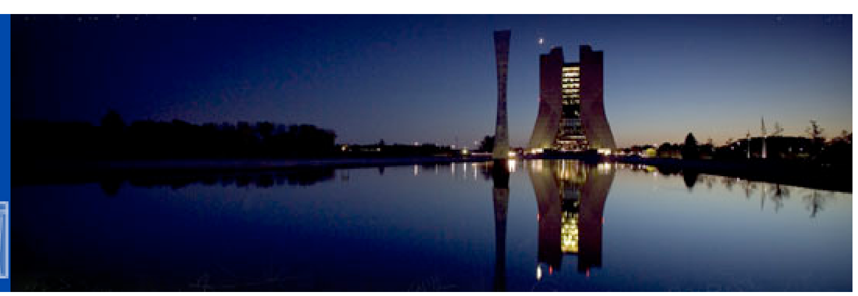
Quantum Chromodynamics

Lecture 2: Leading order and showers

Hadron Collider Physics Summer School 2010

John Campbell, Fermilab



Tasks for today

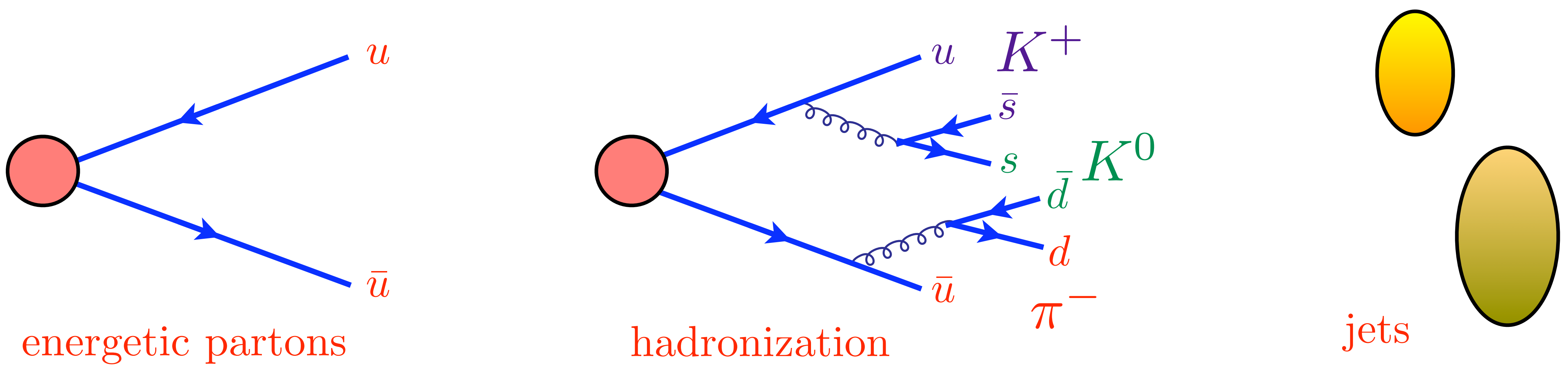
- Discuss a recipe for QCD predictions
    - Leading Order (LO) Monte Carlo.
- Understand the importance of soft and collinear kinematic limits.
    - ... in both matrix elements and phase space.
- Understand how properties of these limits can be used to extend LO predictions.
    - evolution equations and parton showers.

Recipe for QCD cross sections

1. Identify the final state of interest, e.g. leptons, photons, quarks, gluons.
2. Draw the relevant Feynman diagrams and begin calculating.
    - take care of QCD color factors using color algebra.
    - compute the rest of the diagram using spinors, Gamma matrices, etc.
3. This gives us the squared matrix elements.
4. To turn this into a cross section, we need to integrate over momentum degrees of freedom → phase space integration.
    - for final state momenta, this is just like QED.
    - in the initial state, we have the additional complication that we are colliding protons and not quarks/gluons (more on this later).
    - this step almost always performed numerically - "Monte Carlo integration".

Identifying the final state

- From the beginning, we noted that all particles observed in experiments should be color neutral → no quarks or gluons.
- How then can we mesh experimental observations with the QCD Lagrangian, which necessarily involves the fundamental quark and gluon fields?
- A scattering can be described in terms of energetic quarks and gluons (partons) that subsequently hadronize, combining into color-neutral mesons and baryons, without too much loss of energy.
- This concept is often referred to as local parton-hadron duality.



- This naturally accommodates the replacement of jets of particles in the final state by an equivalent number of quarks or gluons.

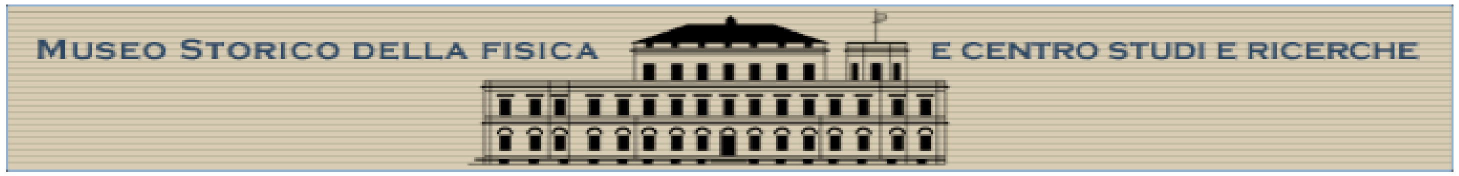
Leading order tools

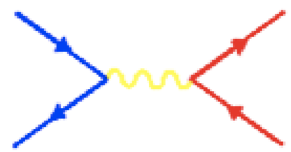
- The leading order estimate of the cross section is obtained by computing all relevant tree-level Feynman diagrams (i.e. no internal loops).
- Nowadays this is practically a solved problem - many suitable tools available.

ALPGEN	M. L. Mangano et al. http://alpgen.web.cern.ch/alpgen/
AMEGIC++	F. Krauss et al. http://projects.hepforge.org/sherpa/dokuwiki/doku.php
CompHEP	E. Boos et al. http://comphep.sinp.msu.ru/
HELAC	C. Papadopoulos, M. Worek http://helac-phegas.web.cern.ch/helac-phegas/helac-phegas.html
Madevent	F. Maltoni, T. Stelzer http://madgraph.roma2.infn.it/

Madgraph

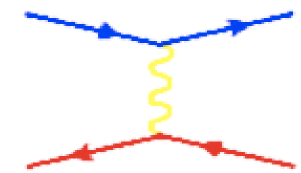
MadGraph / MadEvent is a software that allows you to generate amplitudes and events for any process (with up to 9 external particles) in any model. Implemented models are the Standard Model, Higgs effective couplings, MSSM, the general Two Higgs doublet model, and several minor models, and there is an easy-to-use interface for implementing model extensions. In





MadGraph Version 4

UCL UIUC Fermi
by the MG/ME Development team



Generate Process | Register | Tools | My Database | Cluster Status | Downloads (needs registration) | Wiki/Docs | Admin

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Generate Code On-Line

To improve our web services we now request that you register. Registration is quick and free. You may register for a password by clicking here

Code can be generated either by:

I. Fill the form:

MadGraph Version : MadGraph 4 — What is MadGraph 5?

Model: SM — Model descriptions

Input Process: — Examples/format

Max QCD Order: 99

Max QED Order: 99

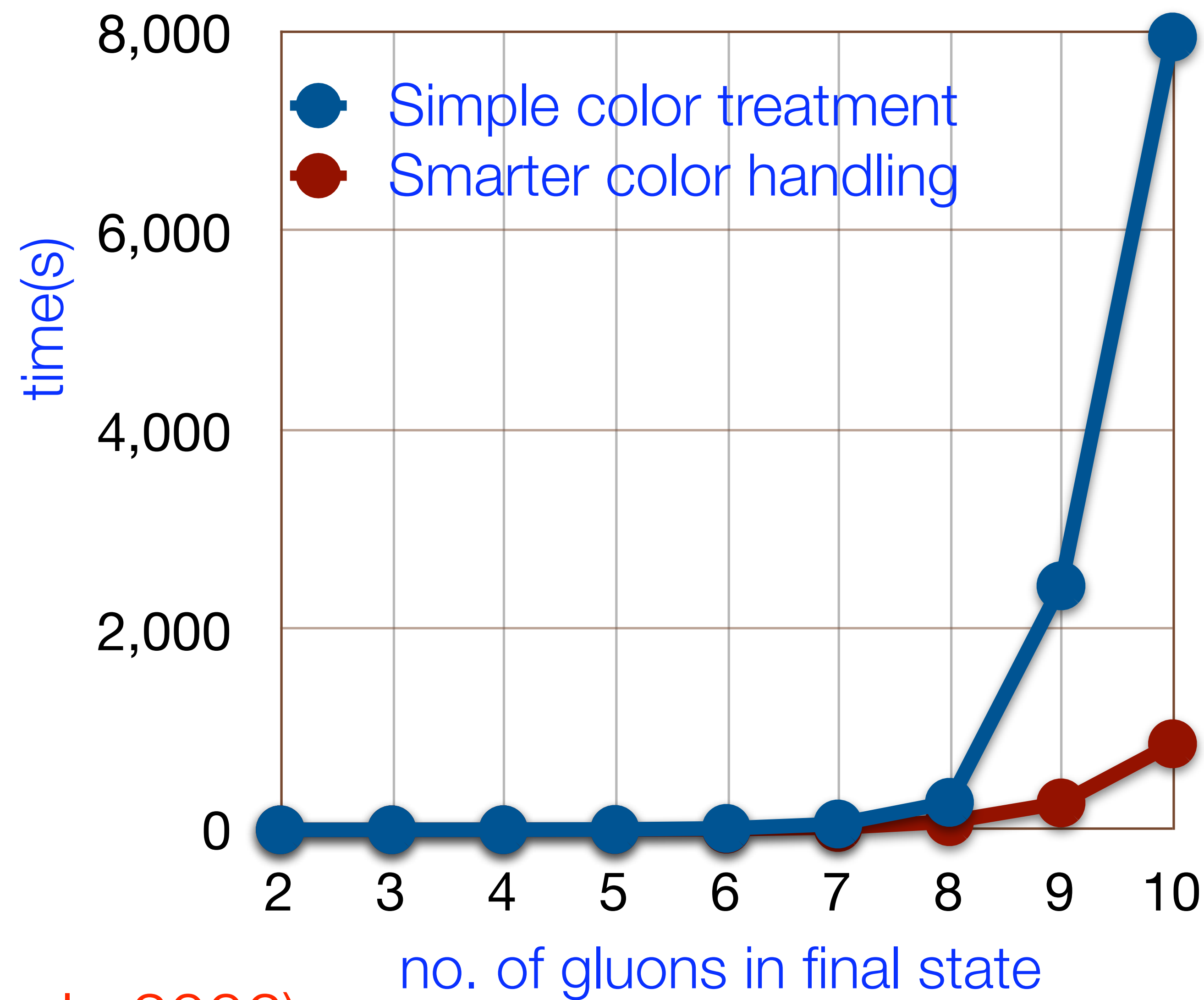
p and j definitions: p=j=d u s c d~ u~ s~ c~ g

sum over leptons: l+ = e+, mu+ ; l- = e-, mu- ; vl = ve, vm ; vl~ = ve~, vm~

Submit

Limiting factors

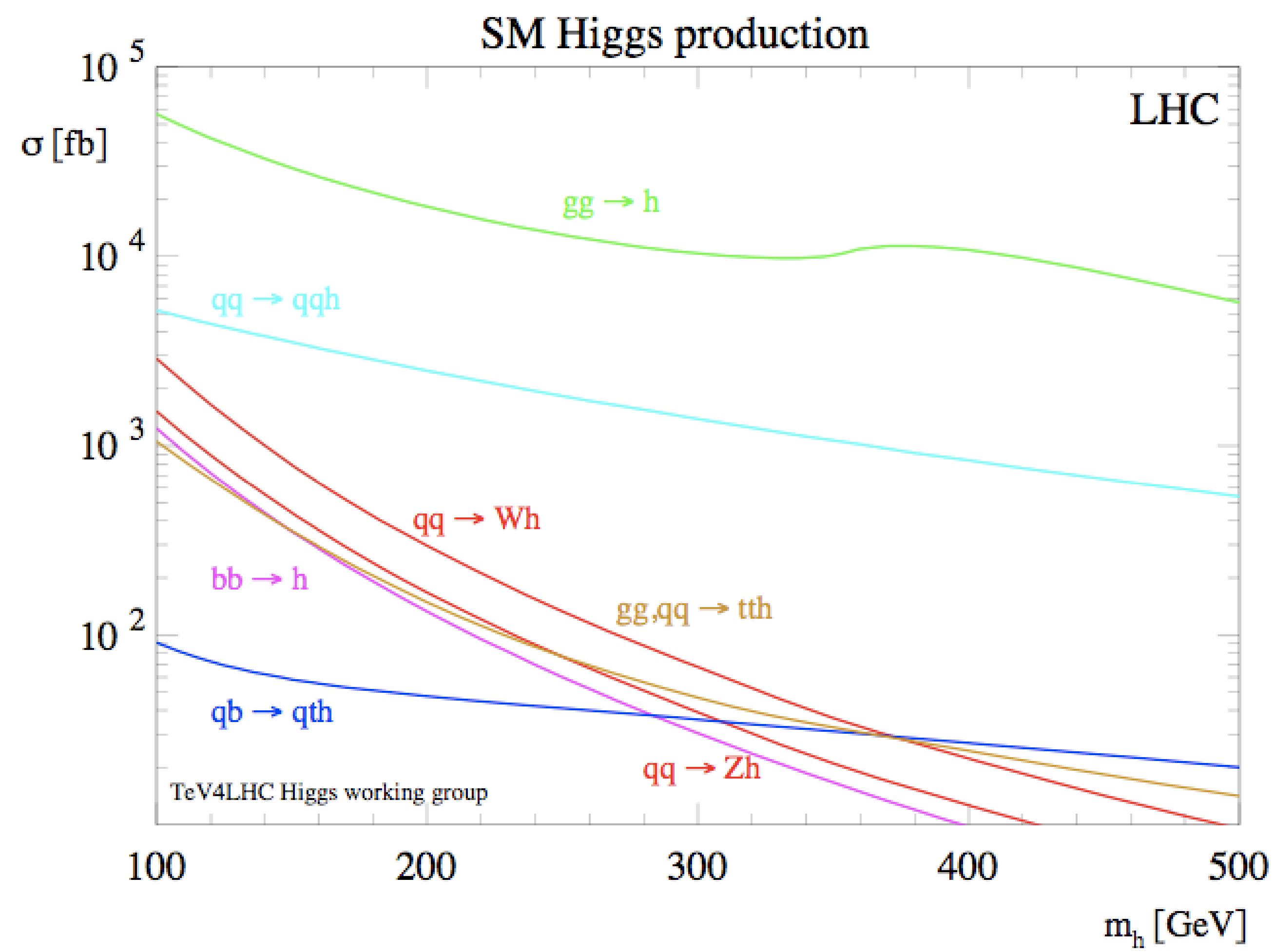
- Solved problem in principle, but computing power is still an issue.
- This is mostly because the number of Feynman diagrams entering the amplitude calculation grows factorially with the number of external particles.
    - hence smart (recursive) methods to generate matrix elements.
- Demonstrated by the time taken to generate 10,000 events involving 2 gluons in the initial state and up to 10 in the final state.
- The lower curve shows a smarter treatment of color factors, which become a limiting factor too.
    - active research area.



(adapted from C. Duhr et al., 2006)

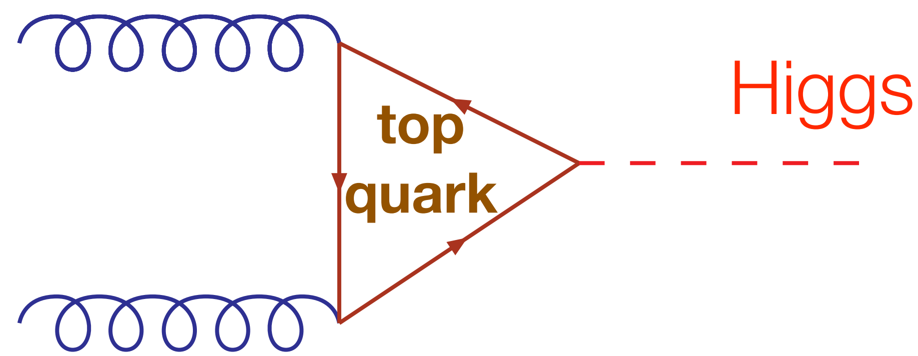
Beyond fixed order

- Ten gluons in the final state is a lot - but doesn't come close to the typical particle multiplicity in a usual event.
- Moreover, we want a tool that says something about hadrons, not partons.
- How can we hope to build something like this from scratch, using QCD?
- Answer: yes! - due to a particular universal behaviour of QCD cross sections.
- To demonstrate this, we start with a short detour into some Higgs physics.
- Shown here are cross sections for different Higgs production modes at the (14 TeV) LHC.
- Here we are interested in the mode with the largest cross section: gluon fusion.



Higgs coupling to gluons

- How does this coupling take place? Certainly not directly!



- The answer is through a loop, with the Higgs coupling preferentially to the heaviest quark available: the top quark.

- In general, loop-induced processes are suppressed compared to tree-level contributions - but at the LHC, gluons will be plentiful (esp. compared to antiquarks - more on that later).

- We're not going to perform this computation here, but note that in the limit that the top mass is infinite the result is formally equivalent to the coupling obtained by adding a term to the Lagrangian:

$$\mathcal{L}_{ggH} = \frac{C}{2}\, H\, F^{A}_{\mu\nu} F^{\mu\nu}_{A}$$

$C = \dfrac{\alpha_s}{6\pi v}$

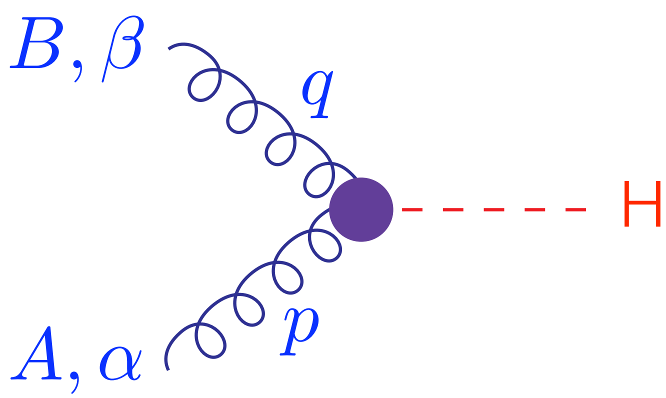
Higgs field

same field strength as before

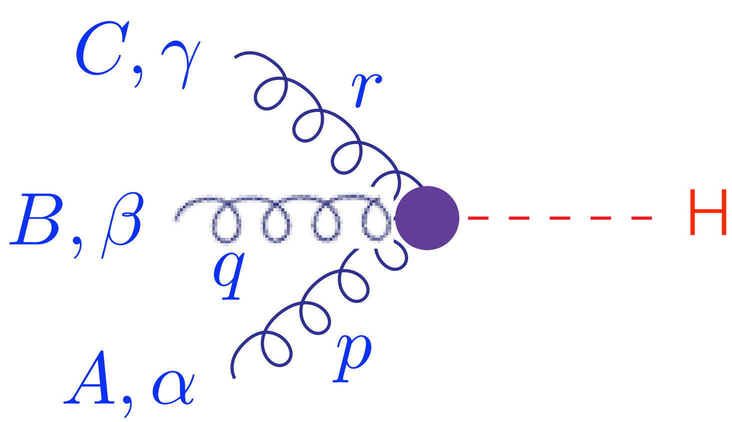
"Effective Theory" gives rise to ggH coupling and new Feynman rules.

Feynman rules: effective theory

- Also get 3- and 4-point vertices that mimic the structure of the pure QCD case.

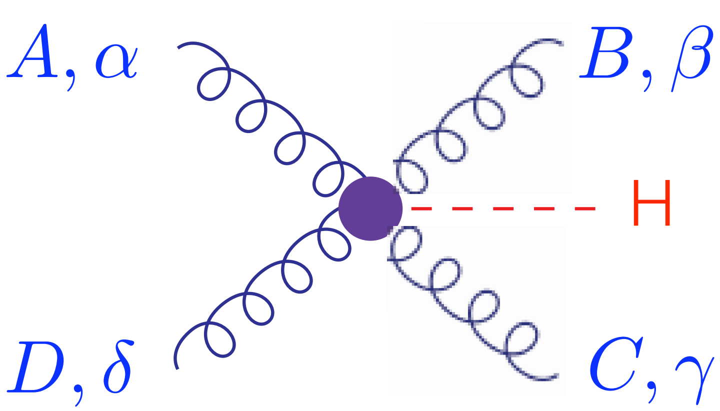


$$iC\delta^{AB}\left(p\cdot q\, g^{\alpha\beta} - p^\beta q^\alpha\right)$$



$$\begin{aligned}-Cg_s f^{ABC}\Big[&g^{\alpha\beta}(p^\gamma - q^\gamma)\\ &+g^{\beta\gamma}(q^\alpha - r^\alpha)\\ &+g^{\gamma\alpha}(r^\beta - p^\beta)\Big]\end{aligned}$$

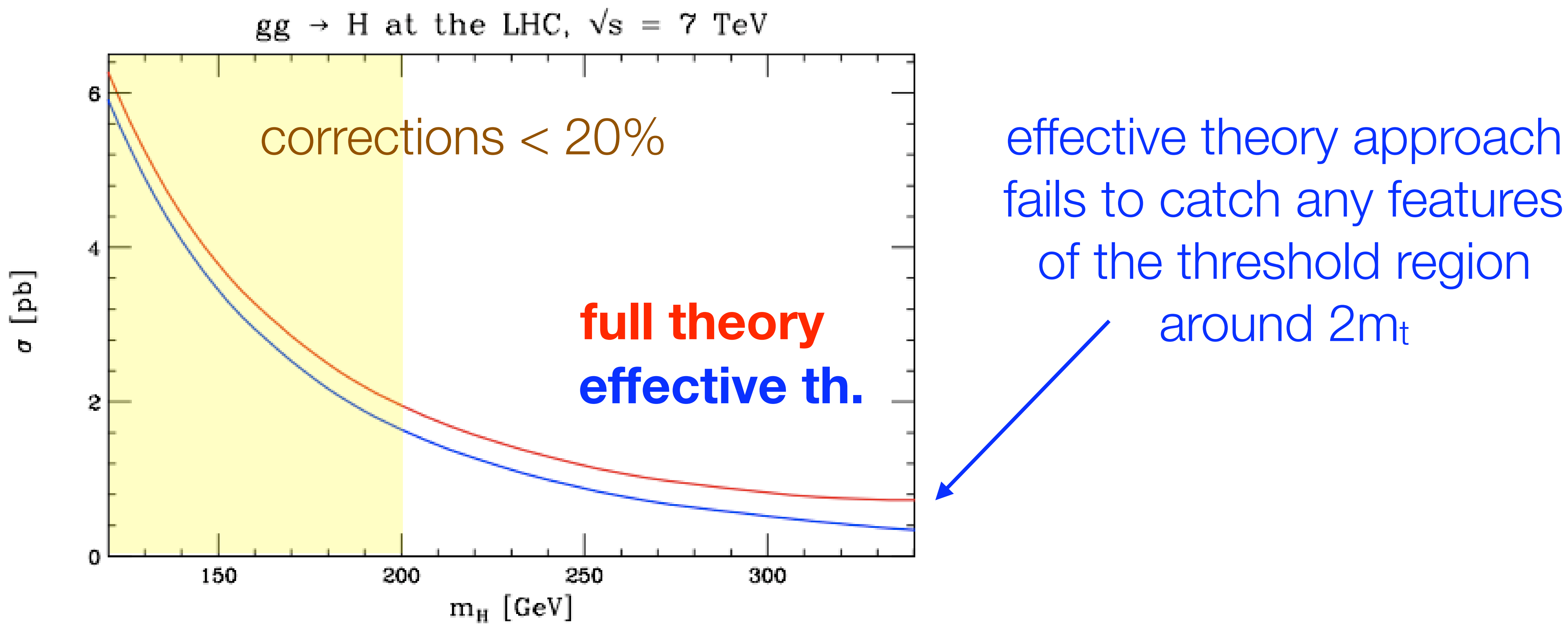
(all momenta incoming)



$$\begin{aligned}&-iCg_s^2 f^{ABX} f^{XCD}\left[g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\gamma\beta}\right]\\ &-iCg_s^2 f^{BCX} f^{XAD}\left[g^{\beta\alpha}g^{\gamma\delta} - g^{\beta\delta}g^{\alpha\gamma}\right]\\ &-iCg_s^2 f^{BCX} f^{XAD}\left[g^{\gamma\beta}g^{\alpha\delta} - g^{\gamma\delta}g^{\beta\alpha}\right]\end{aligned}$$

Effective theory

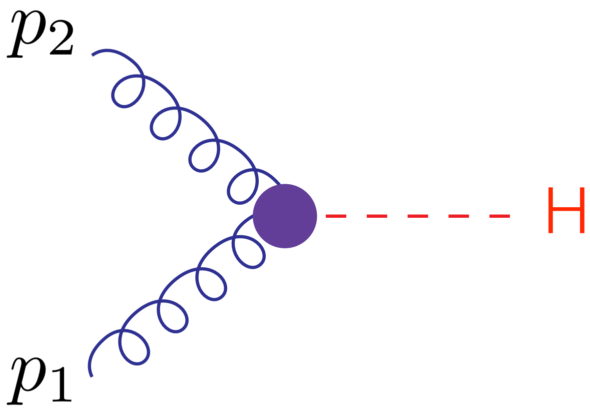
- This effective theory is a good approximation.



- Moreover it is very useful for more complicated calculations
  - chain new vertices together in order to compute cross sections that would be intractable in the full (finite top mass) theory.
  - e.g. producing additional quarks or gluons (i.e. jets).

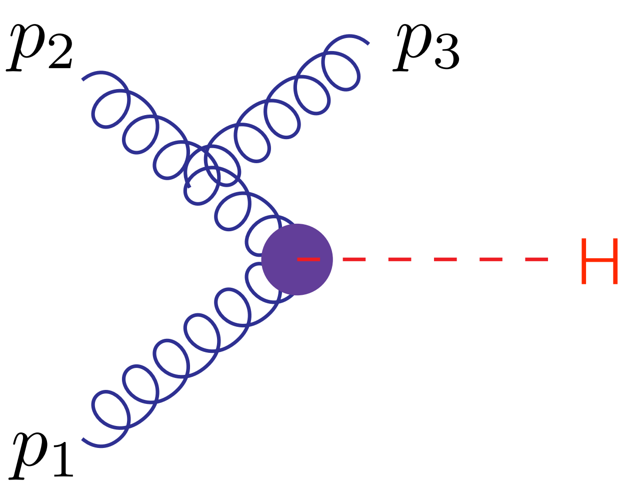
Matrix elements

- First look at the squared matrix elements for this process.



$$|\mathcal{M}_{Hgg}|^2 = 2(N_c^2 - 1)C^2 m_H^4$$

- Now consider adding a gluon (total of 4 diagrams - remember triple-gluon+H).



$$|\mathcal{M}_{Hggg}|^2 = 4N_c(N_c^2 - 1)C^2 g_s^2 \times \left(\frac{m_H^8 + (2p_1.p_2)^4 + (2p_1.p_3)^4 + (2p_2.p_3)^4}{8p_1.p_2\, p_1.p_3\, p_2.p_3}\right)$$

- Inspect this in the limit that gluons 2 and 3 are collinear:

$$p_2 = zP\ , \quad p_3 = (1-z)P$$

Collinear limit: gluons

- Under this transformation we can make the replacements:

$$2p_1.p_2 \to z m_H^2 \ , \quad 2p_1.p_3 \to (1-z) m_H^2 \ , \quad 2p_2.p_3 \to 0 \ ,$$

and simply read off the answer:

$$|\mathcal{M}_{Hggg}|^2 \xrightarrow{\text{coll.}} 4N_c(N_c^2-1)C^2 g_s^2 m_H^4 \left(\frac{1+z^4+(1-z)^4}{2z(1-z)p_2.p_3}\right)$$

- This clearly shares some features with the ggH matrix element squared we just calculated, which we can exploit to write it in a new way.

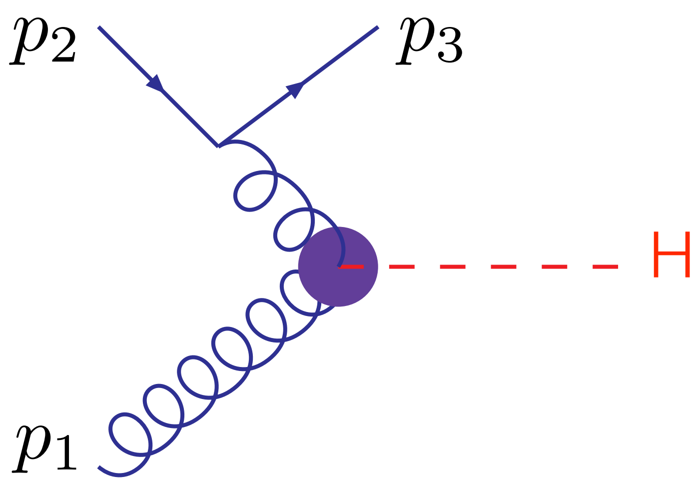
$$|\mathcal{M}_{Hggg}|^2 \xrightarrow{\text{coll.}} \frac{2g_s^2}{2p_2.p_3} |\mathcal{M}_{Hgg}|^2 P_{gg}(z)$$

where the collinear splitting function, which only depends on the relative weight in the splitting (*z*), is defined by:

$$P_{gg}(z) = 2N_c \left(\frac{z^2+(1-z)^2+z^2(1-z)^2}{z(1-z)}\right)$$

Collinear limit: quarks

- Same trick with the two collinear gluons replaced by quark-antiquark pair.



$$|\mathcal{M}_{Hg\bar{q}q}|^2 = 4T_R(N_c^2-1)C^2g_s^2 \times \left(\frac{(2p_1.p_2)^2+(2p_1.p_3)^2}{2p_2.p_3}\right)$$

- We find a similar result. In the collinear limit, the matrix element squared is again proportional to the matrix element with one less parton:

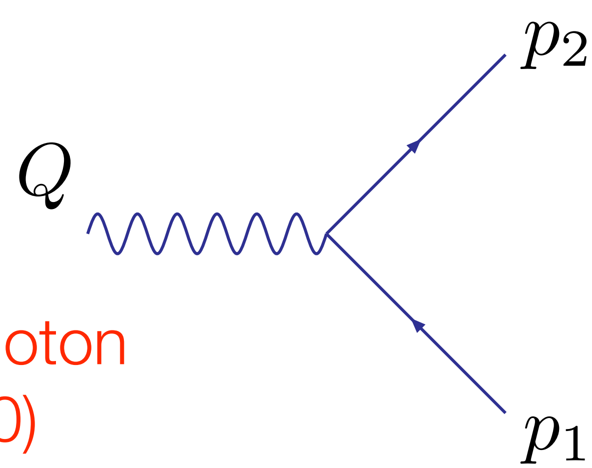
$$|\mathcal{M}_{Hg\bar{q}q}|^2 \xrightarrow{\text{coll.}} \frac{2g_s^2}{2p_2.p_3}|\mathcal{M}_{Hgg}|^2 P_{qg}(z)$$

The splitting function this time is given by:

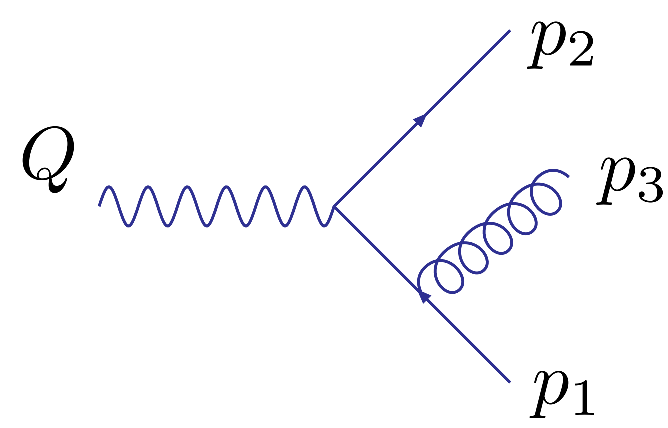
$$P_{qg}(z) = T_R\left(z^2+(1-z)^2\right)$$

Collinear limit: quark-gluon

- To investigate this last case, we need slightly less exotic matrix elements.



virtual photon
(Q^2>0)



$$|\mathcal{M}_{\gamma^*\bar{q}q}|^2 = 4N_c e_q^2 Q^2$$

$$|\mathcal{M}_{\gamma^*\bar{q}qg}|^2 = 8N_c C_F e_q^2 g_s^2 \times \left(\frac{(2p_1.p_3)^2 + (2p_2.p_3)^2 + 2Q^2(2p_1.p_2)}{4\,p_1.p_3\,p_2.p_3}\right)$$

- A similar analysis, with the gluon carrying momentum fraction *(1-z)*, leads to the result:

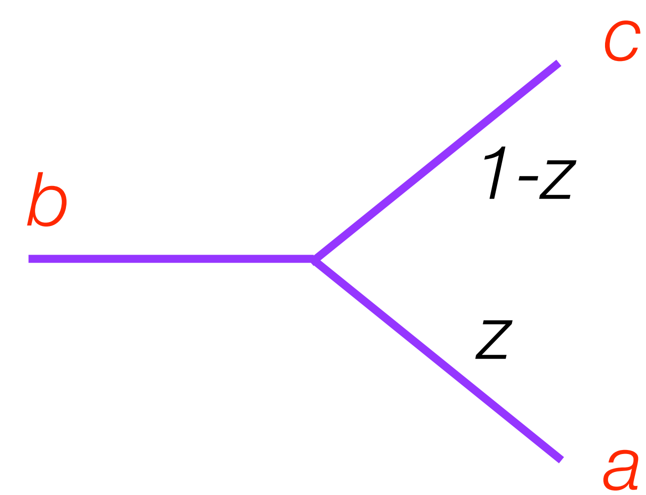
$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)$$

Universal factorization

- The important feature of these results is that they are universal, i.e. they apply to the appropriate collinear limits in all processes involving QCD radiation.
- They are a feature of the QCD interactions themselves.

$$|\mathcal{M}_{ac\ldots}|^2 \xrightarrow{a,\,c\text{ coll.}} \frac{2g_s^2}{2p_a.p_c}|\mathcal{M}_{b\ldots}|^2 P_{ab}(z)$$

collinear singularity



$$P_{qq}(z) = C_F\left(\frac{1+z^2}{1-z}\right)$$

additional soft singularity as $z \to 1$

$$P_{gg}(z) = 2N_c\left(\frac{z^2+(1-z)^2+z^2(1-z)^2}{z(1-z)}\right)$$

soft for $z \to 0$, $z \to 1$

$$P_{qg}(z) = T_R\left(z^2+(1-z)^2\right)$$

Infrared singularities

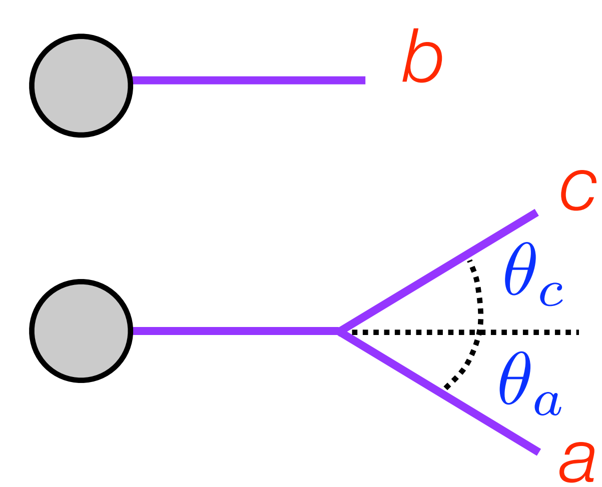
- These are called infrared singularities, which occur when relevant momenta become small.
  - they are thus indicative of long-range phenomena which are, by definition, not well described by perturbation theory.
  - at such scales are approached, hadronization takes over and apparent singularities are avoided.
- In perturbative QCD we must avoid such issues by restricting our attention to infrared safe quantities that are insensitive to such regions.
  - for example: in our leading order calculations, we try to describe jets with large transverse momenta, not arbitrarily soft particles.
  - we shall see later on that it is sometimes useful to regularize such singularities: they can appear in intermediate steps of a calculation, but must disappear at the end (for physical observables).
    - this is a statement of the Kinoshita-Lee-Nauenberg (KLN) theorem.

The silver lining

- On the positive side:
  - we have learned that emission of soft and collinear partons is favoured;
  - we know exactly the form of the required matrix elements when that occurs.
- In fact it's even better than this - it applies to the phase space too.
- Start from the standard phase space formula:

$$d\mathrm{PS}_{(\ldots)b} = (\ldots)\,\frac{d^3\vec{p}_b}{(2\pi)^3 2E_b}$$

$$d\mathrm{PS}_{(\ldots)ac} = (\ldots)\,\frac{d^3\vec{p}_a}{(2\pi)^3 2E_a}\,\frac{d^3\vec{p}_c}{(2\pi)^3 2E_c}$$



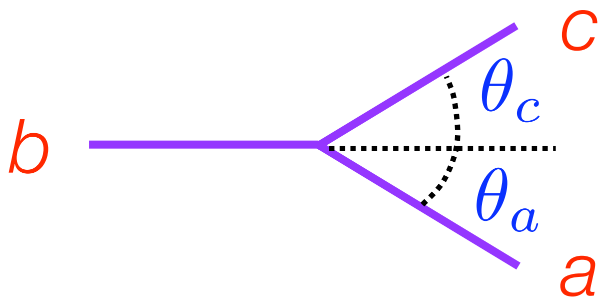
and note that, if we fix the momentum of *a*, we can relate these by:

$$d\mathrm{PS}_{(\ldots)ac} = d\mathrm{PS}_{(\ldots)b}\frac{d^3\vec{p}_a}{(2\pi)^3 2E_a}\,\frac{E_b}{E_c} \approx d\mathrm{PS}_{(\ldots)b}\frac{1}{(2\pi)^2}\,\frac{E_a E_b}{2E_c}\,dE_a\,\theta_a d\theta_a$$

(for $\theta_a \sim 0$)

Small angle approximation

- "Small angle" kinematics of the collinear limit:



$$p_a = z p_b \,, p_c = (1-z) p_b$$

$$\Longrightarrow \; E_a = z E_b \,, E_c = (1-z) E_b$$

$$z\theta_a - (1-z)\theta_c = 0 \qquad \Longrightarrow \; \theta_a = (1-z)(\theta_a + \theta_c)$$

- Introduce new variable *t* to describe virtuality of *b*, related to opening angle:

$$t = (p_a + p_c)^2 = 2E_a E_c (1 - \cos(\theta_a + \theta_c)) = E_b^2 z(1-z)(\theta_a + \theta_c)^2 = \frac{z E_b^2 \theta_a^2}{1-z}$$

- Hence we can write the factorized form in this limit as,

$$d\mathrm{PS}_{(\ldots)ac} = d\mathrm{PS}_{(\ldots)b} \frac{1}{(2\pi)^2} \frac{E_a E_b}{2E_c} \frac{(1-z)E_b}{2zE_b^2} \, dz \, dt \; = d\mathrm{PS}_{(\ldots)ac} \frac{dz \, dt}{16\pi^2}$$

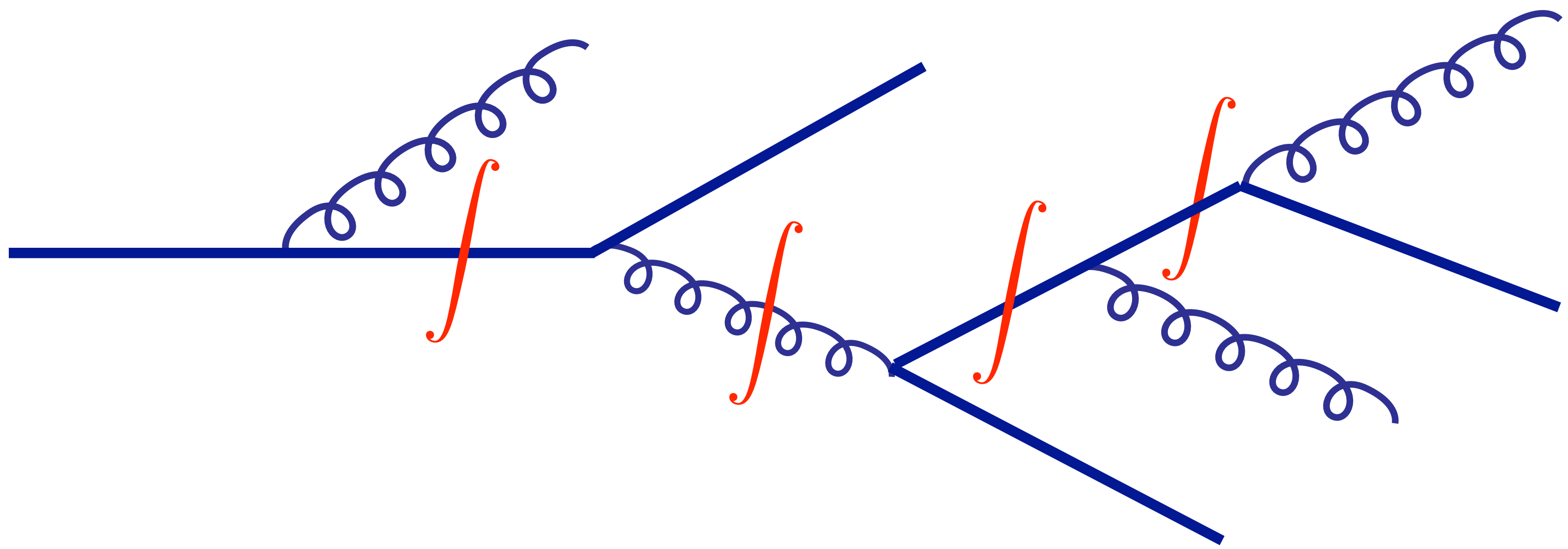
- Combining this with our previous matrix element factorization formula gives:

$$d\sigma_{(\ldots)ac} = |\mathcal{M}_{(\ldots)ac}|^2 d\mathrm{PS}_{(\ldots)ac} = d\sigma_{(\ldots)b} \left(\frac{\alpha_s}{2\pi}\right) \frac{dt}{t} P_{ab}(z) \, dz$$

Parton showers

$$d\sigma_{n+1} = d\sigma_n \left(\frac{\alpha_s}{2\pi}\right) \frac{dt}{t} P_{ab}(z)\, dz$$

- This is an important equation: it tells us how we can generate additional soft and collinear radiation ad infinitum.



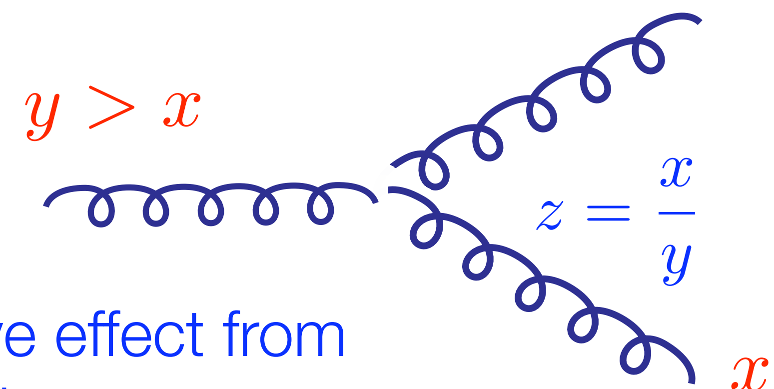
- Technically this is called timelike branching since we have implicitly assumed that all particles are outgoing ($t>0$).
  - extension to the spacelike case (radiation on an incoming line) is similar.
- This is the principle upon which all parton shower simulations are based.

Popular parton shower programs

PYTHIA	T. Sjöstrand et al. http://home.thep.lu.se/~torbjorn/Pythia.html
HERWIG	G. Corcella et al. http://hepwww.rl.ac.uk/theory/seymour/herwig/
HERWIG++	S. Gieseke et al. http://projects.hepforge.org/herwig/
SHERPA	F. Krauss et al. http://projects.hepforge.org/sherpa/dokuwiki/doku.php
ISAJET	H. Baer et al. http://www.nhn.ou.edu/~isajet/

Inside a parton shower

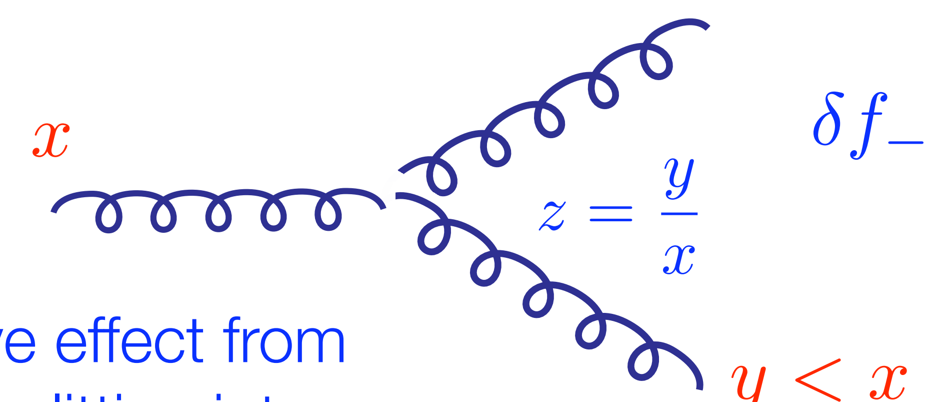
- The defining equation can be interpreted in terms of the probability of having a parton branching with given *(x,t)* at some point in the shower: let's call it *f(x,t)*.
- For simplicity, let's assume that the evolution doesn't change the parton species, e.g. an all-gluon shower (extension is straightforward).
- Now consider a small change from *t* to *t+δt* and its effect on *f(x,t)*.



+ve effect from higher momenta splitting

$$\delta f_+(x,t) = \frac{\delta t}{t} \int_x^1 dy\, dz \left(\frac{\alpha_s}{2\pi}\right) P_{gg}(z) f(y,t) \delta(x - zy)$$

$$= \frac{\delta t}{t} \int_x^1 \frac{dz}{z} \left(\frac{\alpha_s}{2\pi}\right) P_{gg}(z) f(x/z,t)$$



-ve effect from splitting into smaller momenta

$$\delta f_-(x,t) = \frac{\delta t}{t} f(x,t) \int_0^x dy\, dz \left(\frac{\alpha_s}{2\pi}\right) P_{gg}(z) \delta(y - zx)$$

$$= \frac{\delta t}{t} f(x,t) \int_0^1 dz \left(\frac{\alpha_s}{2\pi}\right) P_{gg}(z)$$

The DGLAP equation

- By taking the difference can reinterpret this as a differential equation for *f(x,t)*:

$$t\,\frac{\partial f(x,t)}{\partial t} = \int_0^1 dz\left(\frac{\alpha_s}{2\pi}\right)P_{ab}(z)\left(\frac{1}{z}f(x/z,t) - f(x,t)\right)$$

- This is called the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation.
- It is most convenient to expose a solution to this equation by introducing a Sudakov form factor, Δ(t).

$$\Delta(t) = \exp\left[-\int_{t_0}^{t}\frac{dt'}{t'}\int dz\left(\frac{\alpha_s}{2\pi}\right)P_{ab}(z)\right]$$

- Hence we can rewrite as:

$$t\,\frac{\partial f(x,t)}{\partial t} = \int\frac{dz}{z}\left(\frac{\alpha_s}{2\pi}\right)P_{ab}(z)f(x/z,t) + \frac{f(x,t)}{\Delta(t)}\,\frac{t\,\partial\Delta(t)}{\partial t}$$

$$\Longrightarrow\; t\,\frac{\partial}{\partial t}\left(\frac{f(x,t)}{\Delta(t)}\right) = \frac{1}{\Delta(t)}\int\frac{dz}{z}\left(\frac{\alpha_s}{2\pi}\right)P_{ab}(z)f(x/z,t)$$

The Sudakov form factor

- Integrate up to find solution given boundary condition at $t=t_0$:

$$f(x,t) = \Delta(t) f(x,t_0) + \int_{t_0}^{t} \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \left(\frac{\alpha_s}{2\pi}\right) P_{ab}(z) f(x/z,t)$$

no branching between t_0 and t

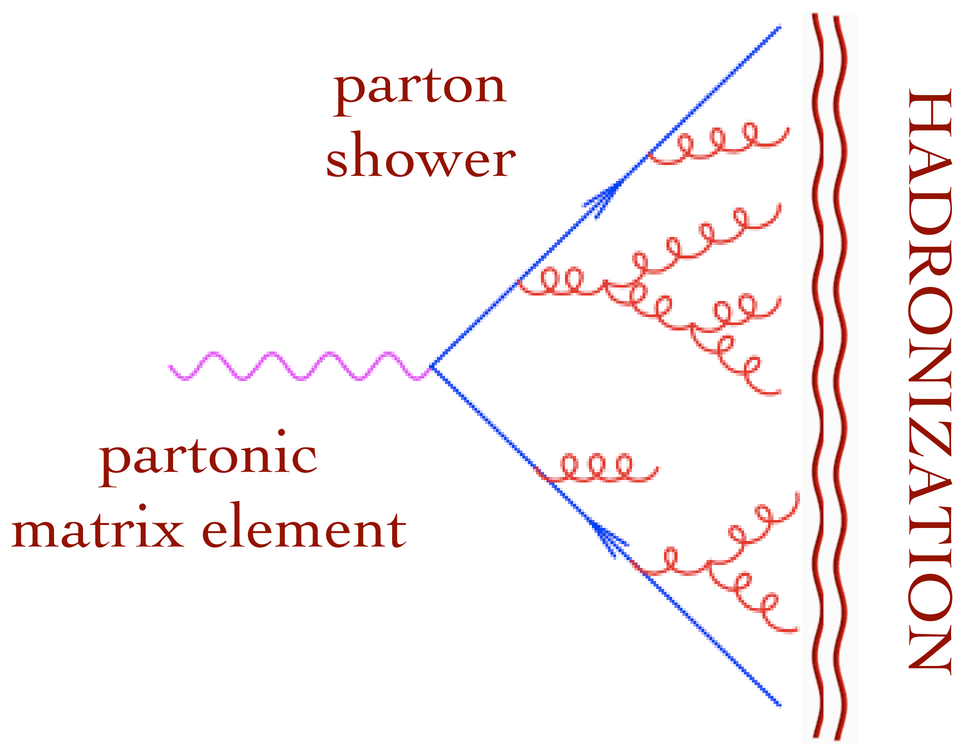
integrate over multiple branchings; for each value of t', no branching between t' and t

- Interpret Sudakov form factor as the probability for no parton emission
  - better: no resolvable parton emission. We must cut off the *z*-integration as $z \rightarrow 1$ to avoid the singularities we found before. Above cutoff unresolvable.
- The Sudakov interpretation lends itself to Monte Carlo methods (universally used in parton showers):
  - pick a random number *r* in [0,1] and determinate t_2 from t_1 from $\dfrac{\Delta(t_2)}{\Delta(t_1)} = r$
  - can generate *z* according to integral over correct P_{ab} for splitting.

Ending the shower

- Eventually the evolution will bring us to a very small scale of *t* at which we no longer believe in the perturbation theory (say ~ 1 GeV). Beyond that point we no longer perform any branching.

- All partons produced in this shower are showered further, until same condition.



- Once this point is reached, no more perturbative evolution possible.

- Partons should be interpreted as hadrons according to a hadronization model.

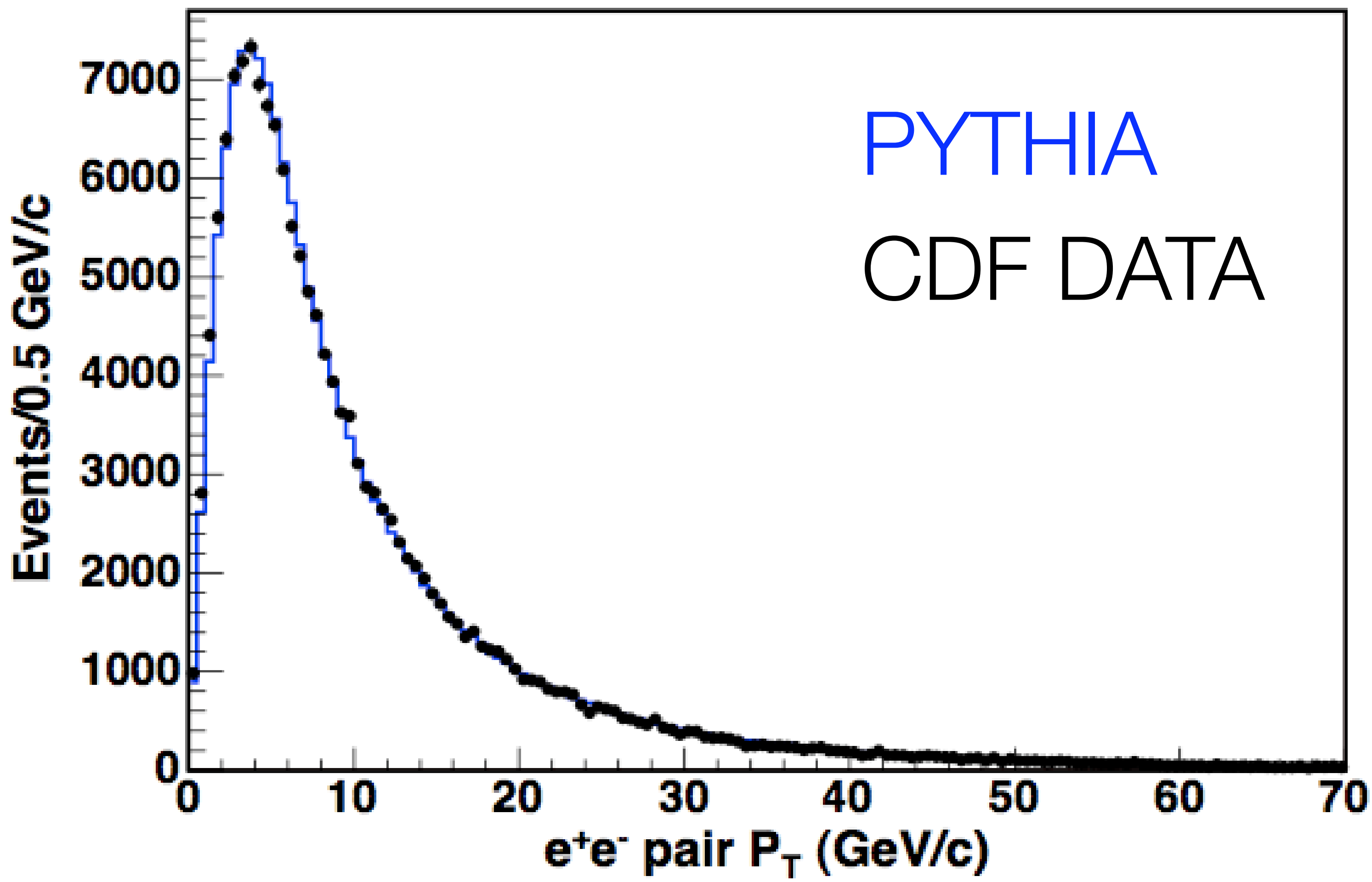
  - examples: string model, cluster model.

- Most importantly: these are all phenomenological models.

- They require inputs that cannot be predicted from the QCD Lagrangian ab initio and must therefore be tuned by comparison with data (mostly LEP).

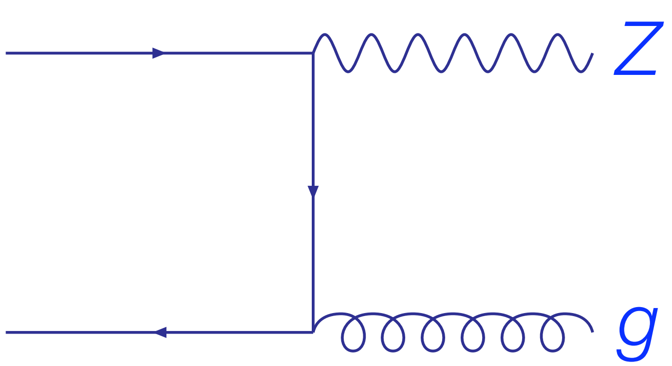
What did we win?

- A parton shower allows us to (attempt to) describe features of the whole event: the output is high multiplicity final states containing hadrons.
- Very flexible framework. In principle, start with any hard scattering (e.g. any theorist's latest and greatest model) and the PS takes care of QCD radiation.



Z boson transverse momentum

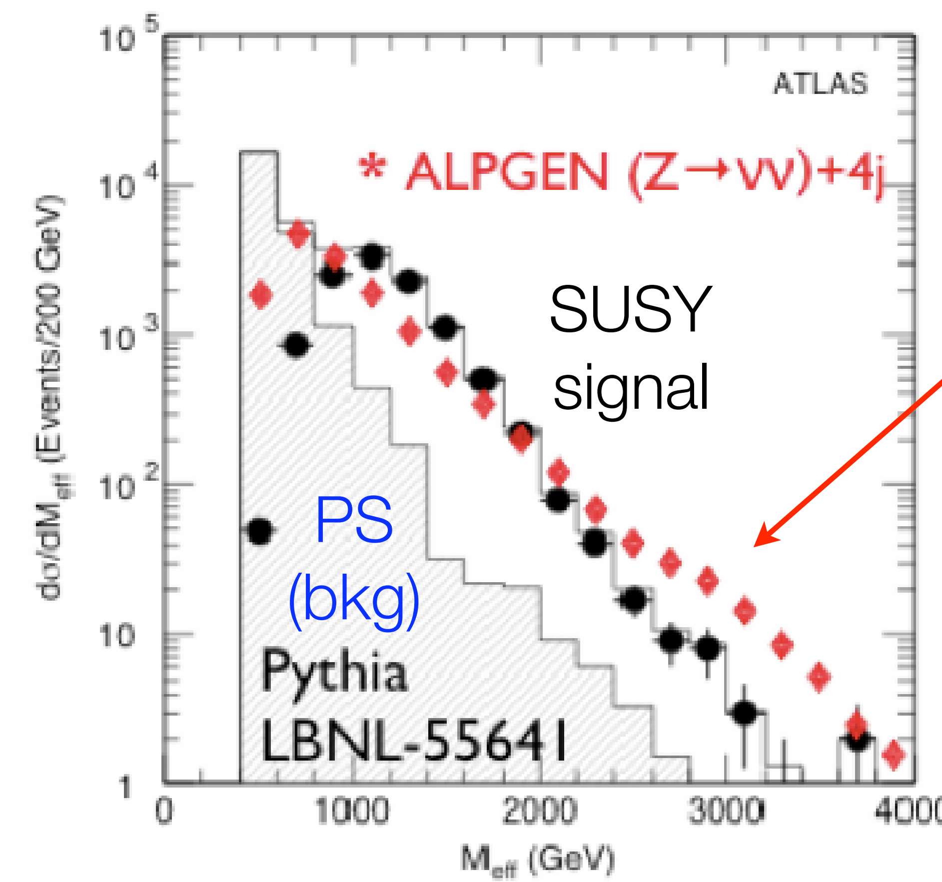
- In contrast to a pure leading order prediction, a parton shower can be matched to data even at low p_T.



- This is true in general: broader region of applicability.

Warnings

- By construction, a parton shower is correct only for successive branchings that are collinear or soft (formally called leading log).
- Should therefore take care when describing final states in which there is either manifestly multiple hard radiation, or its effects might be important.
    - example: simulation of background to a SUSY search in the ATLAS TDR.



improved background calculation

$$M_{\text{eff}} = \Sigma_i |p_{T(i)}| + \not{E}_T$$

- Also: full higher-order corrections are not included (more on this later).
- Uncertainty can only be estimated by comparison with data and/or between different parton shower implementations.
    - the gory details of each shower are often quite different.

Recap

- There are many tools capable of producing leading order cross section predictions from scratch.
- They are limited only by computer power: as a result, cannot generate more than 10 particles in the final state (program/process specific).
- The factorization of both QCD matrix elements and phase space, in the soft and collinear limits, allows us to generate arbitrarily many such branchings.
  - factorization of matrix elements: universal Altarelli-Parisi splitting functions
  - factorization of phase space: small angle approximation.
- Such a formalism leads to a DGLAP evolution equation for the probability of finding a given parton within the branching process.
- Introducing a Sudakov form factor leads to an interpretation which is easy to implement as a parton shower (e.g. Pythia, Herwig, Sherpa).
  - can describe exclusive final states (hadrons), even down to small scales;
  - in regions of hard radiation the soft/collinear approx. may not be sufficient.