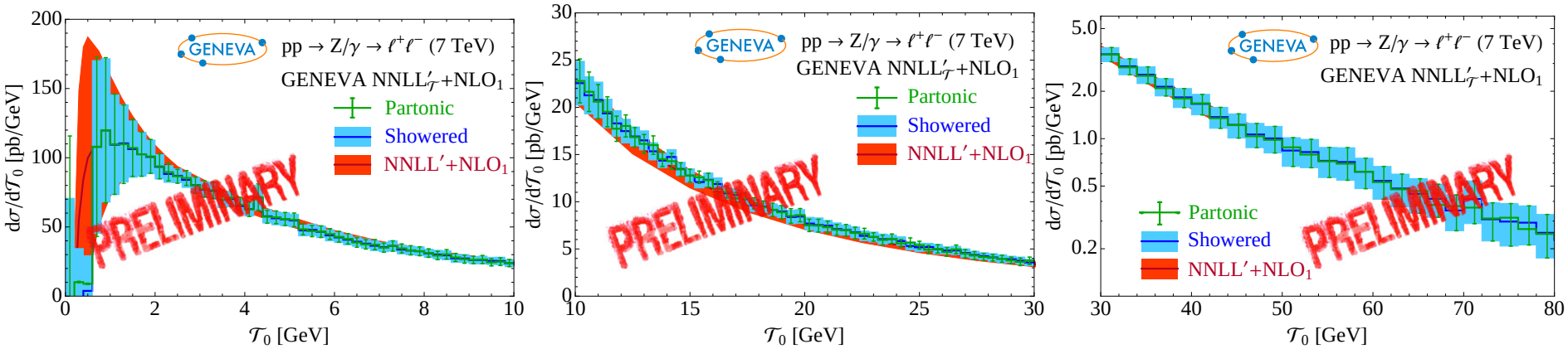

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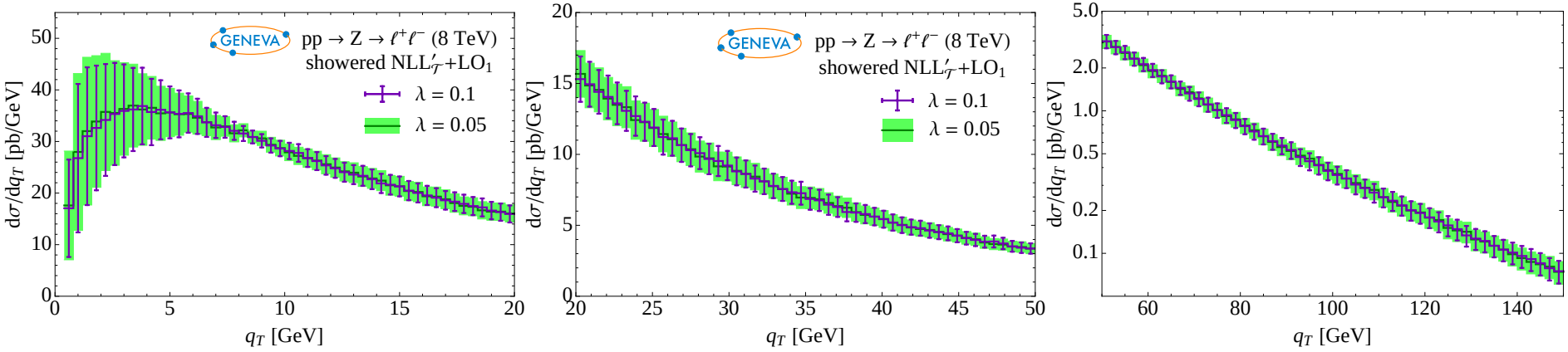

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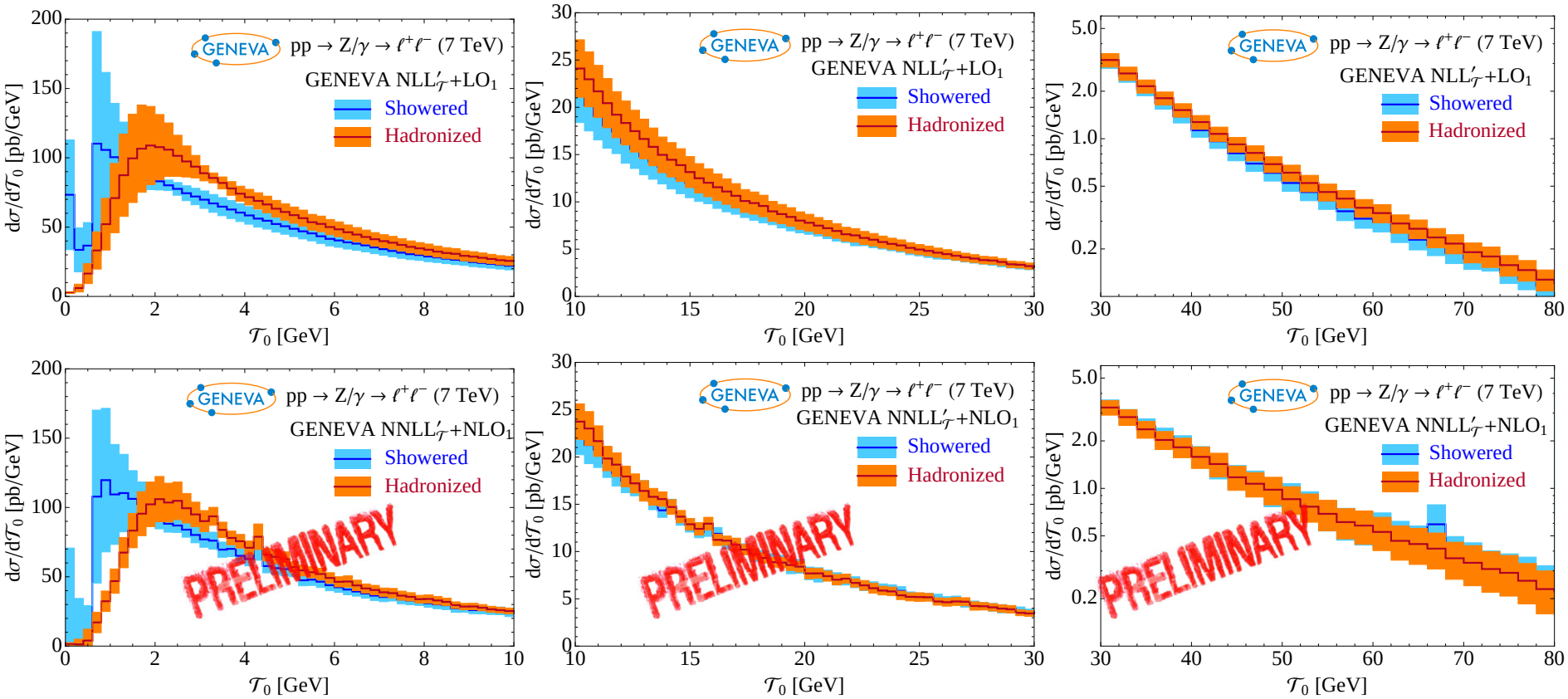
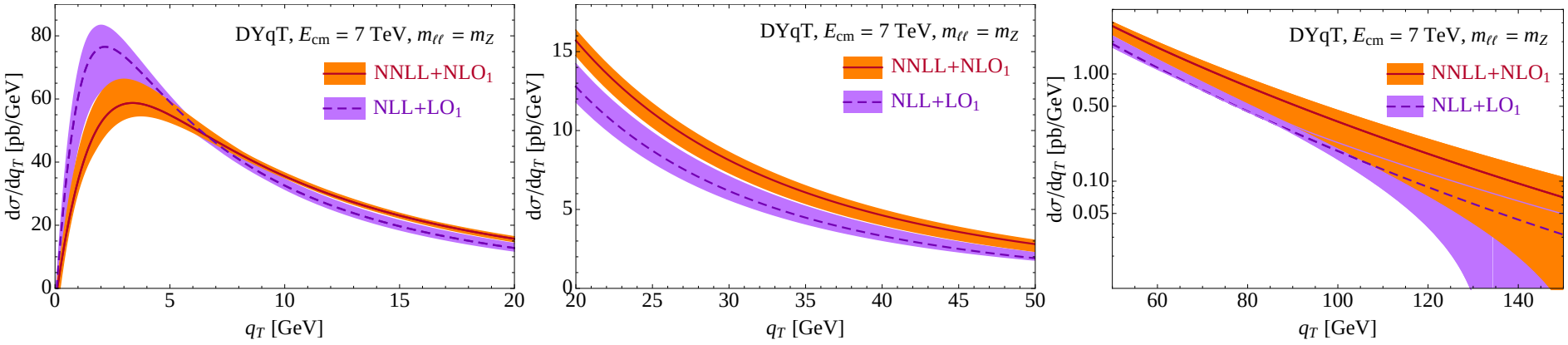
Hadronization



- As expected and observed in e^+e^-, hadronization gives $\mathcal{O}(1)$ shift in peak
- At larger $\mathcal{T}_0$ it reduces to power corrections
- PYTHIA 8 tunes includes shower effects → replacing PYTHIA showering by GENEVA could imply re-tuning of GENEVA+PYTHIA would be necessary

Drell-Yan: predictions for q_T^V and comparison with DYqT



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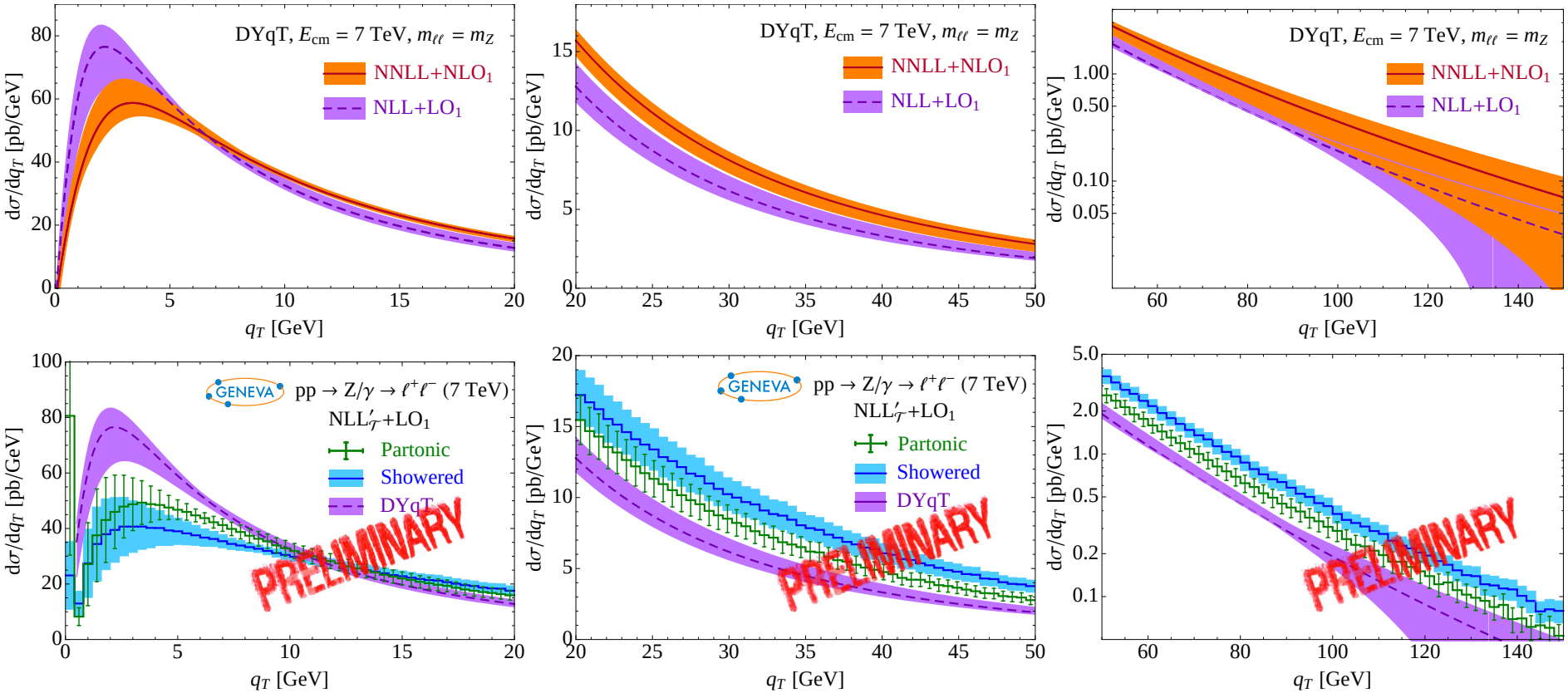

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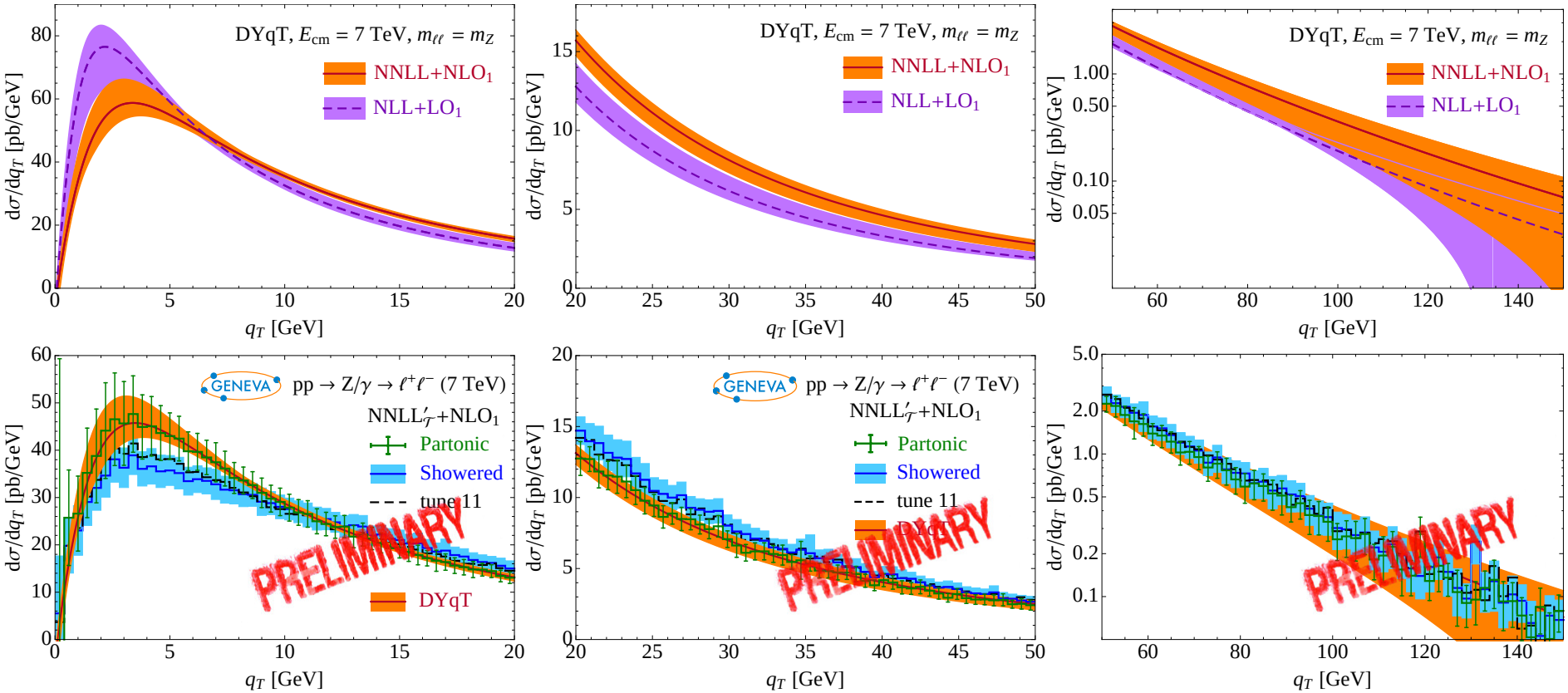

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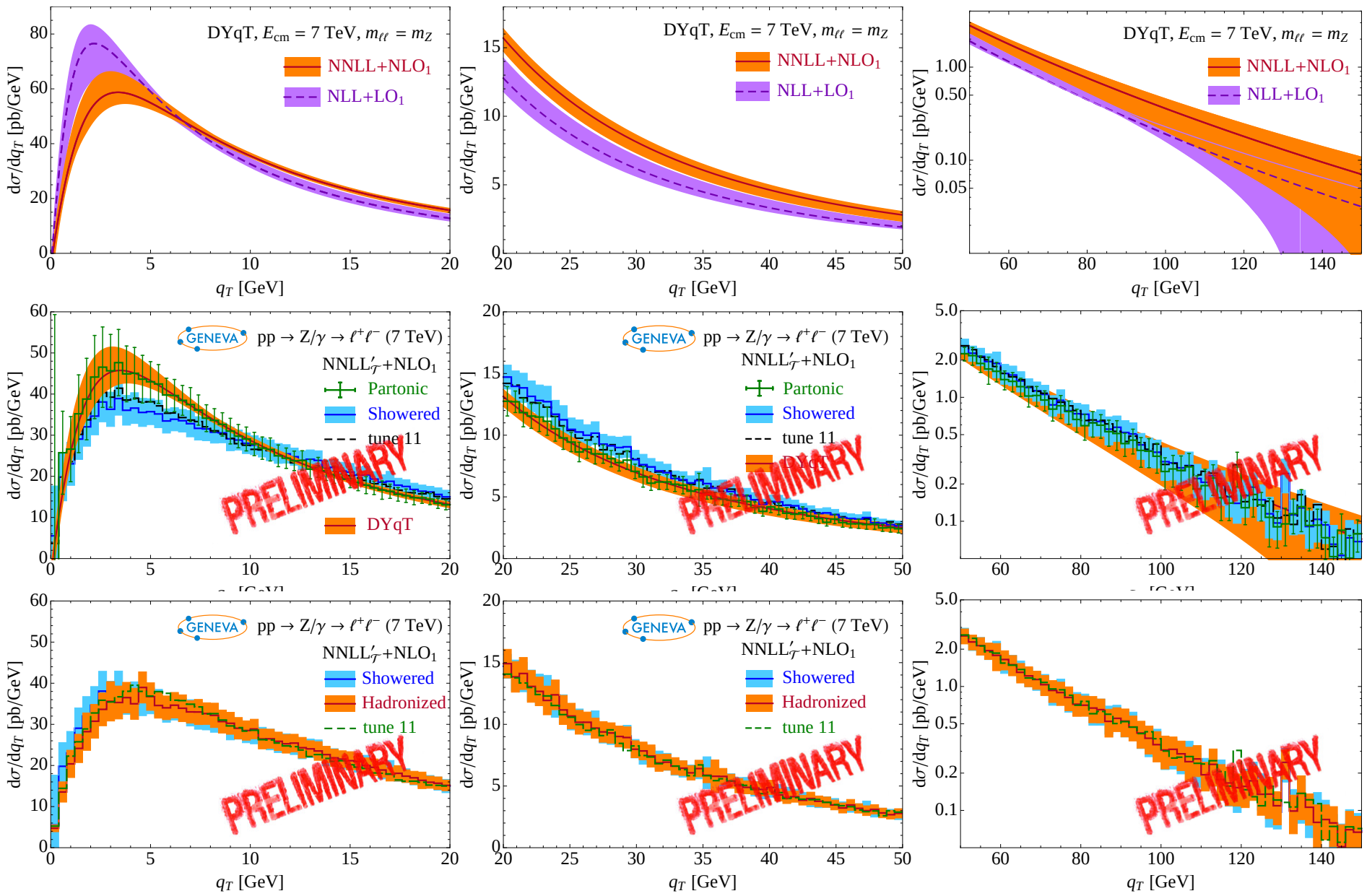

Combining fully exclusive NNLO with parton shower.

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- Sudakov factor, provides (at least) LL resummation of $\mathcal{T}_N^{\mathrm{cut}}$

$$\Delta_N(\Phi_N;\mathcal{T}_N^{\mathrm{cut}}) = \exp\left\{-\int \frac{\mathrm{d}\Phi_{N+1}}{\mathrm{d}\Phi_N}\,\frac{S_{N+1}(\Phi_{N+1})}{B_N(\hat{\Phi}_N)}\,\theta[\mathcal{T}_N(\Phi_{N+1}) > \mathcal{T}_N^{\mathrm{cut}}]\right\}$$

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  - Corrects the finite terms to the exact NNLO cross section.

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- Exclusive N-jet cross section (NNLO+LL)

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- Similar formulation for $N+1$-jet inclusive cross section (NLO+LL) and separation into $N+1$-jet exclusive and $N+2$-jet inclusive

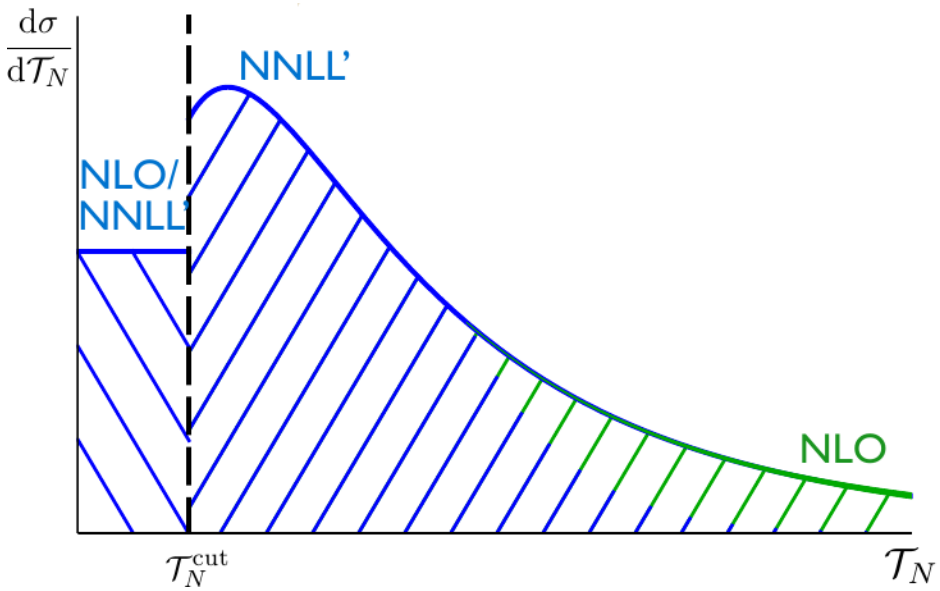
$$\overbrace{\frac{\mathrm{d}\sigma_{\geq N+1}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}})}^{\frac{\mathrm{d}\sigma_{\geq N+1}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}})}$$

$$\frac{\mathrm{d}\sigma_{N+1}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}};\mathcal{T}_{N+1}^{\mathrm{cut}}), \qquad \frac{\mathrm{d}\sigma_{\geq N+2}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+2}}(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}}, \mathcal{T}_{N+1} > \mathcal{T}_{N+1}^{\mathrm{cut}})$$

- Framework general enough to re-derive MiNLO-based NNLO+PS of [Hamilton, Nason, Re, Zanderighi 1309.0017] and UNNLOPS approach of [Hoeche, Li, Prestel 1405.3607]
- Immediate application of this framework in GENEVA to obtain NNLO accuracy . . .

GENEVA @ NNLO+NNLL'$_{\mathcal{T}_0}$

- What do we need to make GENEVA NNLO accurate?



- Inclusive cross section NLO$_0$ accurate
- Perturbative $\mathcal{O}(\alpha_s)$ everywhere
- Logarithms of merging scale ($\mathcal{T}_N^{\rm cut}$) cancel at NNLL' by construction: merging of NLO$_0$ and NLO$_1$ is a by-product

- With NNLL' resummation, NNLO singular contributions are included

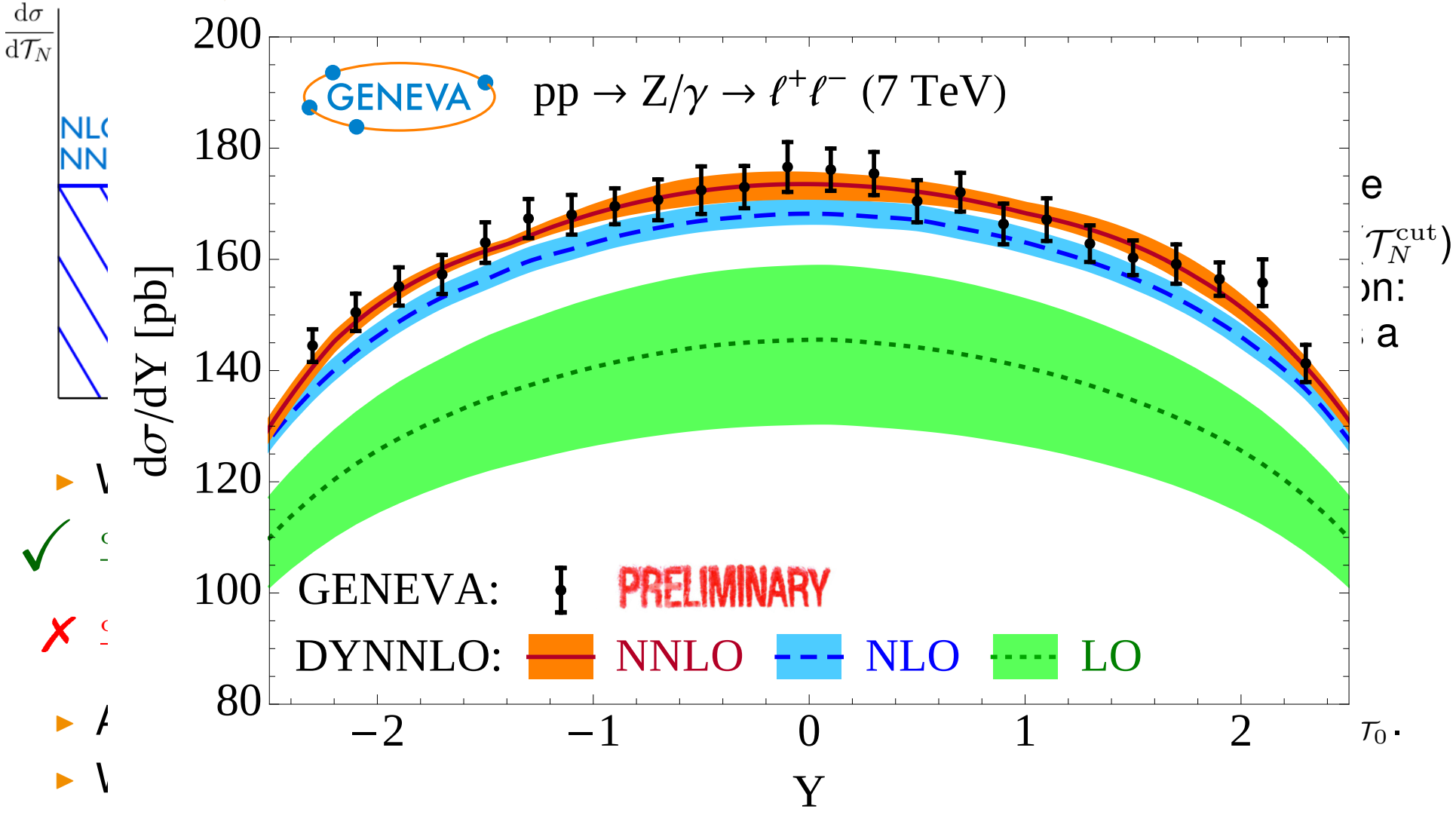
✓ $\frac{d\sigma^C_{\geq N}}{d\Phi_N}\Delta_N(\Phi_N;\mathcal{T}_N^{\rm cut}) \to \frac{d\sigma_N^{\rm resummed}}{d\Phi_N}(\mathcal{T}_N^{\rm cut})$ ✓ $\frac{d\sigma_N^{C-S}}{d\Phi_N}(\mathcal{T}_N^{\rm cut}) = 0$

✗ $\frac{d\sigma_N^{B-C}}{d\Phi_N}(\mathcal{T}_N^{\rm cut}) \to \frac{d\sigma_N^{\rm nonsingular}}{d\Phi_N}(\mathcal{T}_N^{\rm cut}) \neq \frac{d\sigma_N^{\rm NNLO\ ns.}}{d\Phi_N}(\mathcal{T}_N^{\rm cut})$ (suppr. as $(\mathcal{T}_N^{\rm cut})^n$)

- Adding NNLO non-singular to GENEVA 0-jet bin we get NNLO+NNLL'$_{\mathcal{T}_0}$.
- Work in progress . . . stay tuned.

GENEVA @ NNLO+NNLL'$_{\mathcal{T}_0}$

- What do we need to make GENEVA NNLO accurate?



Conclusions and outlook

GENEVA provides a framework for combining higher-order resummation with multiple NLO calculations and shower/hadronization.

- Based on IR-safe, jet-like definitions of events.
- Uses a physics observable, N-jettiness, factorizable and whose resummation is known to NNLL as jet resolution parameter
- e^+e^- and Drell-Yan results looks encouraging.
- Given formulas for jet cross section at the necessary accuracy in both fixed order (NNLO$_N$,NLO$_{N+1}$,LO$_{N+2}$) and resummation regions (LL). Discussed shower matching.
- Path to NNLO+NNLL' clearly defined. **Next steps:**

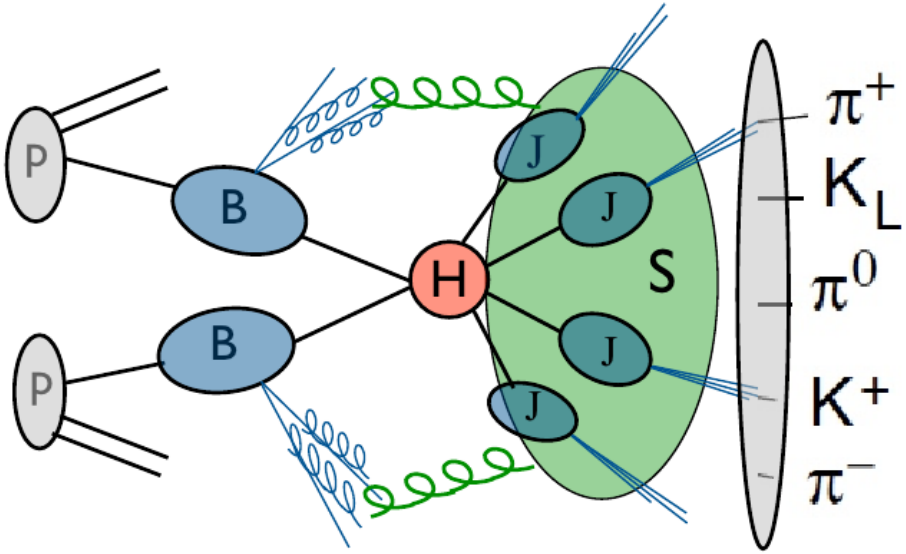
- Adding more jets, e.g. $pp \to V + 0, 1, 2$ and validation with LHC data.
- Looking into other processes, e.g. $gg \to H + 0, 1, 2$ jets.
- Investigate resummation of different resolution parameters
- Study interface to other SMC: `HERWIG++`, `SHERPA` ...
- Assess need for specific tuning of GENEVA + SMC.
- Inclusion of NNLO non singular corrections

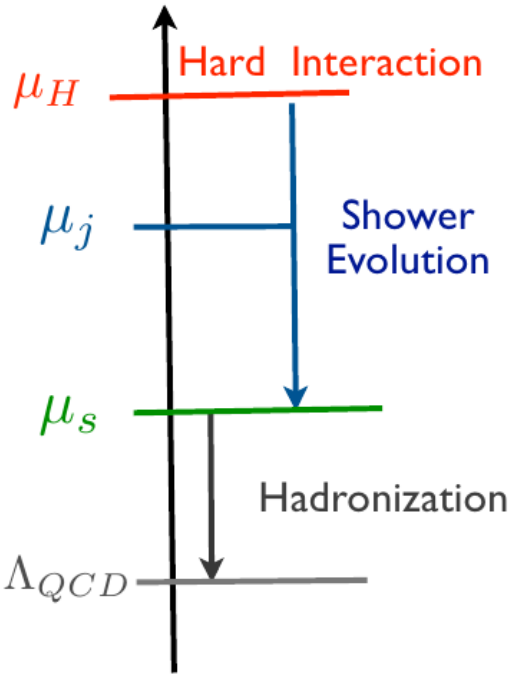
Thank you for your attention!

BACKUP

GENEVA: quick overview

- Monte Carlo is built on the idea of factorization





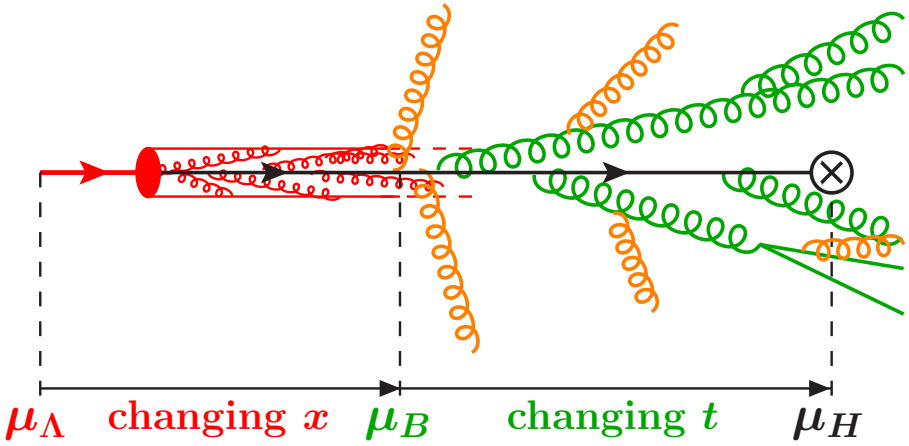
$$d\sigma^{MC} = \text{Hard Interaction} \otimes \underbrace{\boxed{\text{Collinear Evolution} \otimes \text{Soft Radiation}}}_{\text{Parton Shower Evolution}} \otimes \begin{array}{c}\text{PDFs}\\ \text{Hadronization}\\ \text{Underlying Event}\end{array}$$

- Replaces parton-shower evolution with higher order logarithmic resummation from μ_H to $\mu_{\mathbf{B}}, \mu_{\mathbf{J}}, \mu_S$.

Perturbative ingredients for resummation

	Fixed-order corrections		Resummation input		
	singular	nonsingular	γ_x	$\Gamma_{\rm cusp}$	β
LL	LO$_N$	-	-	1-loop	1-loop
NLL	LO$_N$	-	1-loop	2-loop	2-loop
NLL′	NLO$_N$	-	1-loop	2-loop	2-loop
NLL′+LO$_{N+1}$	NLO$_N$	LO$_{N+1}$	1-loop	2-loop	2-loop
NNLL+LO$_{N+1}$	NLO$_N$	LO$_{N+1}$	2-loop	3-loop	3-loop
NNLL′	NNLO$_N$	-	2-loop	3-loop	3-loop
NNLL′+NLO$_{N+1}$	NNLO$_N$	NLO$_{N+1}$	2-loop	3-loop	3-loop

Beam Functions



- Beam functions are perturbative objects, connected to PDF via OPE in SCET

$$B_i(t, x; \mu_B) = \sum_k \int_x^1 \frac{\mathrm{d}\xi}{\xi} \mathcal{I}_{ik}\left(t, \frac{x}{\xi}; \mu_B\right) f_k(\xi; \mu_B)$$

- Calculating these integral on-the-fly is not computationally feasible
- We have prepared interpolation grids for all convolutions we need in GENEVA.
- These grids are LHAPDF6 grids, we can use all the LHAPDF6 machinery, including the interpolator to get the values event-by-event

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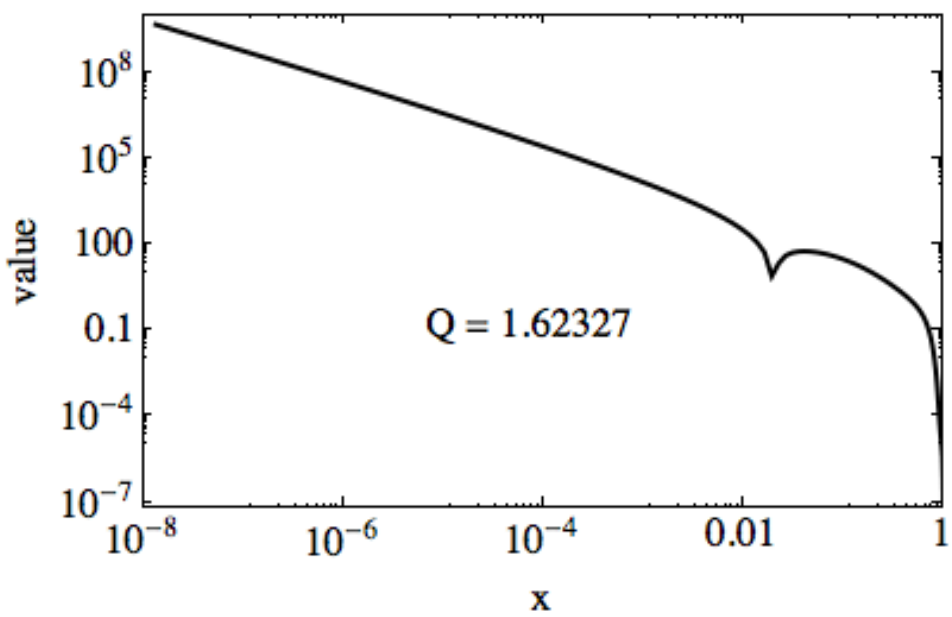
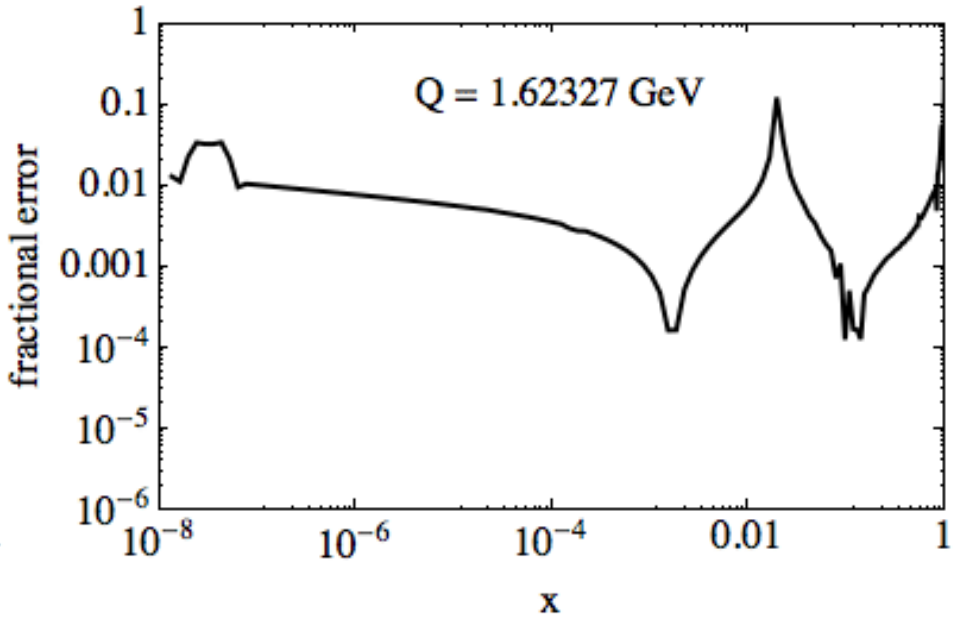
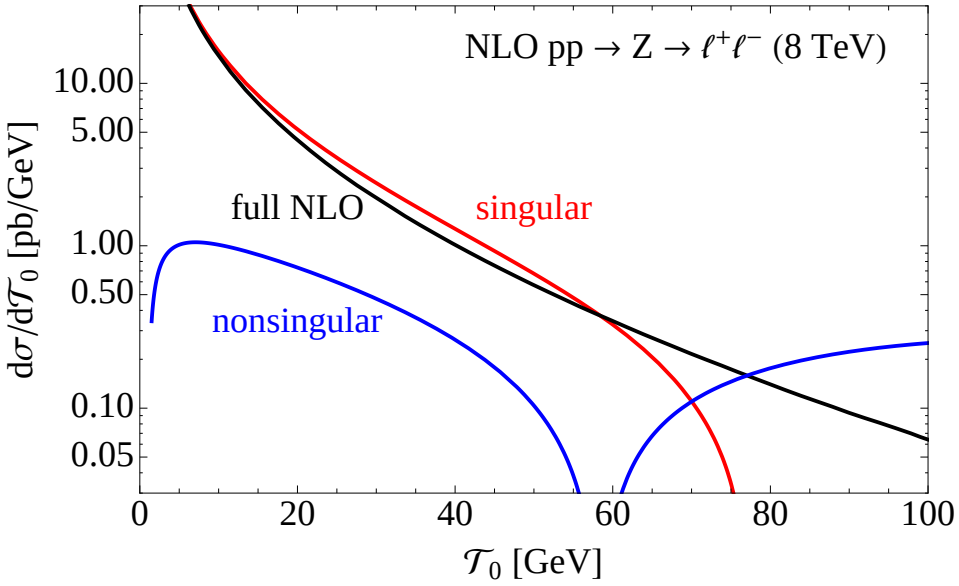
- These grids are LHAPDF6 grids, we can use all the LHAPDF6 machinery, including the interpolator to get the values event-by-event
- Results have been validated against direct integration, e.g. CT10NNLO $P_{gg} \otimes g$

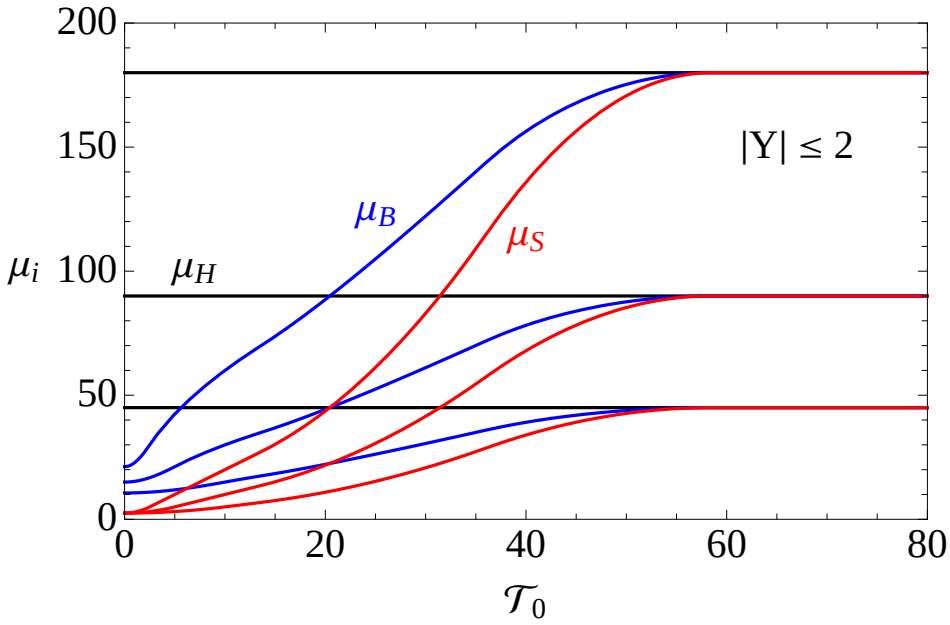


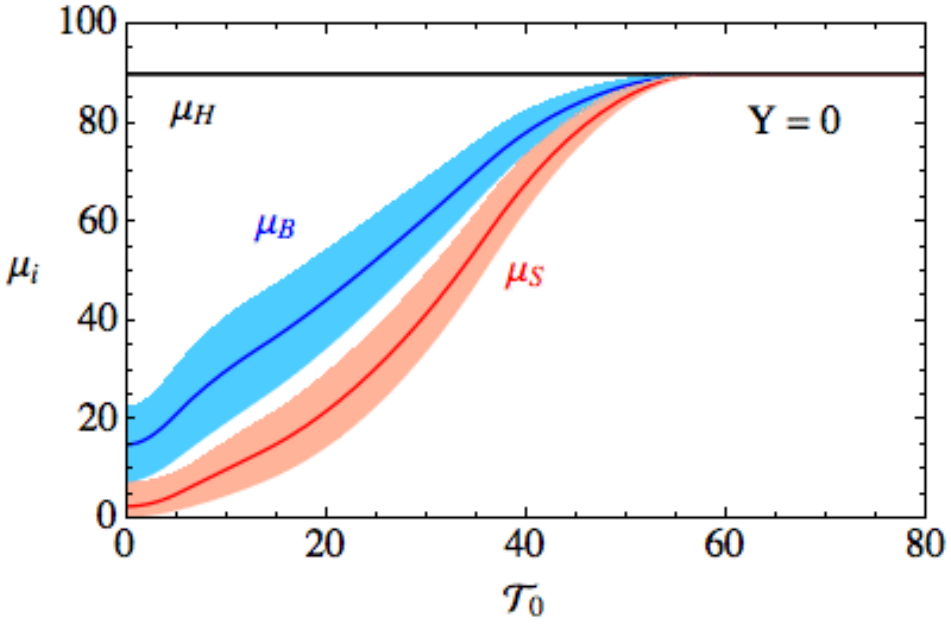


Scale profiles and theoretical uncertainties

- Theoretical uncertainties in resum. are evaluated by independently varying each μ. Final results are added in quadrature
- Range of variations is tuned to turn off the resummation before the nonsingular dominates (Y_V-dependent) and to respect SCET scaling $\mu_H \gtrsim \mu_B \gtrsim \mu_S$
- FO unc. are usual $\{2\mu_H, \mu_H/2\}$ variations. Added linearly.





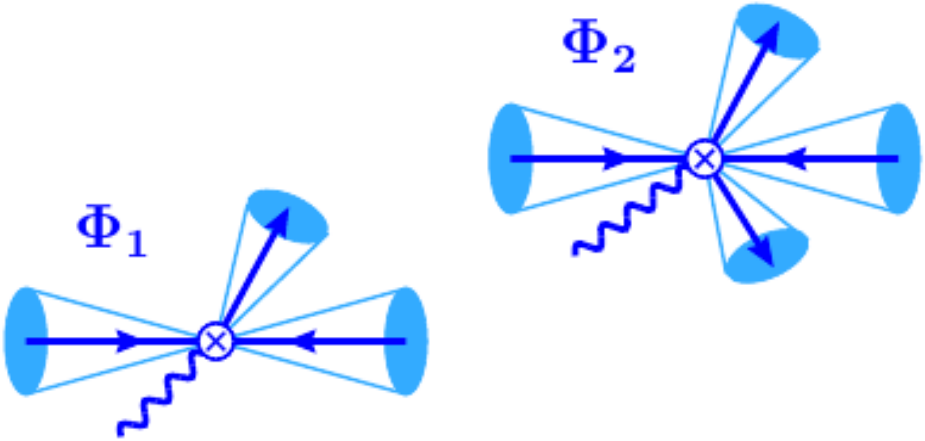


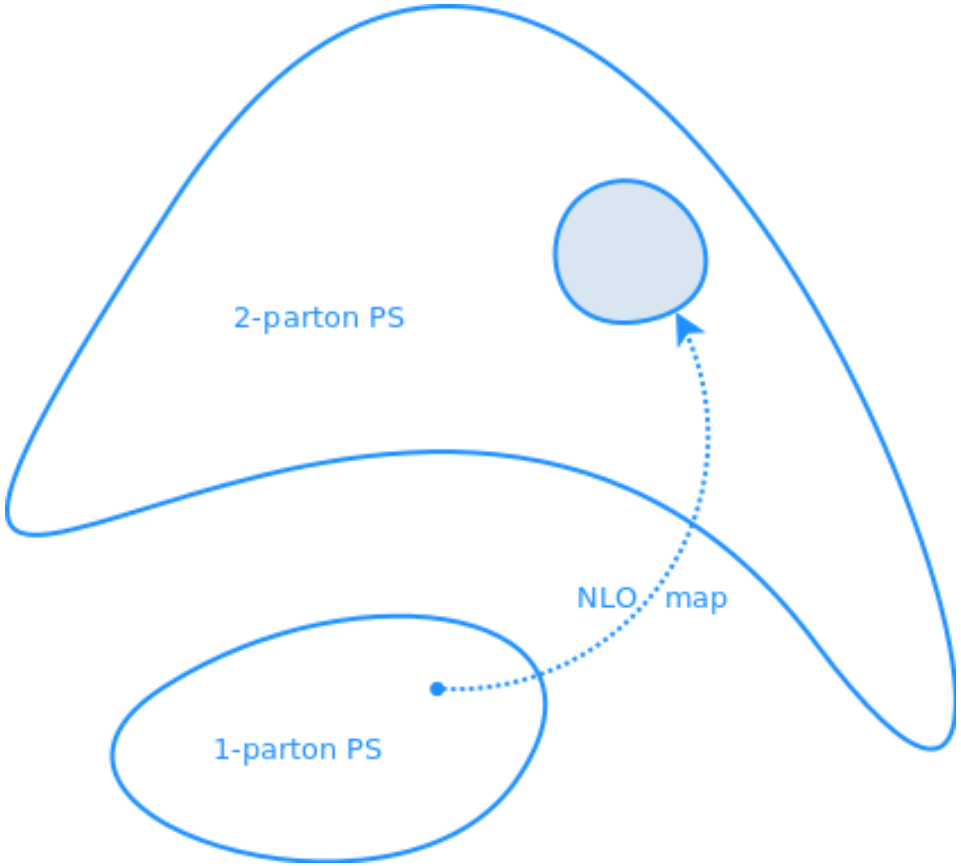
Preserving the $\mathcal{T}_0$ value in $V+1 \to V+2$ partons splittings

$$\frac{\mathrm{d}\sigma^{\mathrm{NLO}}}{\mathrm{d}\Phi_1}(\mathcal{T}_0) = [B_1(\Phi_1) + V_1(\Phi_1)]\,\delta(\mathcal{T}(\Phi_1) - \mathcal{T}_0) + \int \frac{\mathrm{d}\Phi_2}{\mathrm{d}\Phi_1} B_2(\Phi_2)\,\delta\left(\mathcal{T}(\Phi_1(\Phi_2)) - \mathcal{T}_0\right)$$

- When calculating NLO$_1$ we must preserve $\mathrm{d}\Phi_1$
- Real emissions must preserve both $\mathrm{d}^4 q \delta(q^2 - M^2_{\ell^+\ell^-})$ and $\mathcal{T}_0 \equiv \bar{p}_{T,1} e^{-|y_V - \bar{\eta}_1|} = p_{T,1} e^{-|y_V - \eta_1|} + p_{T,2} e^{-|y_V - \eta_2|}$.

- Standard FKS or CS maps are not designed to preserve $\mathcal{T}_0$. They are design to preserve other quantities, like $\hat{s}$ or Y_{tot}. We had to design our own $\mathcal{T}_0$-preserving map. Fixed NLO results are of course map independent.





Iterating the Geneva method to higher multiplicities

- Geneva approach can be iterated to merge several multiplicities at NLO
- Separation of cumulant and spectrum

$$\frac{\mathrm{d}\sigma_{\mathrm{incl}}}{\mathrm{d}\Phi_N} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_N}(\mathcal{T}_N^{\mathrm{cut}}) + \int \frac{\mathrm{d}\Phi_{N+1}}{\mathrm{d}\Phi_N}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_N)\,\theta(\mathcal{T}_N > \mathcal{T}_N^{\mathrm{cut}})\,,$$

$$\frac{\mathrm{d}\sigma_{\mathrm{incl}}}{\mathrm{d}\Phi_{N+1}} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_{N+1}^{\mathrm{cut}}) + \int \frac{\mathrm{d}\Phi_{N+2}}{\mathrm{d}\Phi_{N+1}}\,\frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_{N+2}}(\mathcal{T}_{N+1})\,\theta(\mathcal{T}_{N+1} > \mathcal{T}_{N+1}^{\mathrm{cut}})\,,$$

$$\vdots$$

$$\frac{\mathrm{d}\sigma_{\mathrm{incl}}}{\mathrm{d}\Phi_{N_{\mathrm{max}}}} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_{N_{\mathrm{max}}}}(\mathcal{T}_{N_{\mathrm{max}}}^{\mathrm{cut}} \to \infty)$$

Iterating the Geneva method to higher multiplicities

- Geneva approach can be iterated to merge several multiplicities at NLO
- Resummation factors U replaces shower Sudakovs

$$U_N(\Phi_N, \mathcal{T}_N) = \frac{\mathrm{d}\sigma^{\mathrm{resum}}}{\mathrm{d}\Phi_N \mathrm{d}\mathcal{T}_N} \Big/ \frac{\mathrm{d}\sigma^{\mathrm{resum}}}{\mathrm{d}\Phi_N \mathrm{d}\mathcal{T}_N}\Bigg|_{\mathrm{FO}}$$

$$\begin{aligned}
\frac{\mathrm{d}\sigma_N^{\mathsf{MC}}}{\mathrm{d}\Phi_N}(\mathcal{T}_N^{\mathrm{cut}}) &= \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_N}(\mathcal{T}_N^{\mathrm{cut}})\,, \\
\frac{\mathrm{d}\sigma_{N+1}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_{N+1}^{\mathrm{cut}}) &= \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_{N+1}}(\mathcal{T}_{N+1}^{\mathrm{cut}})\, U_N(\Phi_N, \mathcal{T}_N)\,, \\
&\vdots \\
\frac{\mathrm{d}\sigma_{\geq N_{\max}}^{\mathsf{MC}}}{\mathrm{d}\Phi_{N_{\max}}} &= \frac{\mathrm{d}\sigma}{\mathrm{d}\Phi_{N_{\max}}}(\mathcal{T}_{N_{\max}}^{\mathrm{cut}} \to \infty)\, U_N(\Phi_N, \mathcal{T}_N)\, U_{N+1}(\Phi_{N+1}, \mathcal{T}_{N+1}) \\
&\quad \times \cdots \times U_{N_{\max}-1}(\Phi_{N_{\max}-1}, \mathcal{T}_{N_{\max}-1})
\end{aligned}$$