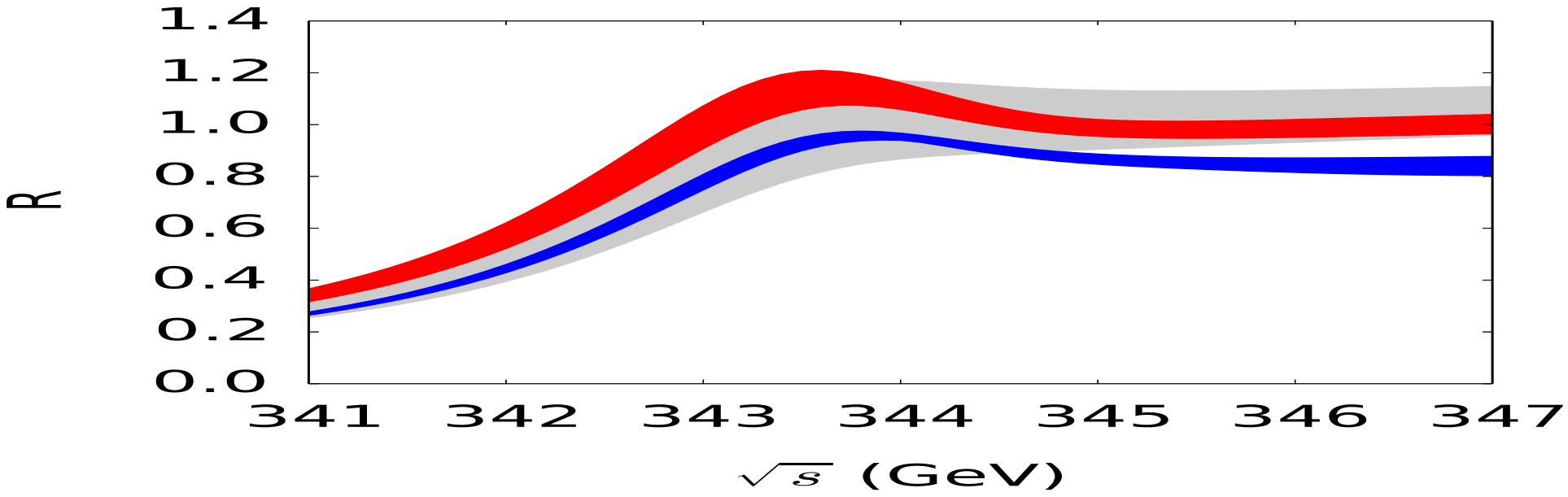
NNNLO corrections to top quark pair production at threshold at the ILC

Matthias Steinhauser | TTP Karlsruhe

Loopfest 2014, June 18-20, 2014

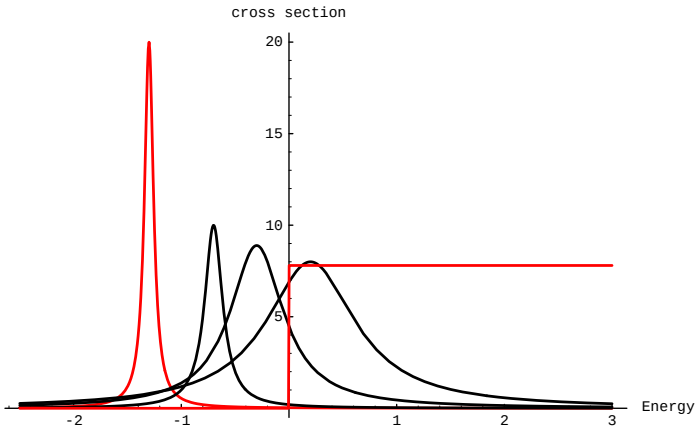


Outline

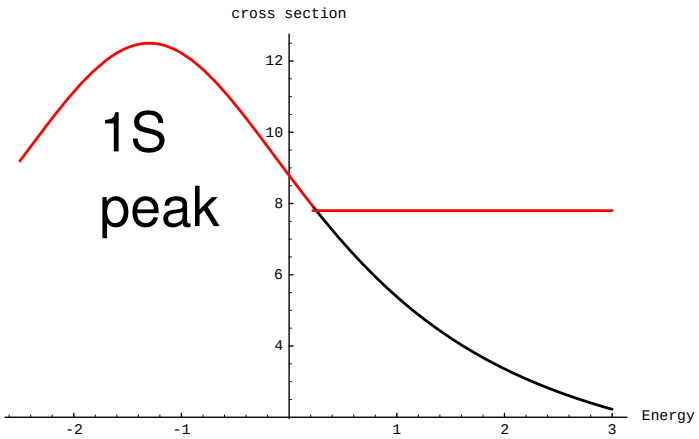
- Motivation/Status
- Framework
- Results

Results obtained in collaboration with
Martin Beneke, Yuichiro Kiyo, Peter Marquard, Alexander Penin, Jan Piclum, Kurt Schuller, Dirk Seidel, Alexander Smirnov, Vladimir Smirnov

“ $e^+e^- \to b\bar{b}$ ”



“ $e^+e^- \to t\bar{t}$ ”



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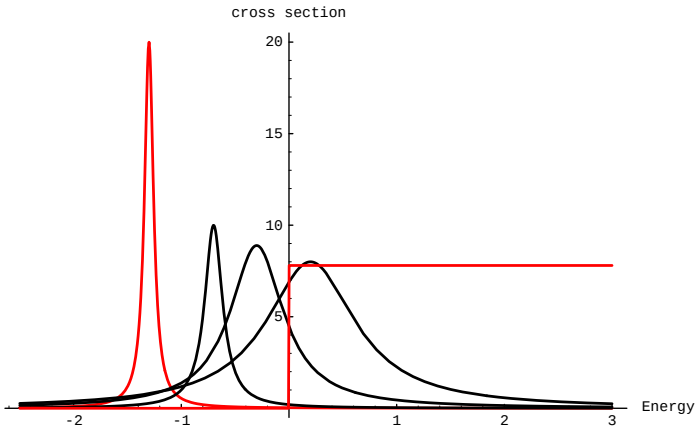

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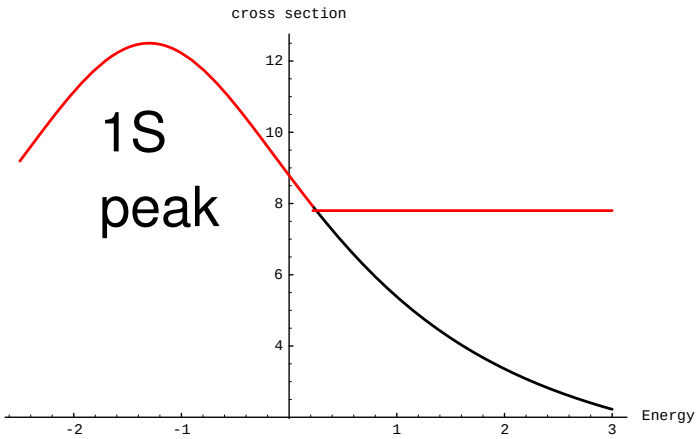
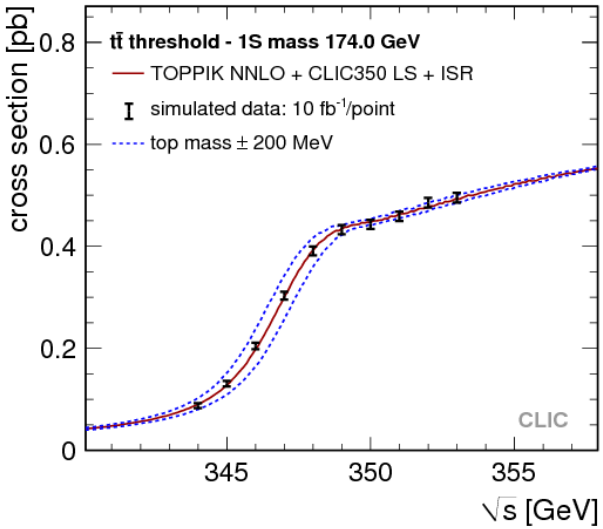




$\delta m_t \sim 100$ MeV

$\delta \Gamma_t \sim 30$ MeV

$\delta \alpha_s \sim 0.001$

$\delta y_t \sim 30\%$

[Martinez,Miquel’02;

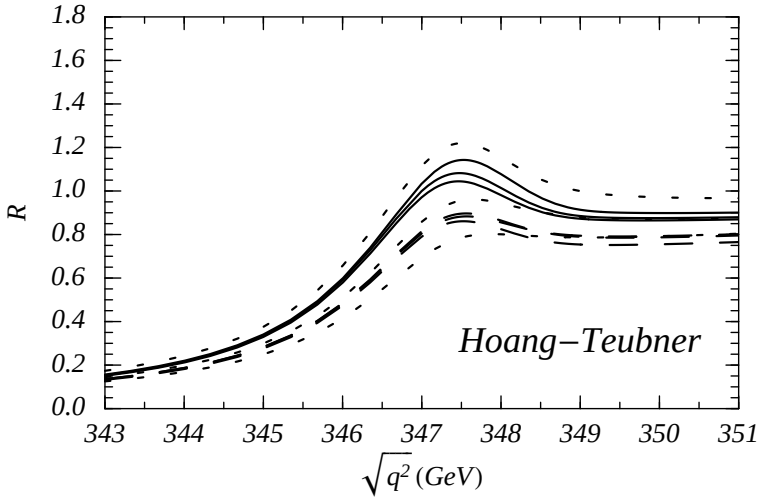
Seidel,Simon,Tesar,Poss’13;

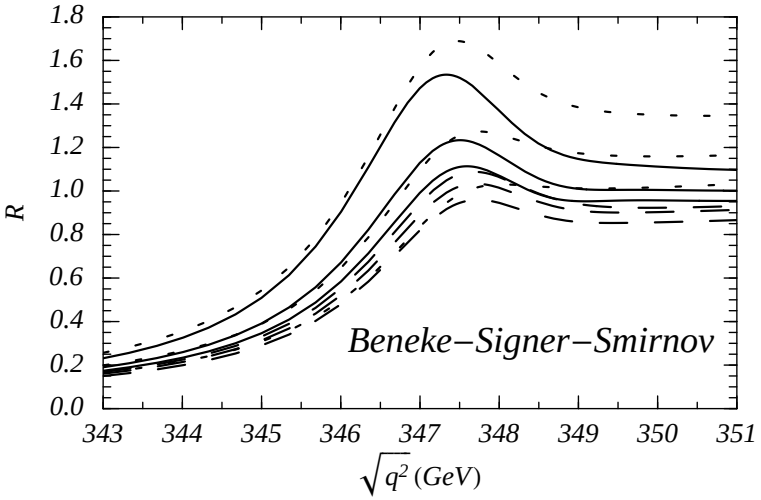
Horiguchi,Ishikawa,Suehara,

Fujii,Sumino,Kiyo,Yamamoto’14]

$e^+e^- \to t\bar{t}$ at NNLO: 2000

[Hoang,Beneke,Melnikov,Nagano,Ota,Penin,Pivovarov,Signer,Smirnov,Sumino,Teubner,Yakovlev,Yelkhovsky'00]

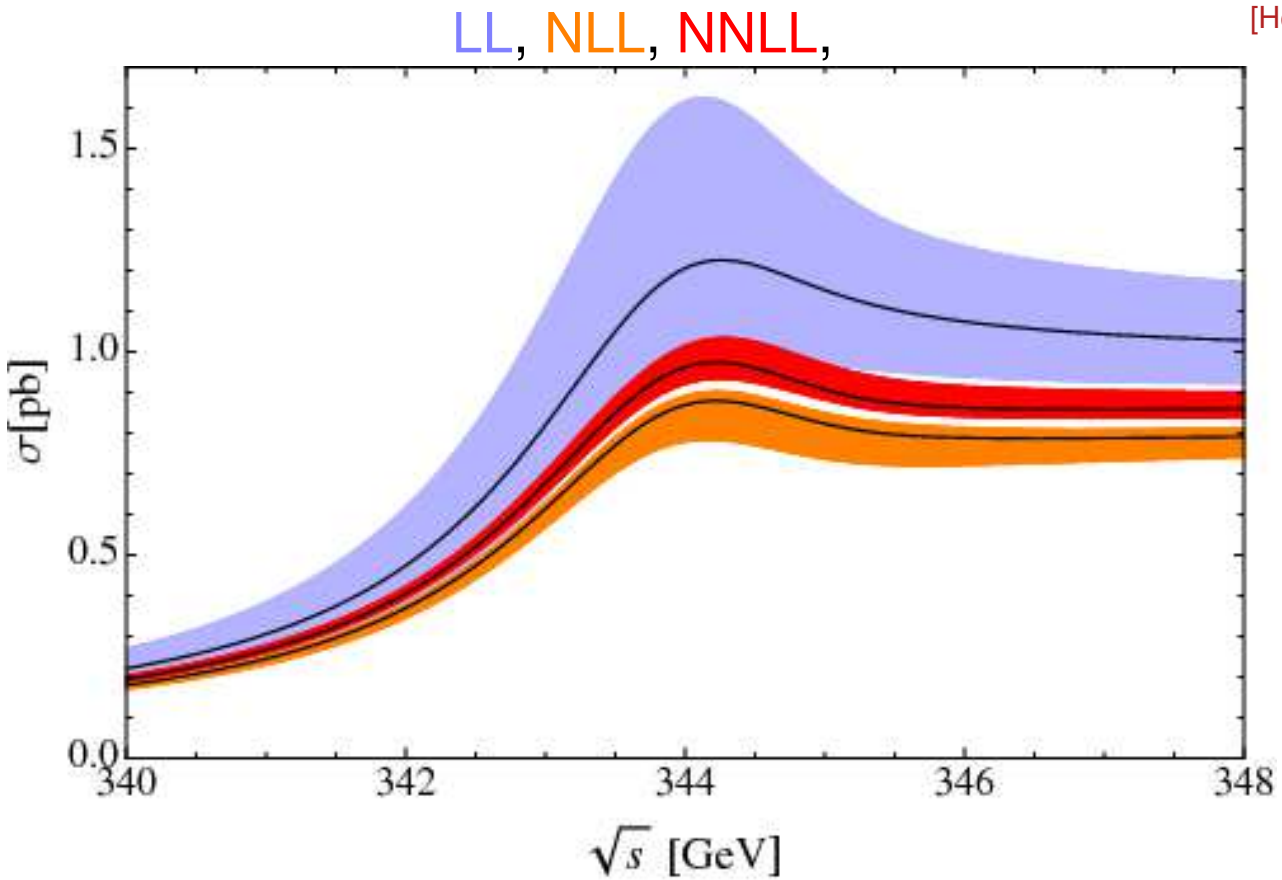




- stabilization of peak position ("threshold mass")
- no stability in normalization of peak
- large differences between different groups

$e^+e^- \to t\bar{t}$ at NNLL: resum $(\alpha_s \ln v)^n$

[Hoang,Stahlhofen'14]



- NNLL not complete
- no overlap of bands from μ variation
- $\delta\sigma/\sigma = \pm 5\%$ (peak)
- see also: [Hoang,Manohar,Stewart,Teubner'02; Pineda,Signer'07]

Framework: potential NRQCD

scales: mass, m: hard $\gg$ momentum, mv: soft $\gg$ energy, mv^2: ultrasoft $\gg \Lambda_{\mathrm{QCD}}$

potential quarks: $\begin{cases} E_{\vec{p}} \sim mv^2 \\ |\vec{p}\,| \sim mv \end{cases}$ $\quad \dfrac{1}{E_{\vec{p}} - \frac{\vec{p}^{\,2}}{2m}}$

ultrasoft gluons: $\begin{cases} E_{\vec{k}} \sim mv^2 \\ |\vec{k}\,| \sim mv^2 \end{cases}$ $\quad \dfrac{1}{E_{\vec{k}}^2 - \vec{k}^{\,2}}$

QCD ⇨ NRQCD ⇨ pNRQCD (potential non-relativistic QCD)

↑ integrate out hard scale "m" from QCD [Caswell,Lepage'86; Bodwin,Braaten,Lepage'95]

↑ integrate out all scales from NRQCD except potential quarks and ultrasoft gluons [Beneke,Smirnov'97; Pineda,Soto'98; Brambilla,Pineda,Soto,Vairo'00]

alternative formulation: velocity NRQCD (vNRQCD)

[Luke,Manohar,Rothstein'00; Hoang,Stewart'03]

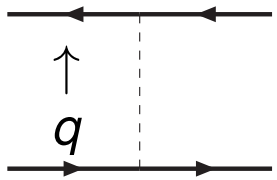
Effective Hamiltonian to N^{3}LO

[Gupta,Radford'81,...,Manohar'97,...,Kniehl,Penin,Smirnov,Steinhauser'02,...,Beneke,Kiyo,Schuller'13]

$$\mathcal{H} = (2\pi)^3\delta(\vec{q})\left(\frac{\vec{p}^{\,2}}{m} - \frac{\vec{p}^{\,4}}{4m^3}\right) + C_c(\alpha_s)V_C(|\vec{q}|) + C_{1/m}(\alpha_s)V_{1/m}(|\vec{q}|)$$
$$+\frac{\pi C_F\alpha_s(\mu)}{m^2}\left[C_\delta(\alpha_s) + C_p(\alpha_s)\frac{\vec{p}^{\,2}+\vec{p}^{\,\prime 2}}{2\vec{q}^{\,2}} + C_s(\alpha_s)\vec{S}^{\,2}\right]$$

Static potential:	$V_C(\lvert\vec{q}\rvert) = -\frac{4\pi C_F\alpha_s(\lvert\vec{q}\rvert)}{\vec{q}^2}$	C_c	3 loops
$1/m$ potential:	$V_{1/m}(\lvert\vec{q}\rvert) = \frac{\pi^2 C_F\alpha_s^2(\lvert\vec{q}\rvert)}{m\lvert\vec{q}\rvert}$	$C_{1/m}$	2 loops
"Breit" potential:	$\propto 1/m^2$	$C_{\delta,p,s}$	1 loop

$$\vec{q} = \vec{p}\,' - \vec{p}$$



$\sigma_{\rm tot}(e^+e^- \to t\bar{t})$ in pNRQCD

$$\sigma(s) = \sigma_0 \,\mathrm{Im}\Big[\,\Pi(q^2 = s + i\epsilon) + Z \text{ boson contr.}\Big]$$

$$\Pi = \frac{N_c}{2m_t^2}\, c_V \left[c_V - \frac{E}{m_t}\left(c_V + \frac{d_V}{3}\right)\right] G(E) + \ldots$$

➪ needed:

1. matching coefficients c_V 1 loop: [Källen, Sarby'55]
   2 loops: [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]; 3 loops: [Marquard,Piclum,Seidel,Steinhauser'06'08'14]
   and d_V 1 loop: [Luke,Savage'98]
2. $G(E)$ 3 loops: [Beneke,Kiyo,Penin'07; Beneke,Kiyo'08; Beneke,Kiyo,Schuller'13]

Ingredients to $G(E)$

Review: [Beneke,Kiyo,Schuller'13]

- LO: Coulomb solution

$$\left(-\frac{\Delta}{m_t}+\frac{C_F\alpha_s}{r}-E\right)G_0(\vec{r},\vec{r}\,',E)=\delta^{(3)}(\vec{r}-\vec{r}\,')$$

- NLO, NNLO, ... ➪ perturbation theory in momentum space:

$$G(E)=\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}\,'}{(2\pi)^{d-1}}\Bigg[G_0(\vec{p},\vec{p}\,';E)$$
$$+\int\frac{d^{d-1}\vec{p}_1}{(2\pi)^{d-1}}\frac{d^{d-1}\vec{p}\,'_1}{(2\pi)^{d-1}}\,G_0(\vec{p},\vec{p}_1;E)\,\delta V(\vec{p}_1,\vec{p}\,'_1)\,G_0(\vec{p}\,'_1,\vec{p}\,';E)+\ldots\Bigg]$$
$$+\delta^{\mathrm{us}}G(E)$$

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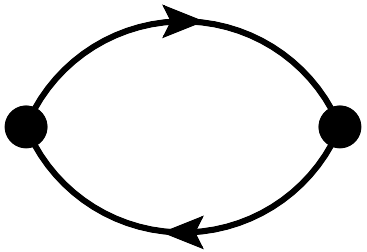

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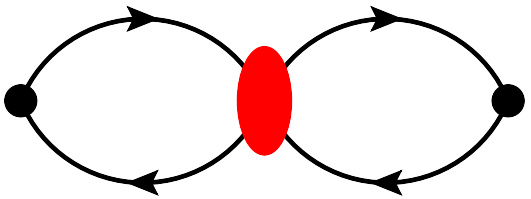

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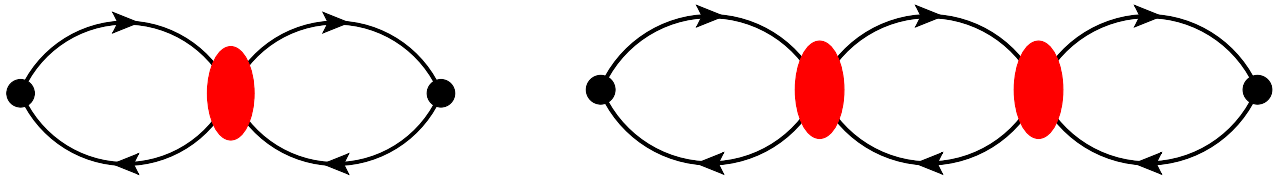

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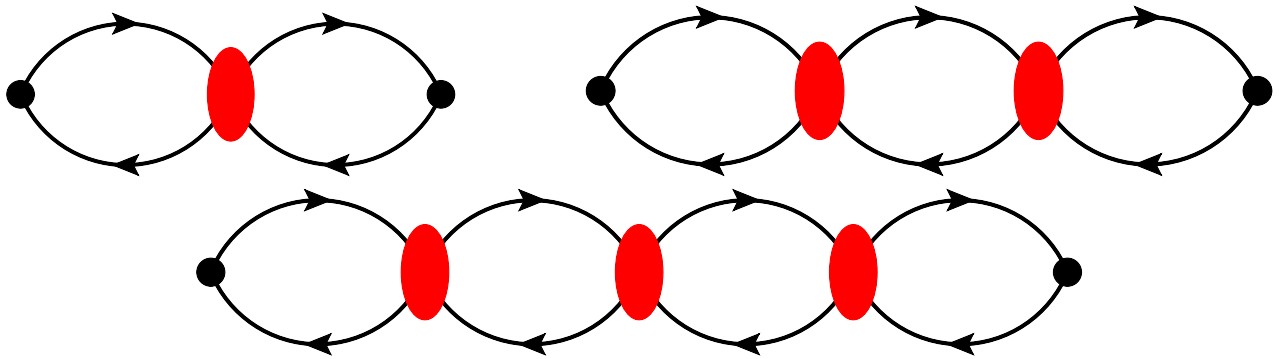

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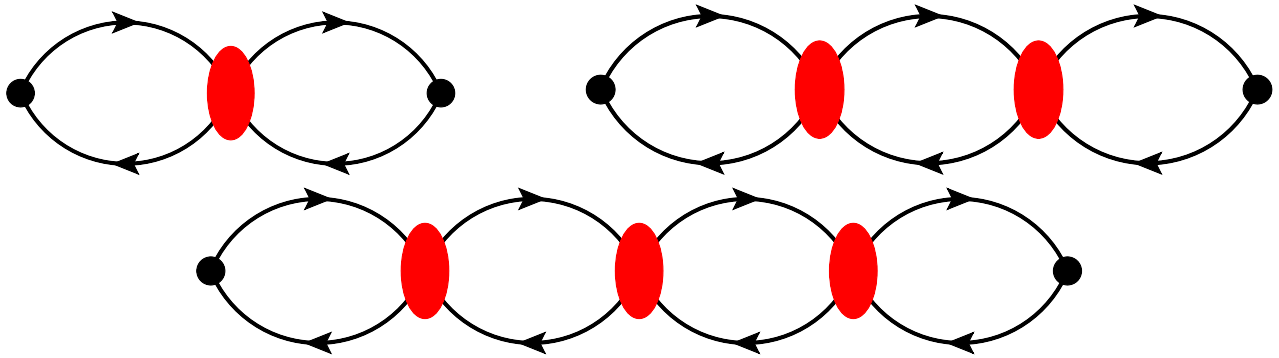
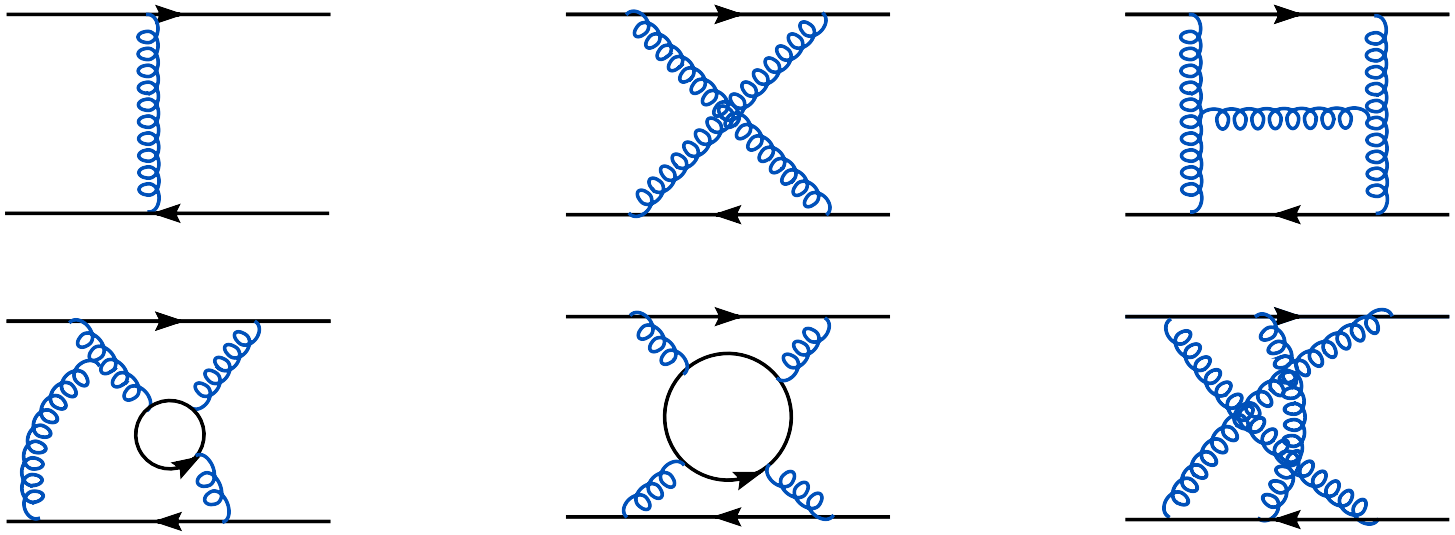


- δV

Static potential:	$V_C=-\frac{4\pi C_F\alpha_s}{\vec{q}^{\,2}}$	3 loops
$1/m$ potential:	$V_{1/m}=\frac{\pi^2 C_F\alpha_s^2}{m\lvert\vec{q}\rvert}$	2 loops
"Breit" potential:	$\propto 1/m^2$	1 loop

Static potential

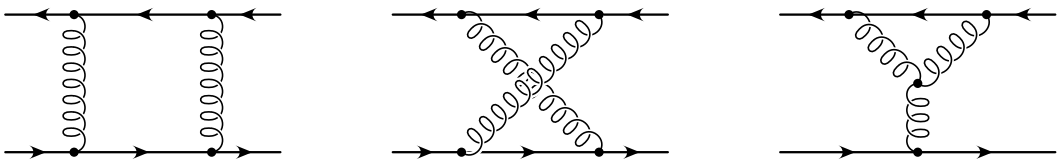
$$V_C = -\frac{4\pi C_F \alpha_s(|\vec{q}\,|)}{|\vec{q}\,|^2}\left[1+\frac{\alpha_s(|\vec{q}\,|)}{4\pi}a_1+\left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\vec{q}\,|)}{4\pi}\right)^3\left(a_3+8\pi^2 C_A^3 \ln\frac{\mu^2}{\vec{q}^{\,2}}\right)+\cdots\right]$$



[Appelquist,Politzer'75,Susskind'77] [Fischler'77;Biloire'80] [Peter'96;Schröder'98]

[Smirnov,Smirnov,Steinhauser'08; Smirnov,Smirnov,Steinhauser'09; Anzai,Kiyo,Sumino'09]

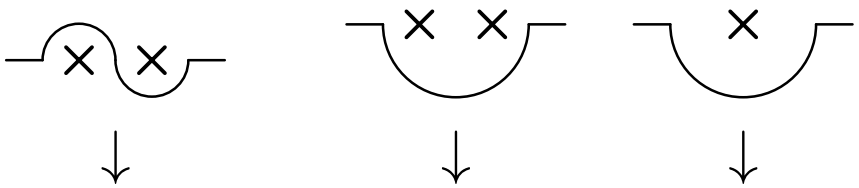
$1/m$ potential: 1-loop calculation



fermion propagator: $$\frac{1}{p_0 - \frac{\vec{p}^{\,2}}{2m} \pm i\epsilon}$$

- soft region: $(p_0, |\vec{p}\,|) \sim (mv, mv)$ ➪ expand in $\frac{\vec{p}^{\,2}}{2m}$
  potential region: $(p_0, |\vec{p}\,|) \sim (mv^2, mv)$ ➪ no expansion allowed

- poles in p_0 plane:



no expansion possible — expansion possible

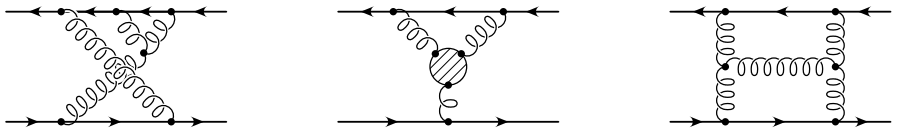
$1/m$ potential: 2 loops

[Kniehl,Penin,Smirnov,Steinhauser'01; Penin,Smirnov,Steinhauser'13; Beneke,Kiyo,Marquard,Penin,Seidel,Steinhauser'14]

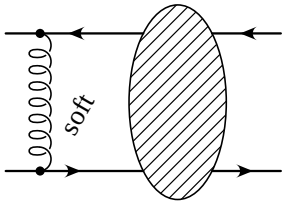
classify according to colour factor: $C_F^2 C_A$, $C_F^2 T n_l$, $C_F C_A^2$, $C_F C_A T n_l$

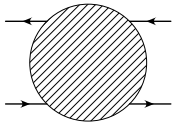
- "most non-abelian": $C_F C_A^2$, $C_F C_A T n_l$

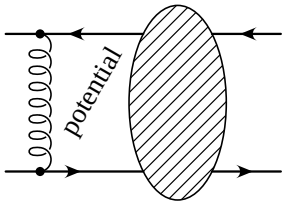
  ➪ expansion possible

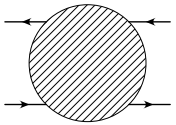


- "reducible" $C_F^2 C_A$, $C_F^2 T n_l$



➪ expand  up to $1/m$ ➪ soft integration

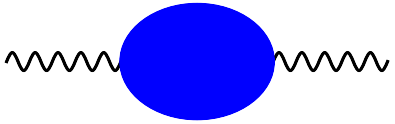


➪ expand  up to $1/m^2$ ➪ potential integration $\sim m$

- for $G(E)$ to NNNLO: $\mathcal{O}(\epsilon)$ part is needed at 2 loops

Residue Z_t

$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\, \Pi(q^2) = i \int \mathrm{d}x\, e^{iqx} \langle 0| T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$



$$\Pi(q^2) \overset{E\to E_n}{=} \frac{N_c}{2m_t^2} \frac{Z_n}{E_n-(E+i0)} + \dots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t}\, c_v\, \left(c_v + \frac{d_v}{3}\right)\right] |\psi_1(0)|^2$$

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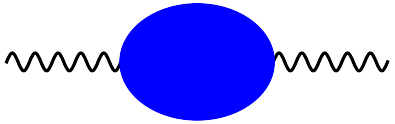

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$$\begin{aligned}
Z_t &= \frac{(C_F m_t^{\mathrm{PS}} \alpha_s)^3}{8\pi} \Big[1 + (-2.131 + 3.661 L)\, \alpha_s \\
&\quad + \left(8.38 + 1.27 x_f - 7.26 \ln \alpha_s - 13.40 L + 8.93 L^2\right) \alpha_s^2 \\
&\quad + \left(5.46 + (-2.23 + 0.78 L_f)\, x_f + 2.21_{a_3} + 21.48_{b_2\epsilon}\right. \\
&\quad + 37.53_{c_f} - 134.8(0.1)_{c_g} + (-9.79 - 44.27 L) \ln \alpha_s - 16.35 \ln^2 \alpha_s \\
&\quad \left. + (53.17 + 4.66 x_f)\, L - 48.18 L^2 + 18.17 L^3\right) \alpha_s^3 + \mathcal{O}(\alpha_s^4)\Big] \\
&= \frac{(C_F m_t^{\mathrm{PS}} \alpha_s)^3}{8\pi} \left[1 - 2.13 \alpha_s + 23.66 \alpha_s^2 - 113.0(0.1) \alpha_s^3 + \mathcal{O}(\alpha_s^4)\right]
\end{aligned}$$

$$x_f = \mu_f / (m_t^{\mathrm{PS}} \alpha_s), \quad L = \ln\left(\mu / (m_t^{\mathrm{PS}} C_F \alpha_s)\right), \quad L_f = \ln(\mu^2/\mu_f^2)$$

[Marquard,Piclum,Seidel,Steinhauser'14]

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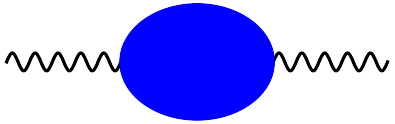
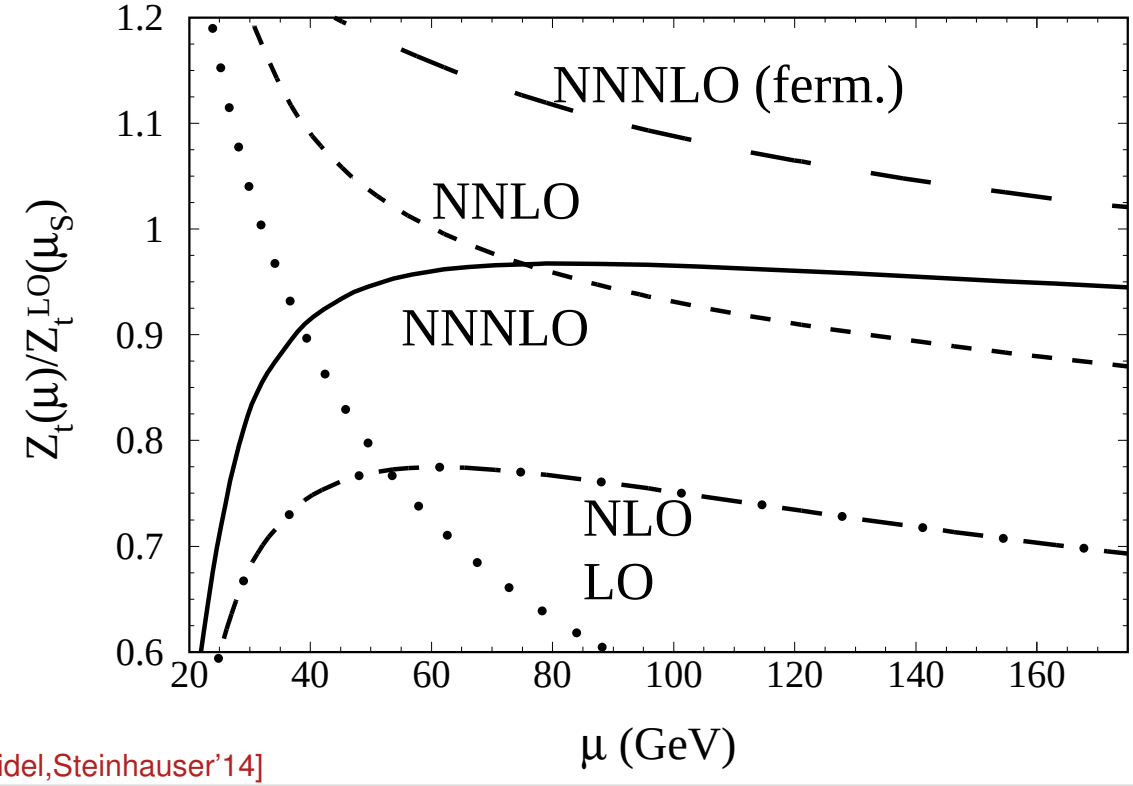
$$\left(-q^2 g_{\mu\nu} + q_\mu q_\nu\right)\,\Pi(q^2) = i\int \mathrm{d}x\, e^{iqx}\langle 0|\,T j_\mu(x) j_\nu^\dagger(0)|0\rangle$$

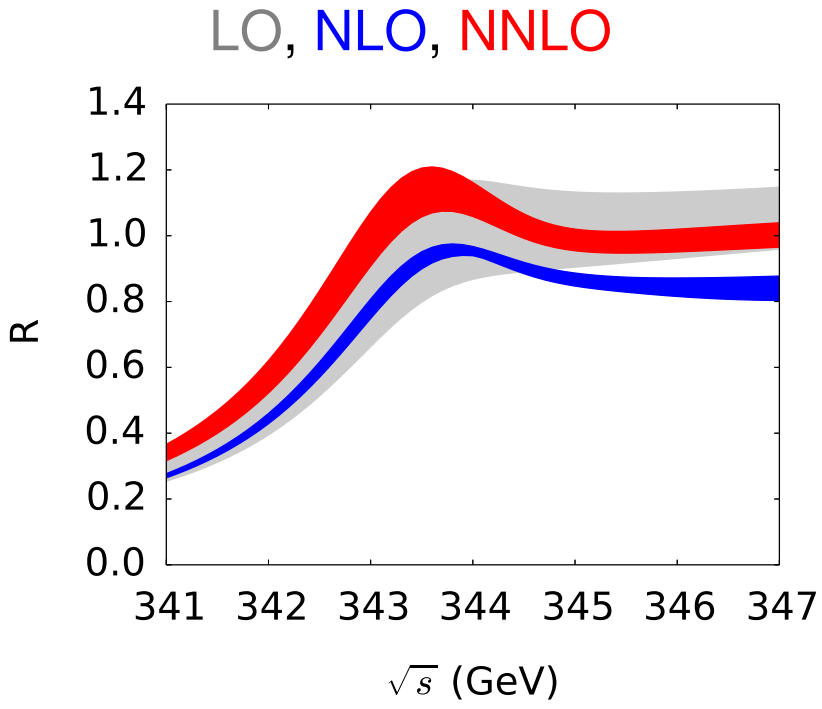


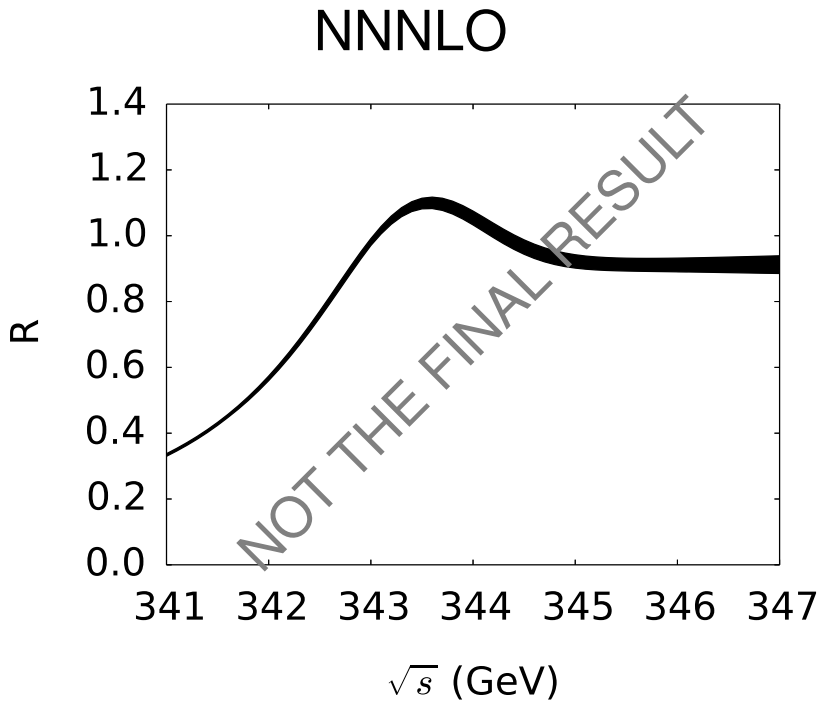
$$\Pi(q^2) \overset{E\to E_n}{=} \frac{N_c}{2m_t^2}\frac{Z_n}{E_n-(E+i0)} + \ldots \qquad Z_t = \left[c_v^2 - \frac{E_1}{m_t}\,c_v\,\left(c_v + \frac{d_v}{3}\right)\right]\,|\psi_1(0)|^2$$



[Marquard,Piclum,Seidel,Steinhauser'14]

$\sigma_{\mathrm{tot}}(e^+e^- \to t\bar{t})$



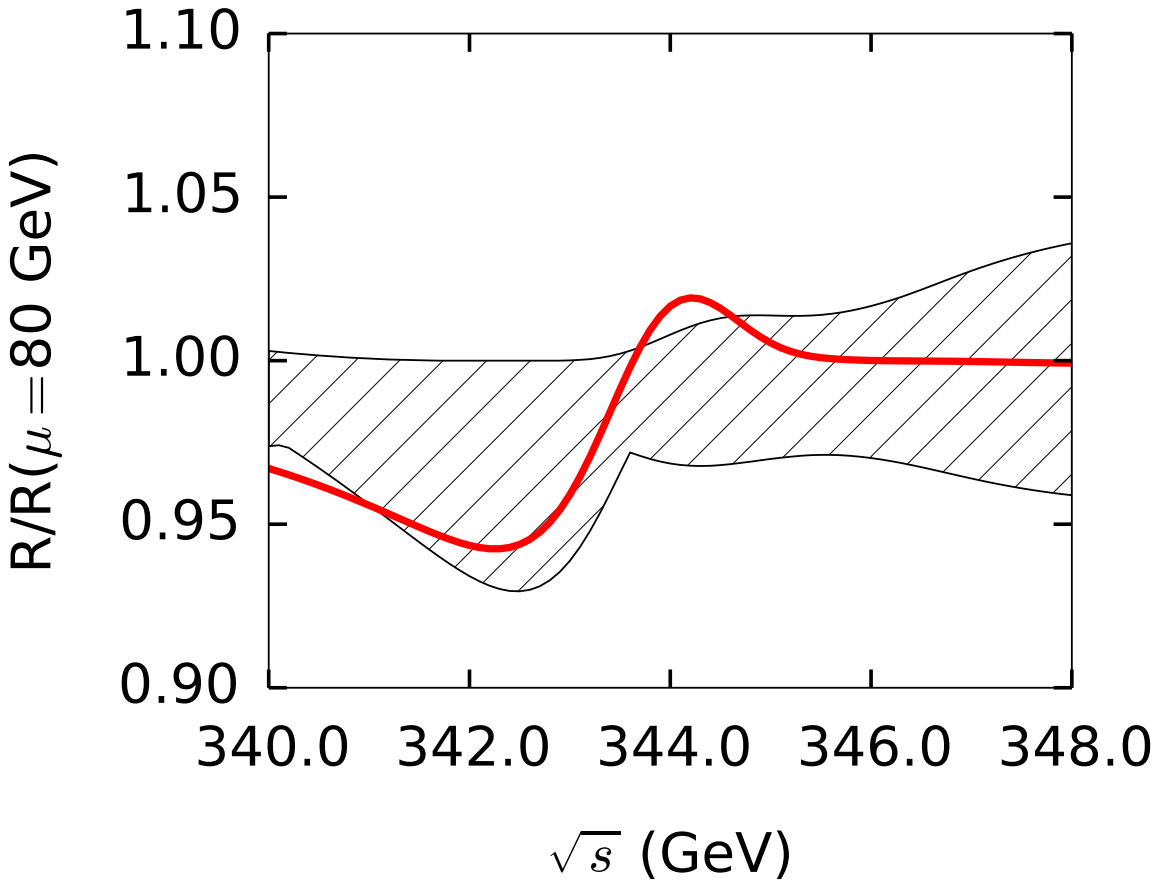


50 GeV $\leq \mu \leq$ 350 GeV

$m_t^{\mathrm{PS}} =$ 171.3 GeV

[Beneke et al.]

Parameter variation:
$m_t^{PS} \to m_t^{PS} + 50$ **MeV**



[Beneke et al.]

Conclusions

- 3rd order effective Hamiltonian ✓
- 3-loop matching coefficient ✓
- large NNNLO corrections to c_V and $G(E)$
- reduction of theoretical uncertainty