Lattice-QCD progress in hadronic contributions to muon g-2

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Snowmass on the Mississippi
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“Tough questions” on muon g-2

24. Improvements in the muon g-2 measurement need to be accompanied with improvements in the Standard Model prediction for the term involving the hadronic vacuum polarization. What are the prospects for improvement of the current estimate, including via lattice gauge theory?

    To reach the parts per billion level in the error, the contribution from light-by-light scattering must also be improved with input from low-energy data. How can this be done?

25. Why should g-2 be measured more precisely when the theoretical error is so large and uncertain?

    How will lattice calculations evolve and what cross-checks of them will be available?

"Tough questions" on muon g-2

24. Improvem th
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including

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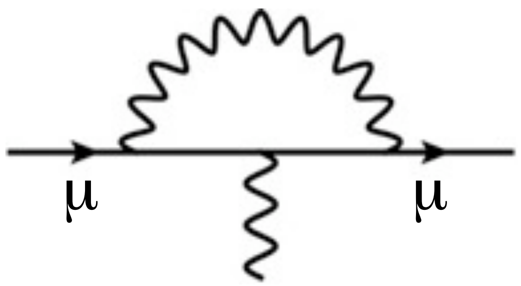
Muon g-2 in the Standard Model



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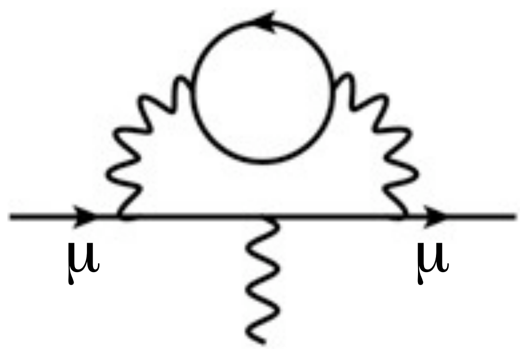
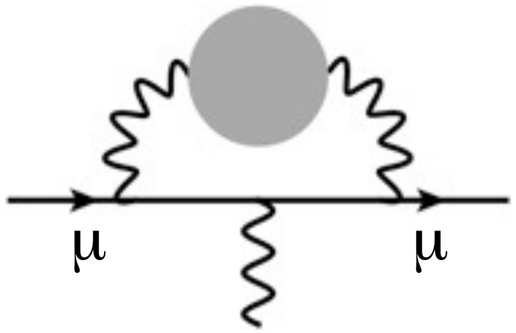


QED (4 loops) & EW (2 loops)

+

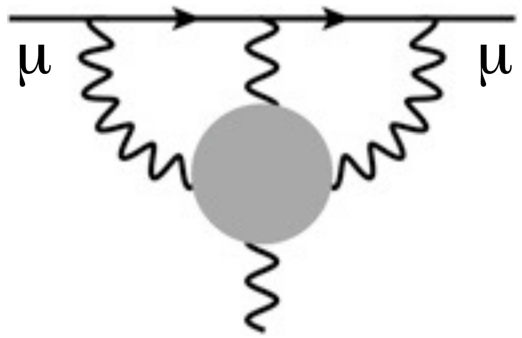
Hadronic vacuum polarization (HVP):



from experimental result for $e^+e^- \rightarrow$ hadrons plus dispersion relation

+

Hadronic light-by-light (HLbL):



estimated from models such as large N_c, vector meson dominance, χPT, etc…

+ …

Contribution	Result ($\times 10^{11}$)		Error
QED (leptons)	116 584 718 $\pm$ 0.14	$\pm\, 0.04_\alpha$	0.00 ppm
HVP(lo) [1]	6 923 $\pm$ 42		0.36 ppm
HVP(ho)	-98 $\pm\, 0.9_{\rm exp}$	$\pm\, 0.3_{\rm rad}$	0.01 ppm
HLbL [2]	105 $\pm$ 26		0.22 ppm
EW	154 $\pm$ 2	$\pm$ 1	0.02 ppm
Total SM	116 591 802 $\pm$ 49		0.42 ppm

[1] Davier, Hoecker, Malaescu, Zhang, Eur.Phys.J. C71 (2011) 1515
[2] Prades, de Rafael, Vainshtein, 0901.0306

HVP from $e^+e^-\to$hadrons

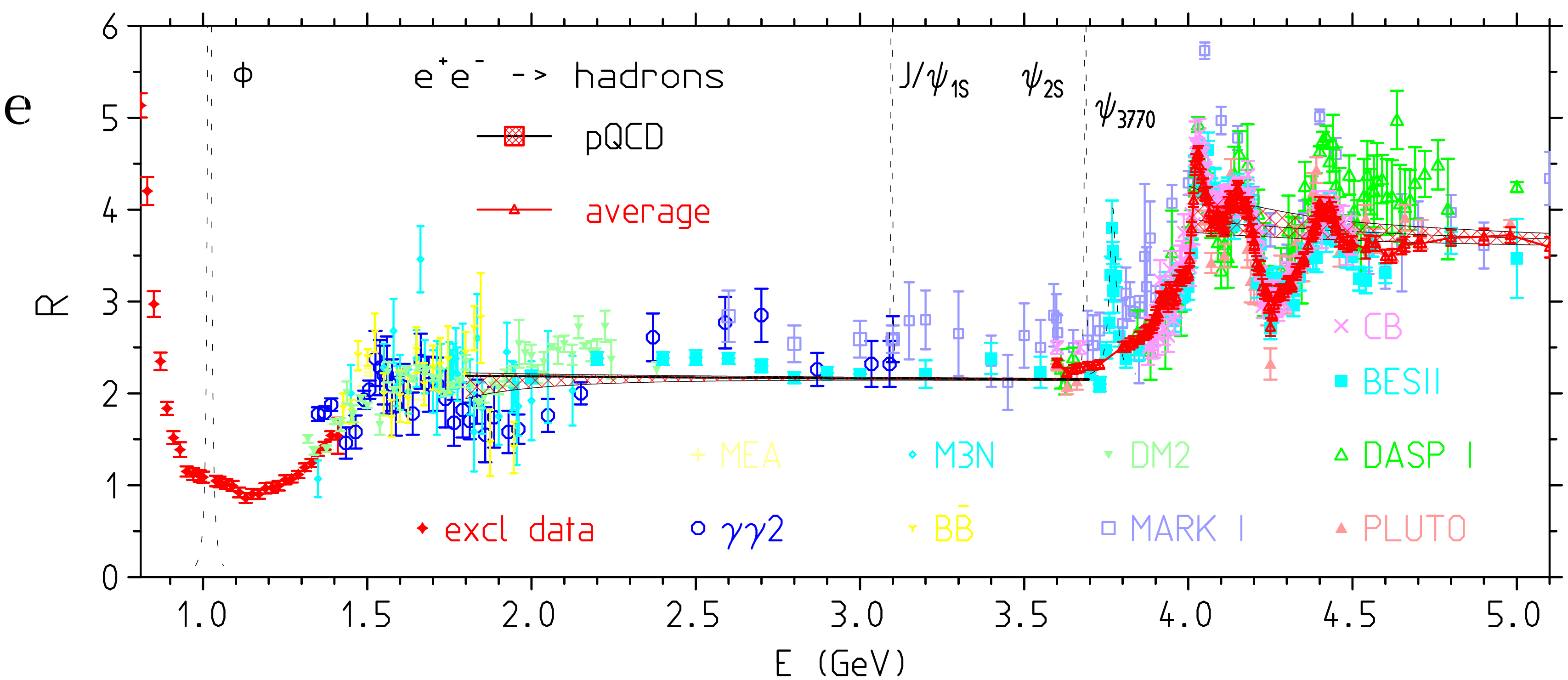
- Standard-Model value for a_μ^{HVP} obtained from experimental measurement of $\sigma_{total}(e^+e^-\to \text{hadrons})$ via optical theorem:

$$a_\mu^{\rm HVP} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} ds \frac{R(s)K(s)}{s^2} \qquad R \equiv \frac{\sigma_{\rm total}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

  - (Away from quark thresholds, use four-loop pQCD)

- Includes >20 multi-particle channels with up to six final-state hadrons

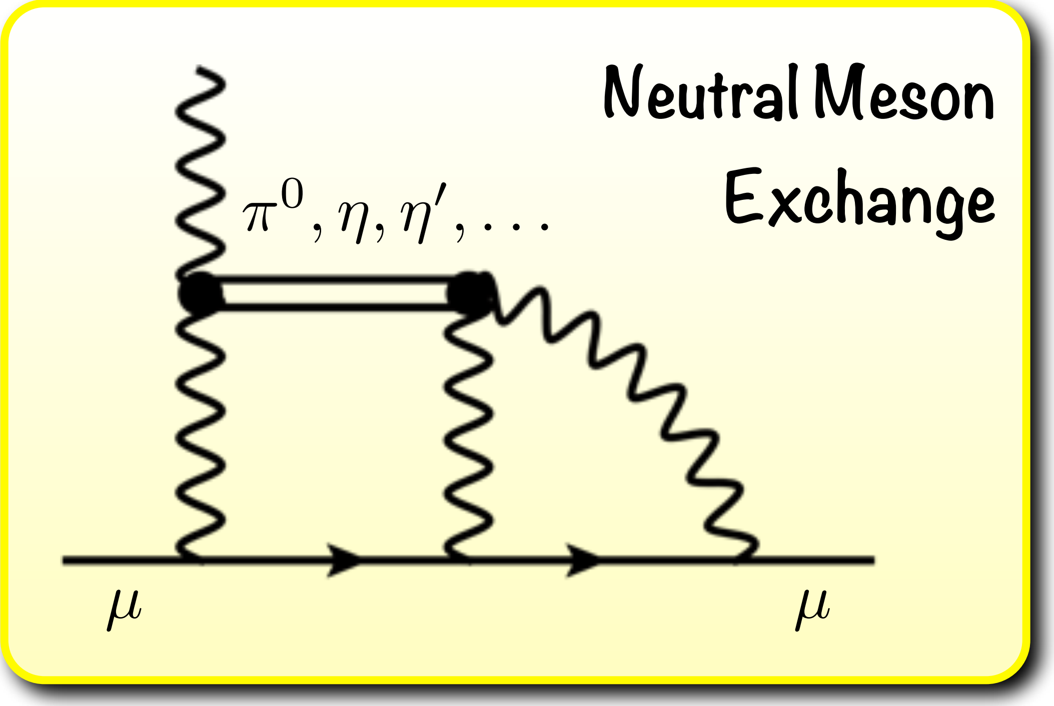
- Multi-hadron channels represent a small absolute contribution to a_μ^{HVP}, but contribute a significant fraction of the total uncertainty

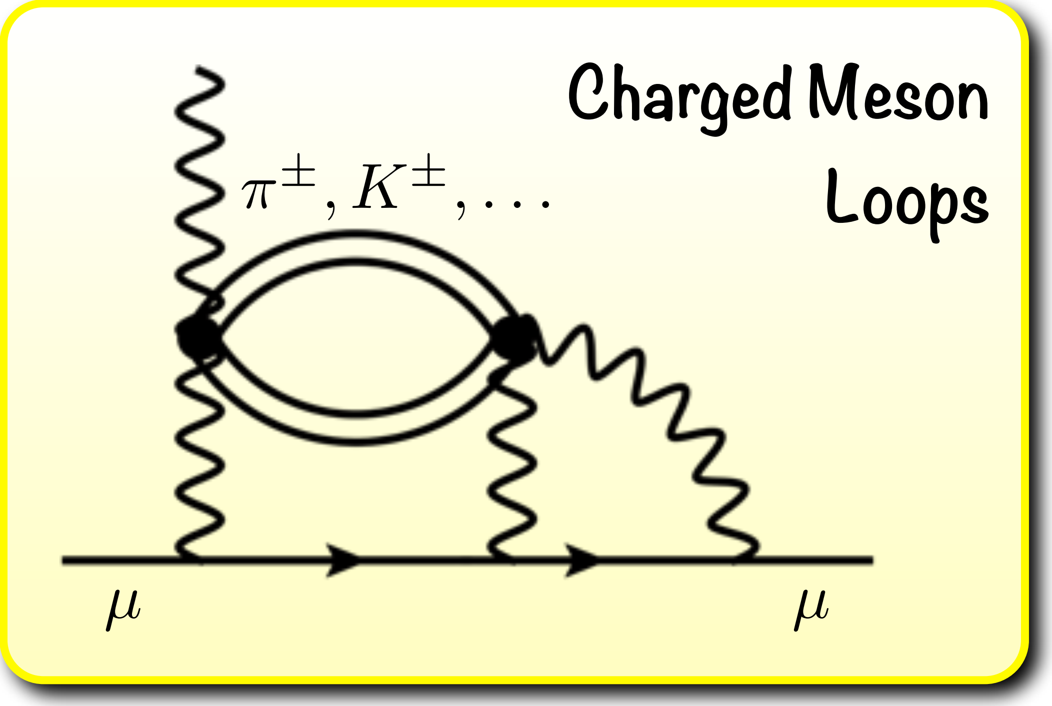


[Jegerlehner and Nyffeler, Phys.Rept. 477 (2009) 1-110]

HLbL from QCD models

- Hadronic light-by-light contribution cannot be expressed in terms of experimental quantities and must be obtained from theory [cf. Jegerlehner and Nyffeler, Phys.Rept. 477 (2009) 1-110 and Refs. therein]
  - All recent calculations compatible with constraints from large-N_c and chiral limits
  - All normalize dominant π^0-exchange contribution to measured $\pi^0 \to \gamma\gamma$ decay width
  - Differ for form factor shape due to different QCD-model assumptions such as vector-meson dominance, chiral perturbation theory, and the large N_c limit





The Glasgow consensus for HLbL

[Prades, de Rafael, Vainshtein, 0901.0306]

- Quoted error for a_μ^{HLbL} is based on model estimates, but does not cover spread of values
  - π^0-exchange contribution estimated to be ~10 times larger than others
  - Largest contribution to uncertainty ($\pm 1.9\times10^{-10}$) attributed to charged pion and kaon loop contributions

Table 1: *Contribution to a^{HLbL} from π^0, η and η' exchanges*

Result	Reference
$(8.5 \pm 1.3) \times 10^{-10}$	[7, 8]
$(8.3 \pm 0.6) \times 10^{-10}$	[4, 5, 6]
$(8.3 \pm 1.2) \times 10^{-10}$	[1]
$(11.4 \pm 1.0) \times 10^{-10}$	[9]

$\Rightarrow$ $a^{\mathrm{HLbL}}(\pi, \eta, \eta') = (11.4 \pm 1.3) \times 10^{-10}$

➡ Error could easily be underestimated (and comparable to that from HVP!), and is not systematically improvable

Target precision for hadronic contributions

- Lattice QCD can provide first-principles calculations of a_μ^{HVP} and a_μ^{HLbL} from QCD first principles with controlled uncertainties that are systematically improvable

- Muon g-2 currently measured experimentally to 0.54 ppm

a_μ^{exp} = 116 592 089(54)(33) x 10^{-11} [E821]

$a_\mu^{exp} - a_\mu^{SM}$ = 287(80) x 10^{-11} [3.6σ]

  - A >3σ discrepancy with the Standard Model if you trust the SM prediction...

- **Muon g-2 Experiment** aims to reduce the error to 0.14 ppm

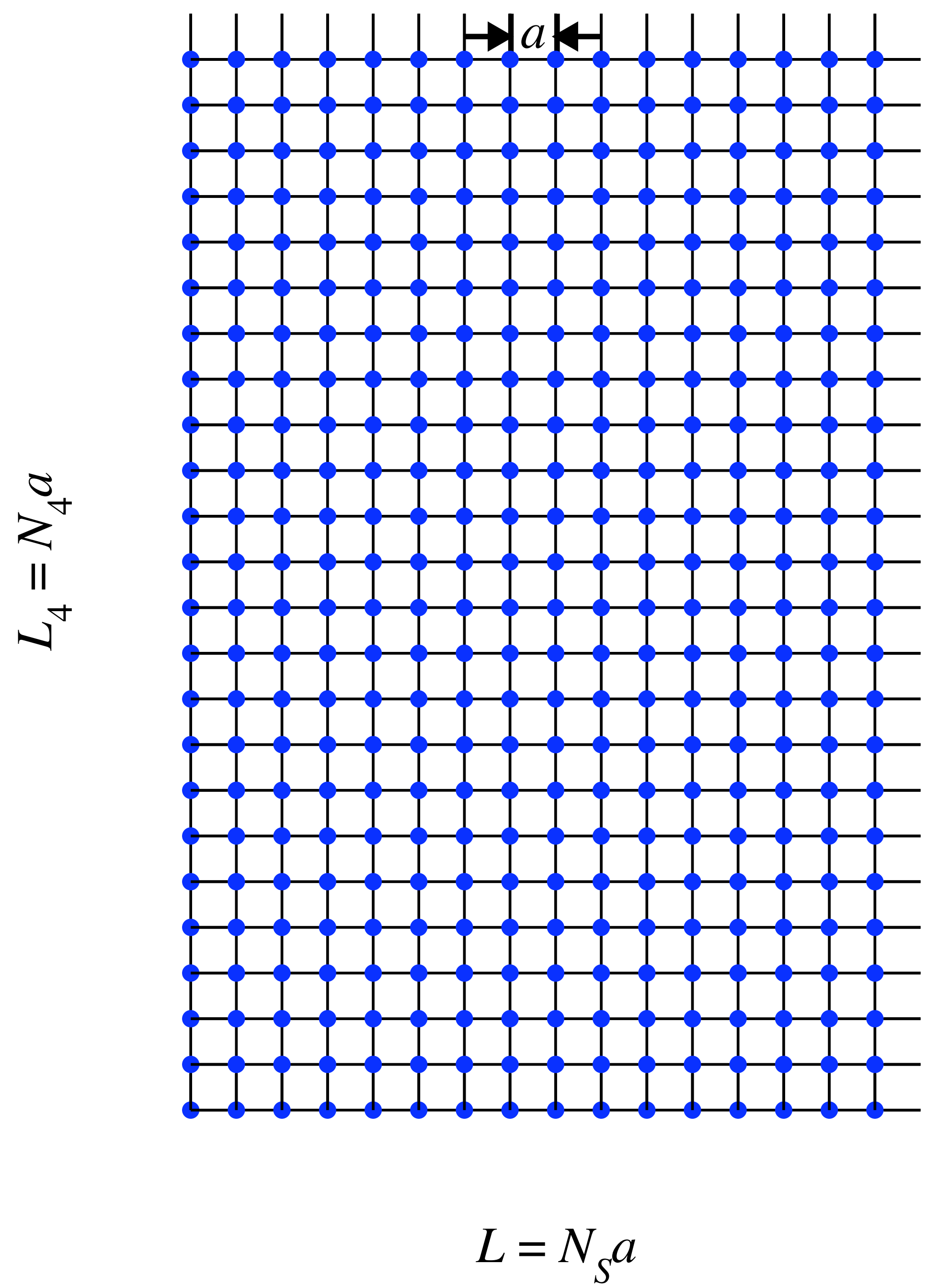
  ➡ Given this target precision, the uncertanty goals are (assuming fixed central values of HVP and HLbL):

$$\delta(a_\mu^{HVP}) \sim 0.2\%, \quad \delta(a_\mu^{HLbL}) \sim 15\%$$

  ➡ *So what can we expect from lattice QCD on the time scale of New g-2?*

Lattice Quantum Chromodynamics

- Systematic method for calculating hadronic parameters from QCD first principles



- Define QCD on a (Euclidean) spacetime lattice
- Replace derivatives by discrete differences and integrals by sums, e.g.:

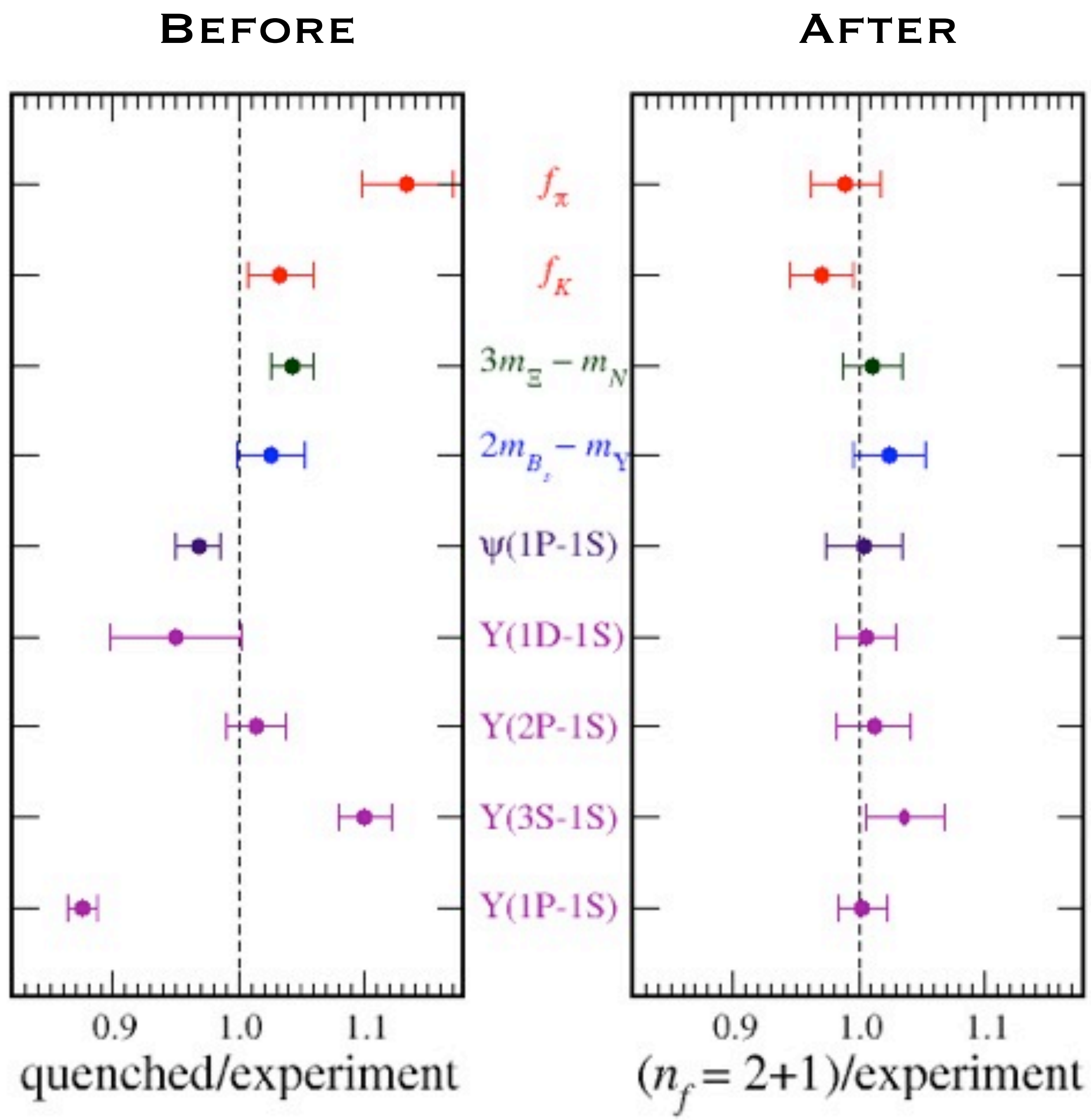
$$\partial\psi(x) \longrightarrow \frac{\psi(x+a) - \psi(x-a)}{2a}$$

$$\psi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\tilde{\psi}(k) \longrightarrow \sum_k e^{-ik\cdot x}\tilde{\psi}(k)$$

- Simulate numerically using **Monte Carlo methods and importance sampling**
- Many choices for how to discretize QCD action
  - Different lattice fermion formulations are optimal for different physical quantities
- All recover continuum QCD when lattice spacing a→0 and box size L→∞

Lattice QCD in the 21st century

- For the past decade, it has been possible to simulate realistic QCD including the effects of the dynamical u, d, & s quarks in the vacuum
- Over this time, lattice methods have been used to calculate many simple quantities with **controlled uncertainties and complete error budgets**
- Most precise results are for matrix elements with only hadron in initial state and at most one hadron in final state, where all hadrons are stable under QCD



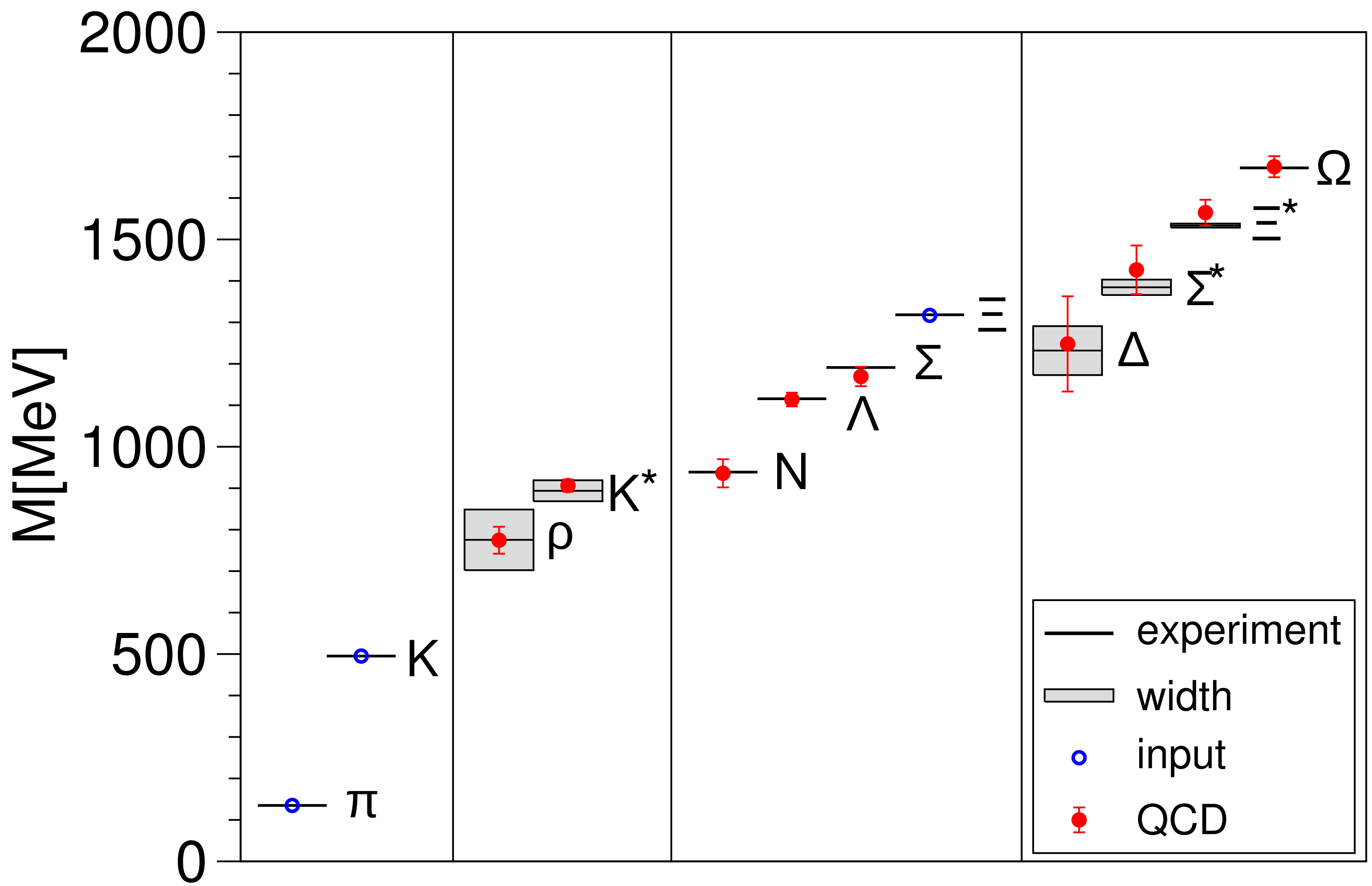
Lattice methods tested and errors verified by (i) comparison with experiment, and (ii) comparison of independent lattice calculations sensitive to different systematics

[HPQCD, MILC, & Fermilab Lattice Collaborations Phys.Rev.Lett.92:022001,2004]

Light-hadron spectrum

[BMW Collaboration, Science 322 (2008) 1224-1227]

- Light-hadron masses much larger than constituent quark masses, so primarily due to energy stored in gluon field and to quarks' kinetic energy

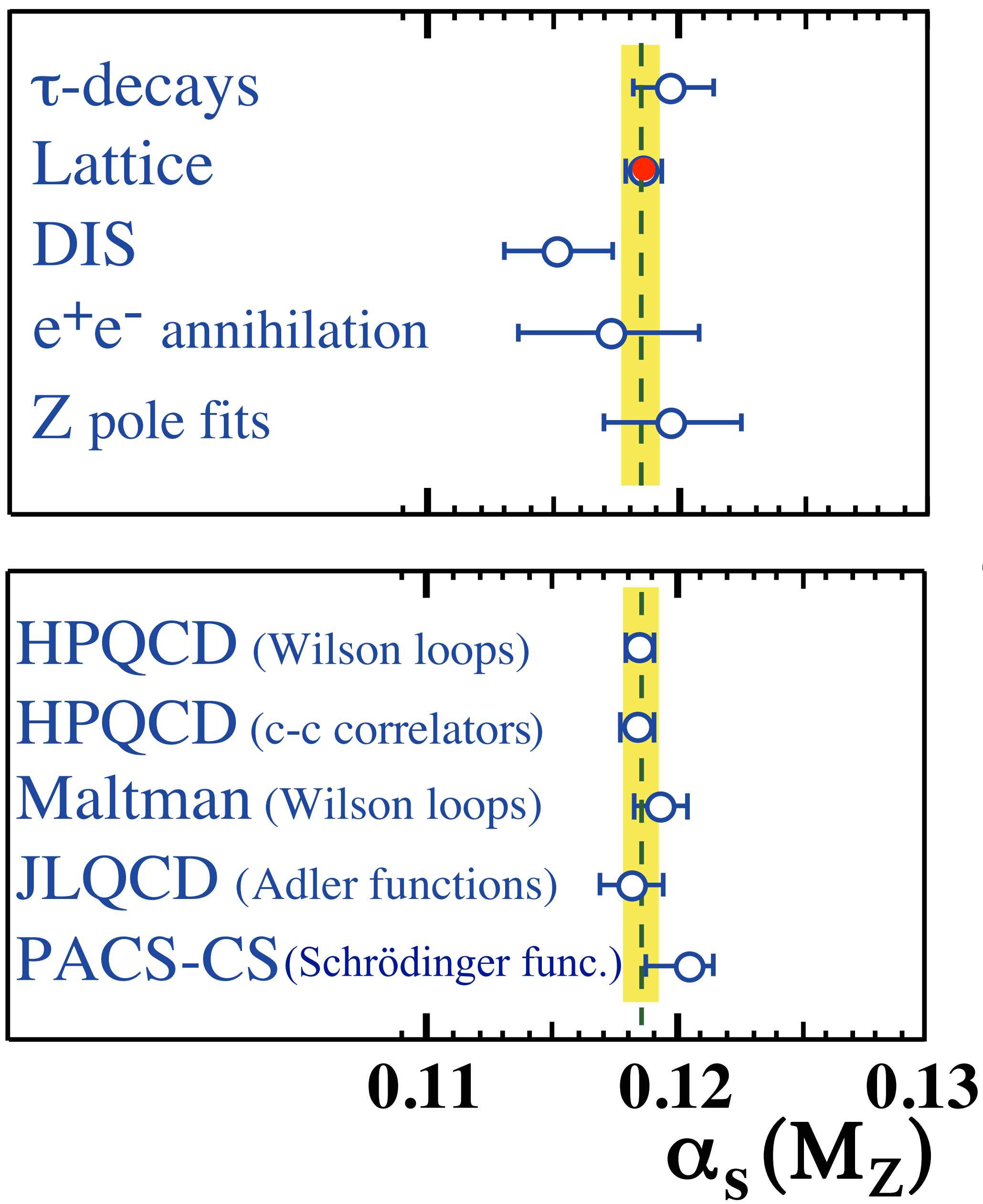


- ***Agreement within 1% of experiment a nontrivial test of nonperturbative QCD dynamics***

The strong coupling constant

✦ Lattice result (**RED**) agrees with non-lattice determinations, with smaller uncertainties

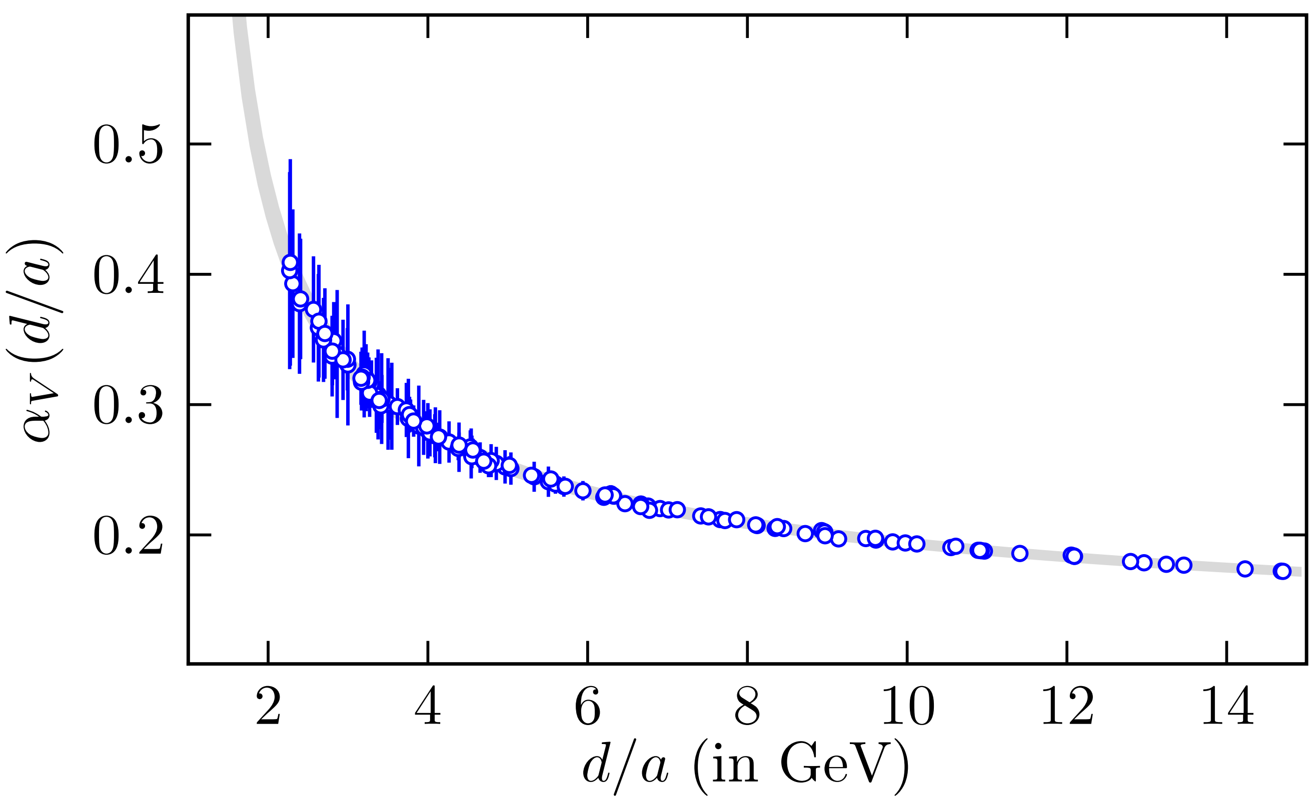
[Particle Data Group (2013)]



❖ Several independent lattice approaches obtain consistent results with similar precision

❖ Most precise lattice result from fitting NNNLO QCD β-function to 22 short-distance lattice quantities built from Wilson loops

[Phys.Rev. D78 (2008) 114507]



Lattice calculations of HVP

- Several independent efforts ongoing

Collaboration	N_f	Fermion action	$a_\mu^{\rm HVP} \times 10^{10}$
Aubin & Blum	2+1	Asqtad staggered	$713(15)_{\rm stat}(31)_{\chi\rm PT}(??)_{\rm other}$
ETMC	2	twisted-mass	$572(16)_{\rm total}$
ETMC (*preliminary*)	2+1+1	twisted-mass	$674(21)_{\rm stat}(18)_{\rm sys}(??)_{\rm disc}$
Edinburgh	2+1	domain-wall	$641(33)_{\rm stat}(32)_{\rm sys}(??)_{\rm disc}$
Mainz	2	$\mathcal{O}(a)$ improved Wilson	$618(64)_{\rm stat+sys}(??)_{\rm disc}$

- Use same general method, but introduce different improvements to address some of the most significant sources of systematic uncertainty

[1] Aubin & Blum, Phys.Rev. D75 (2007) 114502
[2] Feng *et al.*, Phys.Rev.Lett. 107 (2011) 081802
[3] Hotzel *et al.*, Lattice 2013
[4] Boyle *et al.*, Phys.Rev. D85 (2012) 074504
[5] Della Morte *et al.*, JHEP 1203 (2012) 055

General approach

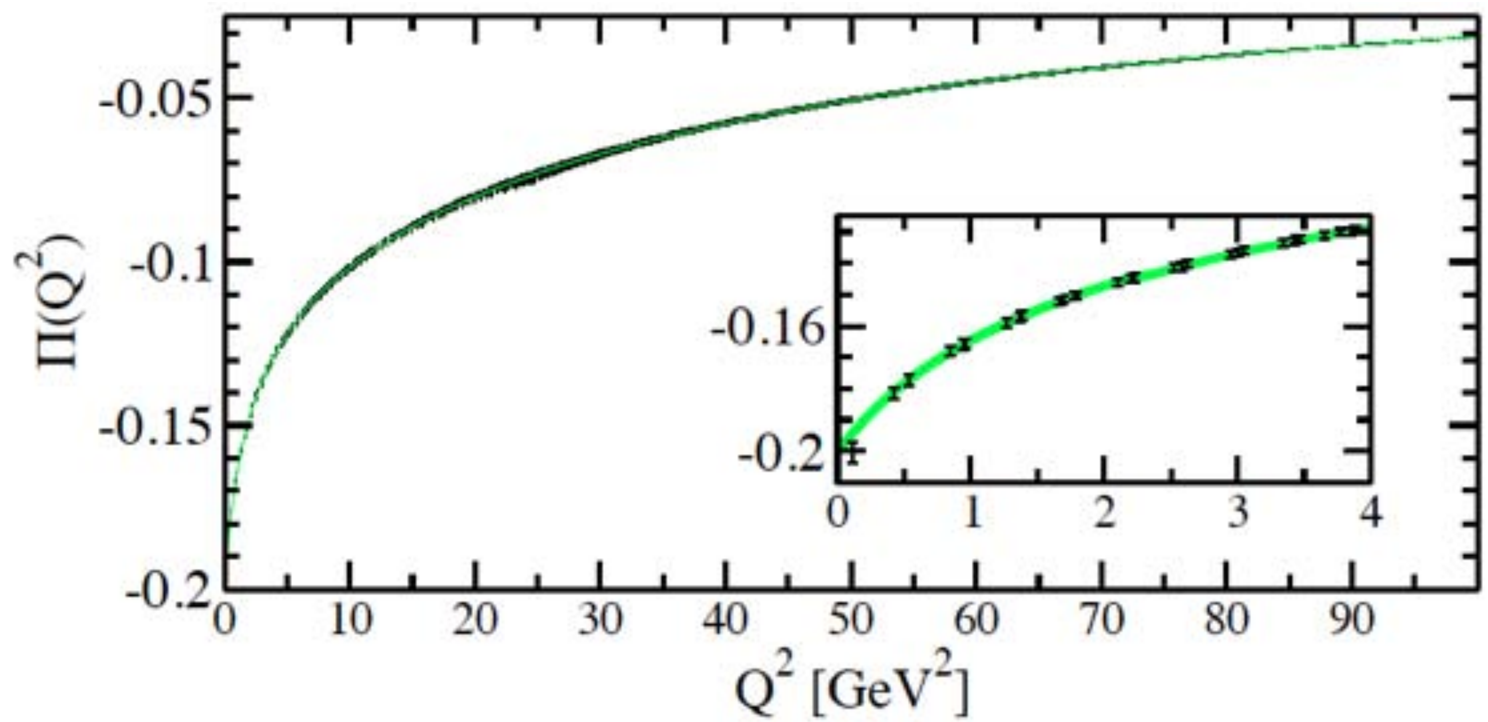
[Blum, Phys.Rev.Lett. 91 (2003) 052001]

- Calculate a_μ^{HVP} directly in from the Euclidean space vacuum polarization function:

$$a_\mu^{\mathrm{HVP(LO)}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2) \left[\Pi(Q^2) - \Pi(0)\right]$$

$$i\Pi_{\mu\nu}(q^2) =$$

[plot from Dru Renner]



- $\Pi(Q^2)$ is a simple correlation function of two electromagnetic currents
- In Euclidean space, $\Pi(Q^2)$ has a smooth Q^2 dependence with no resonance structure

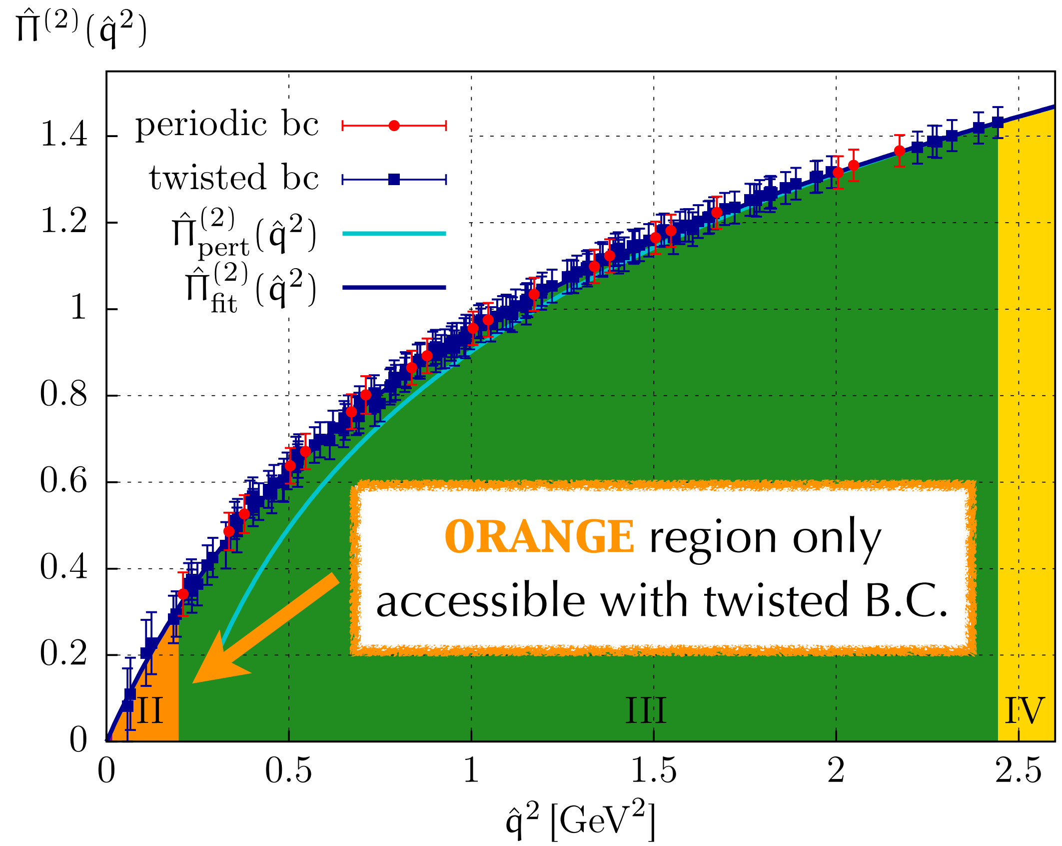
Recent developments

TWISTED BOUNDARY CONDITIONS [Della Morte et al., JHEP 1203 (2012) 055]

- Because of finite spatial lattice size (volume=L^3), simulations with periodic boundary conditions can only access discrete momentum values in units of $(2\pi/L)$ [**RED** points]

  ➡ Lattice data sparse and noisy in low-Q^2 region where contribution to a_μ^{HVP} is largest

- Introduce twisted B.C. for fermion fields to access momenta below $(2\pi/L)$ [**BLUE** points]



PADÉ APPROXIMANTS [Aubin et al.,Phys.Rev. D86 (2012) 054509]

- Even with twisted B.C., contributions to a_μ^{HVP} from $\Pi(Q^2)$ for momenta below the range directly accessible in current lattice simulations are significant

  ➡ Must assume functional form for Q^2 dependence and extrapolate $Q^2 \to 0$

- Use model-independent fitting approach based on analytic structure of $\Pi(Q^2)$ to eliminate systematic associated with vector-meson dominance fits

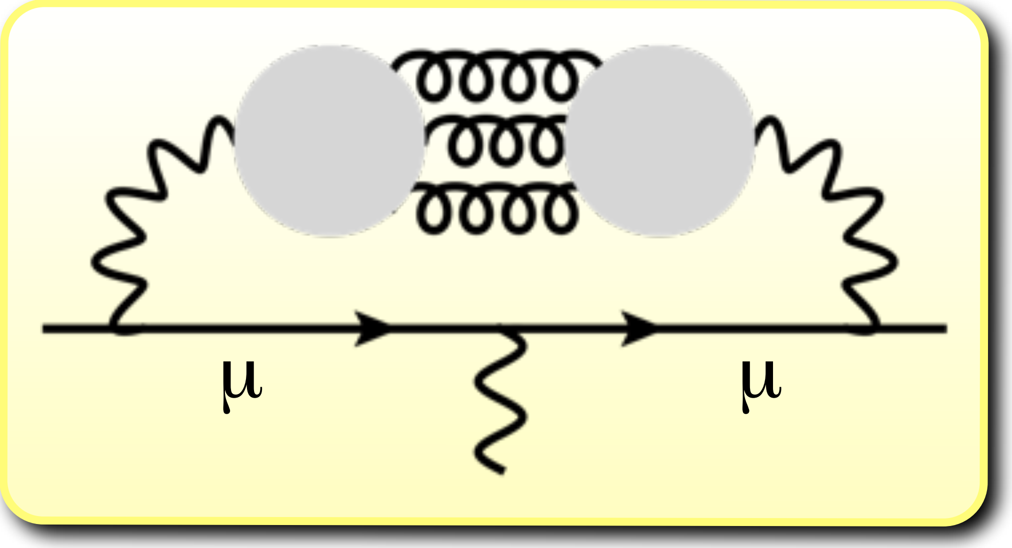
Remaining issues

(1) Chiral extrapolation

- Simulations at the physical pion mass are underway

(2) Quark-disconnected contributions

- Noisy and difficult to compute with good statistical accuracy
- Chiral Perturbation Theory estimate suggests that they could be of O(10%) [Della Morte & Jüttner, JHEP 1011 (2010) 154]



(3) Charm sea-quark contributions

- Simulations with dynamical charm quarks are underway
- Perturbative QCD estimate suggests that charm contribution could be comparable to entire size of HLbL or EW contributions [Bodenstein *et al.*, PRD85 (2012) 014029]

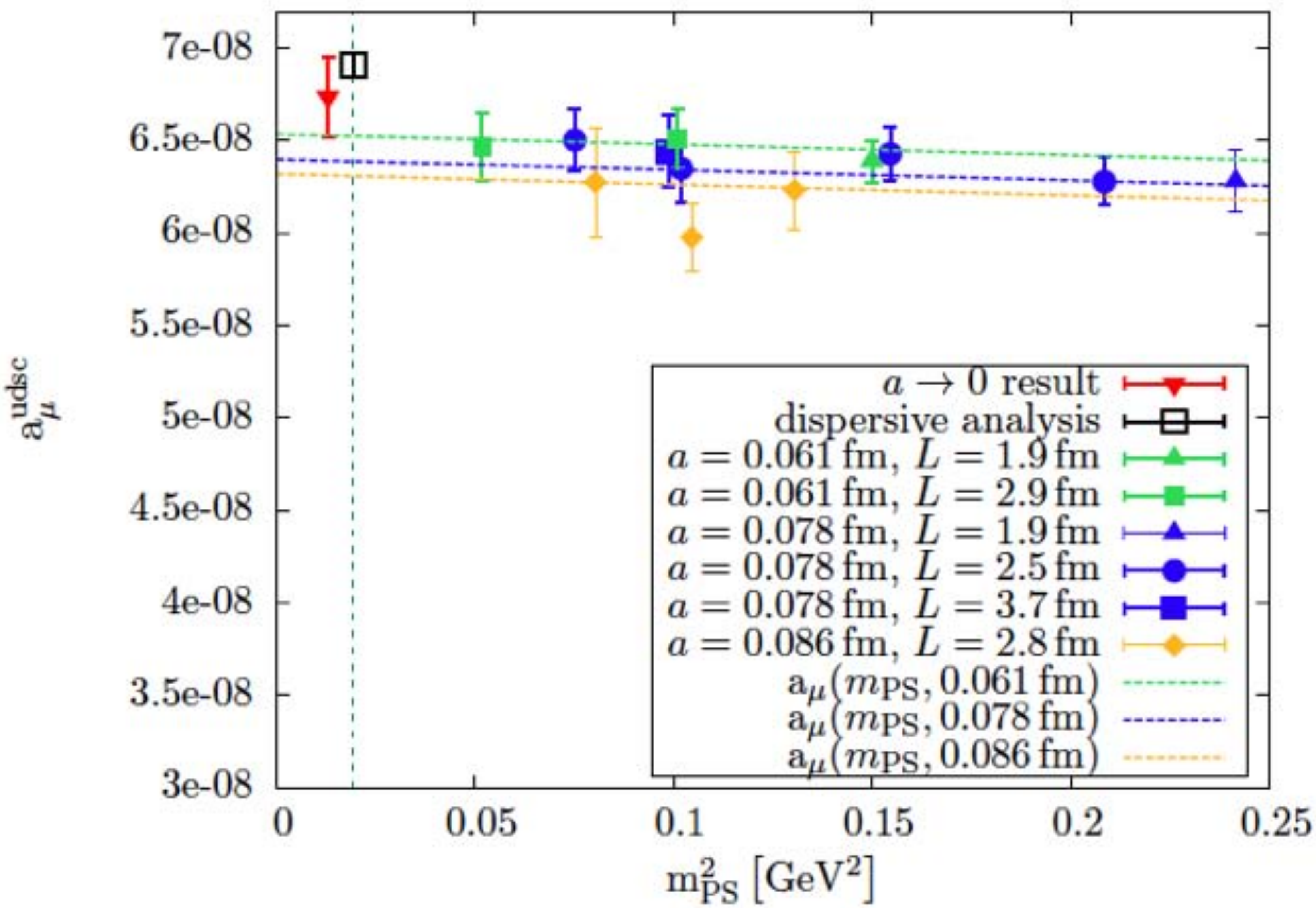
(4) Isospin breaking

- Will become relevant once the precision reaches the percent level

✦ Can all be addressed straightforwardly with sufficient computing resources

First four-flavor result (PRELIMINARY)

[G. Hotzel for ETM Collaboration, Lattice 2013]



$a_\mu^{HVP} = 6.74(21)_{stat}(18)_{sys} \times 10^{-10}$

- Error estimate does not yet include sea-quark mass mistuning (small) or quark-disconnected contributions (as much as ~10%?)

Lattice efforts on HLbL

- Several efforts ongoing to compute all or part of the light-by-light contribution with different methods

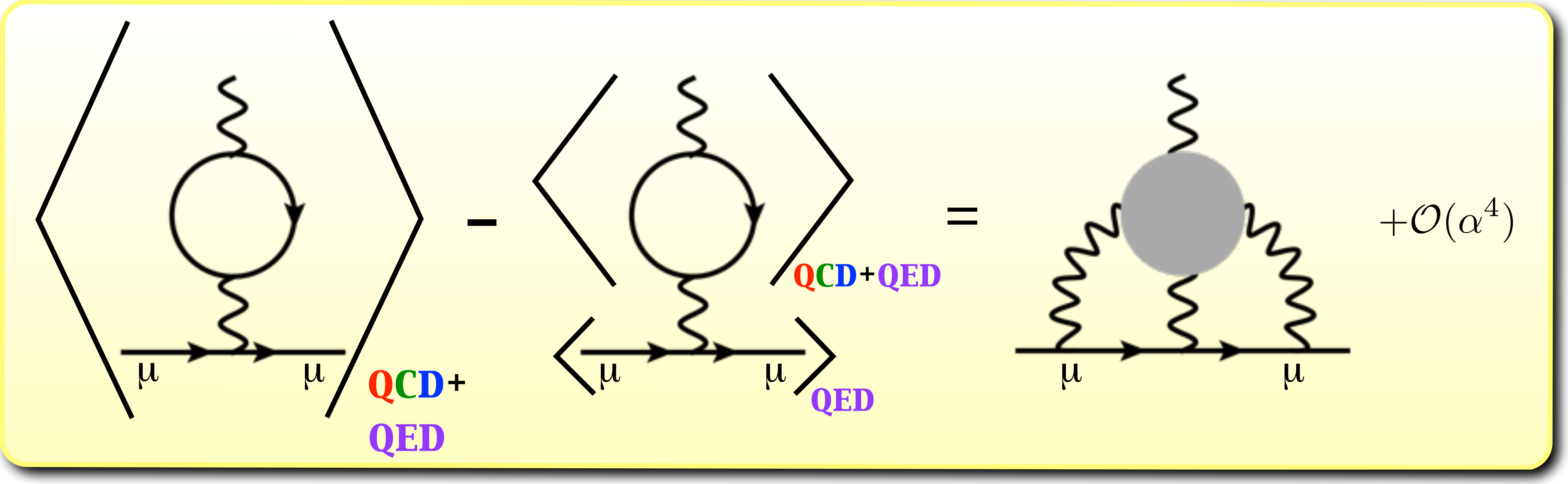
Collaboration	Method	N_f	Fermion action
RBC	QCD+QED	2+1	domain-wall
JLAB	$\pi^0 \to \gamma\gamma$ form factor	2+1	Clover
JLQCD	$\pi^0 \to \gamma\gamma$ form factor	2	overlap
QCDSF	direct $\langle JJJJ \rangle$	2	Clover

- None of them yet have results for a_μ^{HLbL}

[1] Hayakawa *et al.*, PoS LAT2005 (2006) 353; Blum *et al.*, PoS LATTICE2012 (2012) 022; ...
[2] Cohen *et al.*, PoS LATTICE2008 (2008) 159
[3] Feng *et al.*, Phys.Rev.Lett. 109 (2012) 182001
[4] Rakow, Lattice 2008

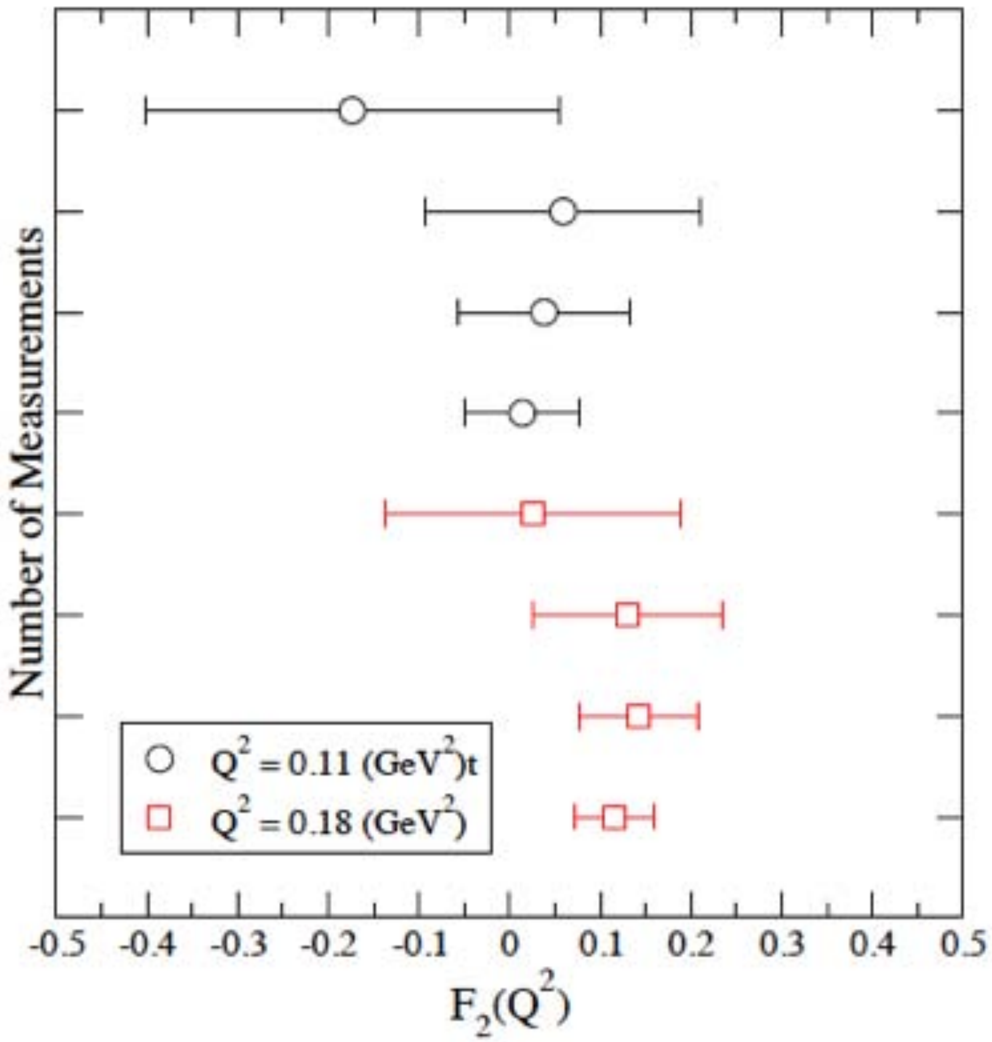
QCD + QED simulations

- Most promising method introduced by Blum and collaborators in which one computes the full hadronic amplitude, including the muon and photons, nonperturbatively [Hayakawa *et al.*, PoS LAT2005 (2006) 353]
- Treat photon field in parallel with gluon field and include in gauge link, so the simulation and analysis follows a conventional lattice-QCD calculation
- In practice, must insert a single valence photon connecting the muon line to the quark loop "by hand" into the correlation function, then perform correlated nonperturbative subtraction to remove the dominant $O(\alpha^2)$ contamination



Preliminary tests

- **Early results appear promising** [Blum *et al.*, PoS LATTICE2012 (2012) 022]
- Stable, statistically-significant signal emerging in the ballpark of model estimates



- $a = 0.114$ fm; $V=(24 \times a)^3$
- $Q^2 = 0.11$ and 0.18 GeV^2
- $m_\pi = 329$ MeV
- $m_\mu = 190$ MeV
- $\alpha=1/4\pi$ to enhance signal

$$a_\mu^{HLbL} = F_2(Q^2 \rightarrow 0) \times (\alpha/\pi)^3$$

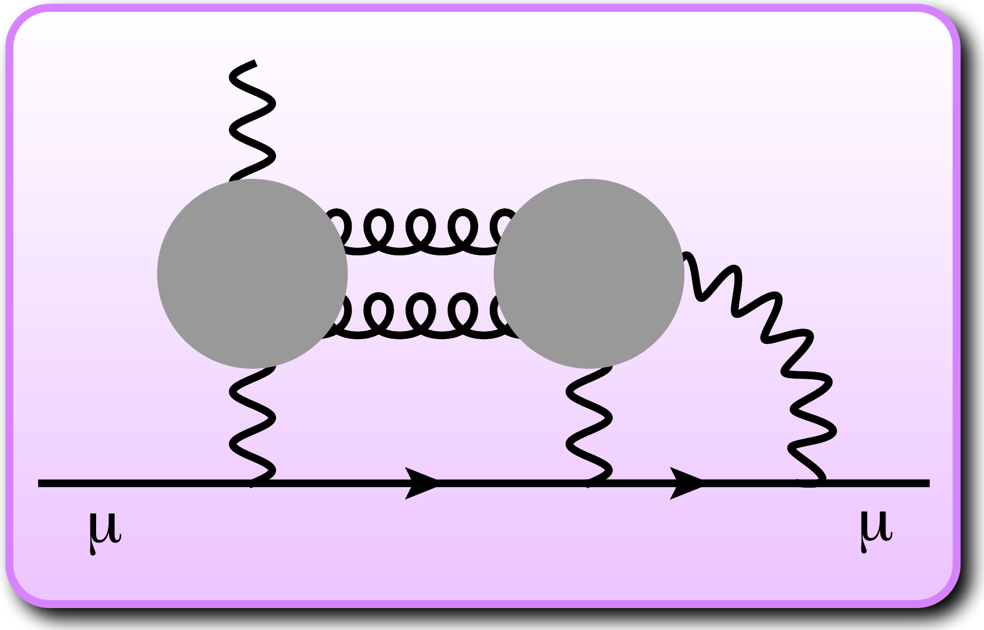
Other outstanding issues

(1) Finite-volume effects

- QED-only calculations suggest that errors due to the finite lattice size may be significant, but increased computing power is allowing the generation of larger lattices

(2) Quark-disconnected contributions

- Preliminary calculations work in the quenched approximation of QED, so contributions from diagrams with two quark loops only connected by a pair of gluons are not included
- Studying various approaches to include these such as directly simulating dynamical photons



(3) Chiral ($m_q \rightarrow m_q^{phys}$) and continuum ($a \rightarrow 0$) extrapolations

- New large-volume lattices being generated have close-to-physical pion masses

(4) Momentum extrapolation ($Q^2 \rightarrow 0$)

Still quite a bit of work to do...

Outlook

Hadronic vacuum polarization

- Theoretical improvements + increased computing resources should enable a lattice-QCD determination with **few-percent error on the timescale of Muon g-2 Experiment**
- Will have independent cross-checks from several collaborations
- With this precision may already be able to weigh in on e^+e^- versus τ discrepancy
- No remaining theoretical barriers to eventually reducing uncertainty to sub-percent level, at which point the lattice determination can supplant the experimentally-based value

Hadronic light-by-light

- Calculations still in early stages and future errors are difficult to predict
- **Determination in next five years with ~15% precision possible, but not guaranteed**
- Significant computing (*and human*) resources will be devoted to this high-priority calculation
- May need further theoretical developments, and independent cross-checks will be essential

☆ Continued support for lattice-QCD hardware and software is essential for computations needed to interpret muon g-2 as well as measurements throughout the experimental HEP program

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- Will have independent cross-checks from several collaborations
- With this precision may already be able to weigh in on e^+e^- versus τ discrepancy
- No remain[...]cent level, at which p[...] value

> For more details see USQCD Collaboration white paper
> http://www.usqcd.org/documents/g-2.pdf
> and Project X Physics Book
> http://arxiv.org/abs/arXiv:1306.5009

Hadronic li[...]

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Extras

Scope of lattice QCD

✦ Lattice calculations are needed throughout the entire current and future U.S. experimental high-energy physics program

Quark flavor physics

- CKM matrix elements
- Rare kaon and B decays

Higgs branching fractions

- Charm- and bottom-quark masses
- Strong coupling constant

Nucleon matrix elements

- Proton & neutron EDMs
- Proton & neutron decay matrix elements
- Neutron-antineutron oscillations

Neutrino physics

- Nucleon axial form factor

Strong dynamics at the LHC

- Composite-Higgs model building

Muon g-2

- Hadronic vacuum polarization
- Hadronic light-by-light contribution

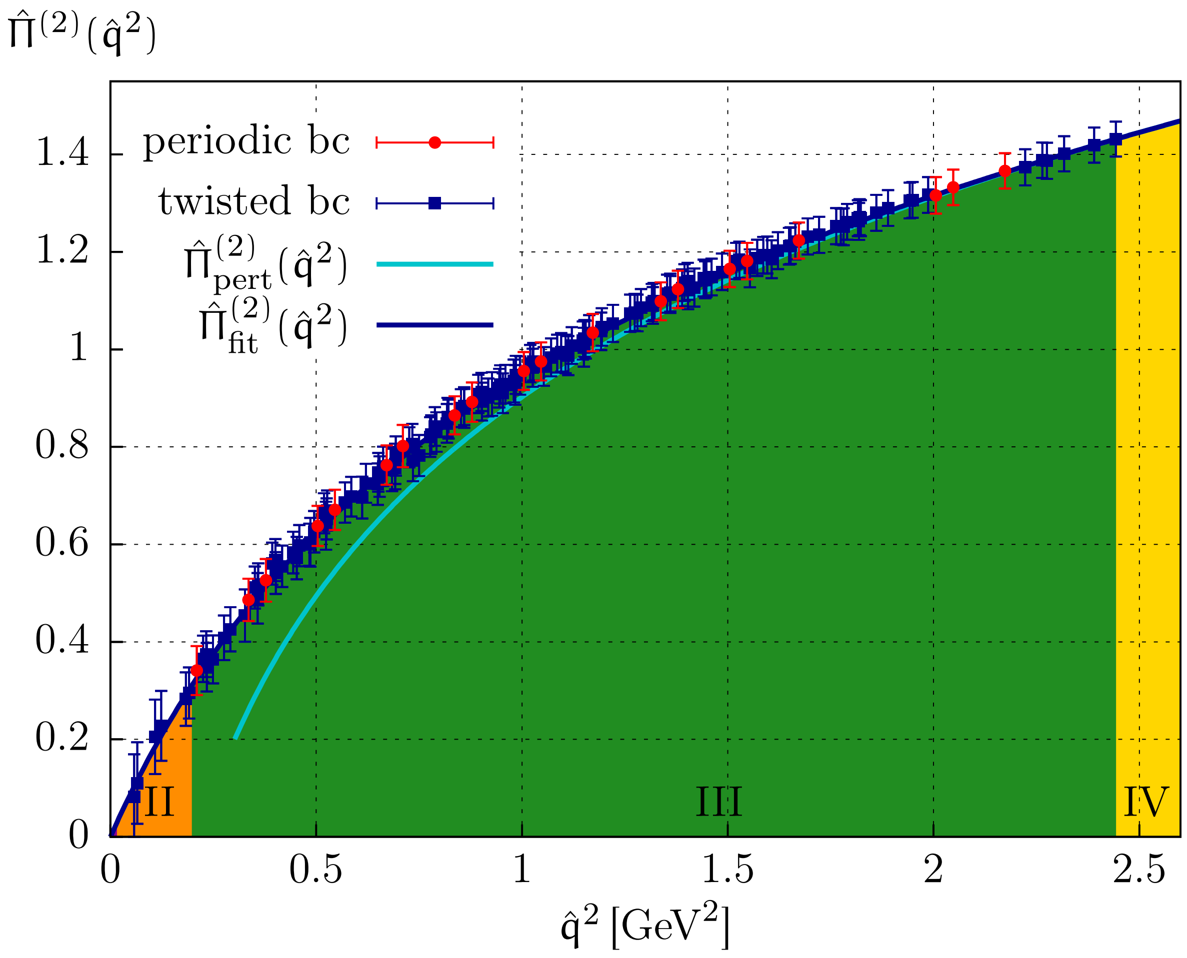
Mu2e, Dark-matter searches

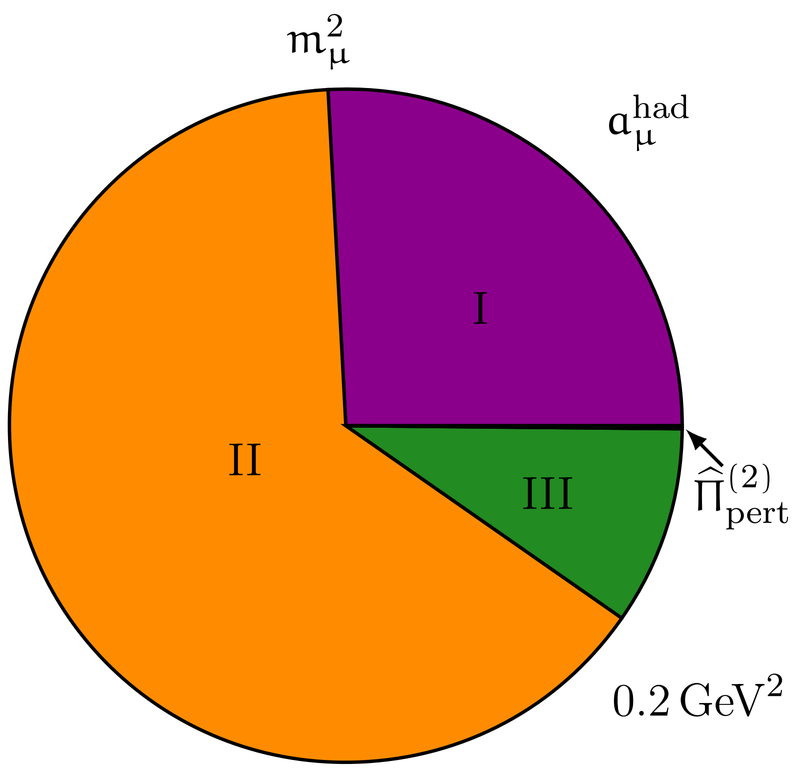
- Light-and strange-quark contents of nucleon

Twisted boundary conditions for HVP

[Della Morte et al., JHEP 1203 (2012) 055]

- Mainz group introduced use of twisted boundary conditions for the fermion fields to access momenta below $(2\pi/L)$ [**BLUE** points]





- **ORANGE** region with $m_\mu^2 < Q^2 < (2\pi/L)^2$ only accessible with twisted B.C

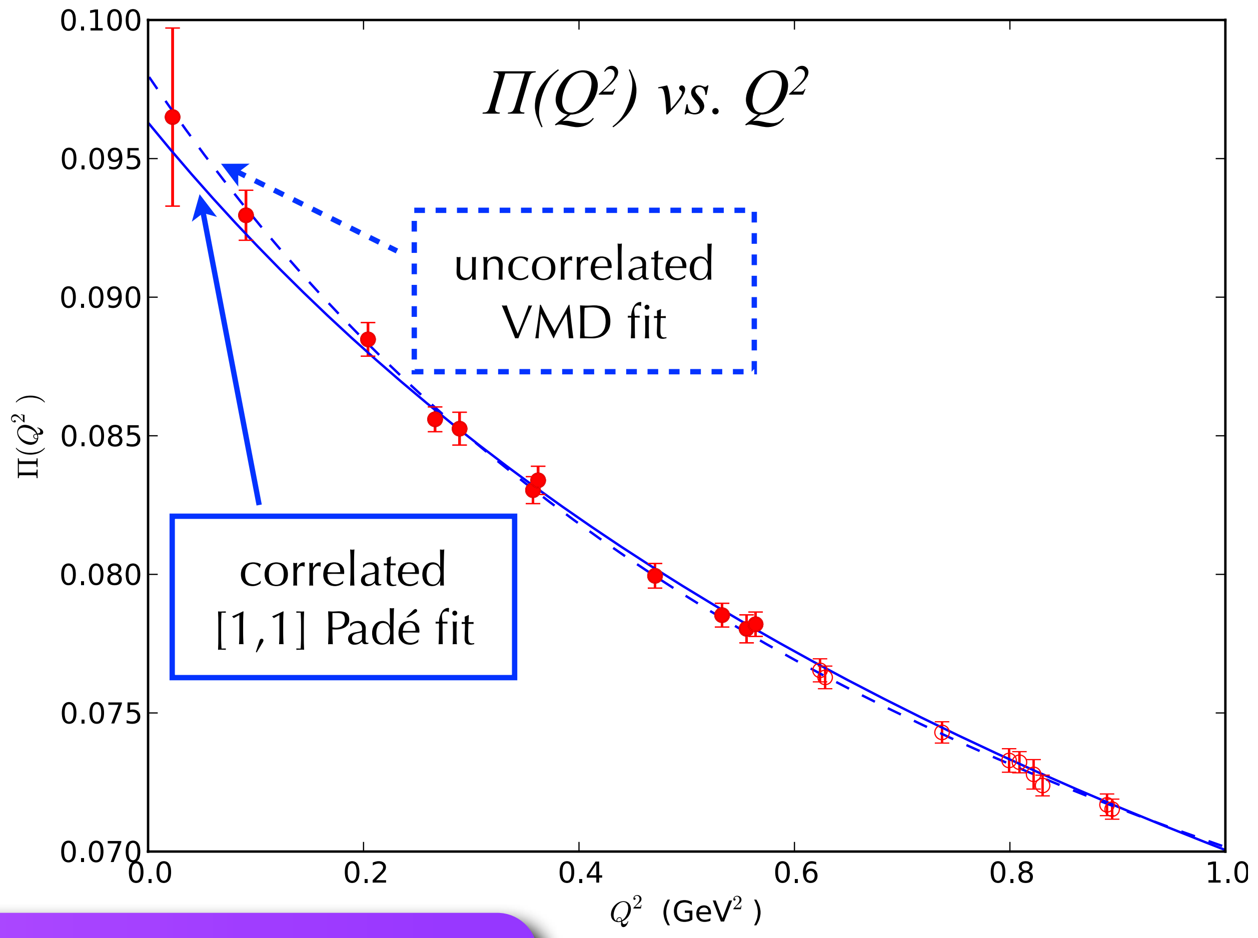
- **PURPLE** region with $0 < Q^2 < m_\mu^2$ only constrained by smoothness requirements on function

Padé approximants for HVP

[Aubin et al.,Phys.Rev. D86 (2012) 054509]

- Most lattice calculations use a form inspired by vector-meson dominance (VMD) which includes the ρ pole, and possibly a few additional parameters to absorb other contributions

- To eliminate systematic associated with VMD fits, use model-independent fitting approach based on analytic structure of $\Pi(Q^2)$



	χ^2/dof	$10^{10} a_\mu^{\mathrm{HLO},Q^2\leq 1}$	$\Pi(0)$	a_i	b_i	a_0
VMD	38.6/18	646(8)	0.1222(6)	0.0595(8)	0.64 (fixed)	–
[0, 1]	14.3/17	550(20)	0.1203(7)	0.0646(16)	0.83(5)	–
[1, 1]	13.9/16	572(41)	0.1206(8)	0.052(16)	0.68(20)	0.005(7)
[1, 2]	13.9/15	572(37)	0.1206(8)	0.052(14)	0.68(19)	–
				1(6)	$0.3(1.0)\times 10^3$	
[2, 2]	13.9/14	572(38)	0.1206(8)	0.052(14)	0.68(18)	0.003(27)
				1(31)	$0.4(6.0)\times 10^3$	

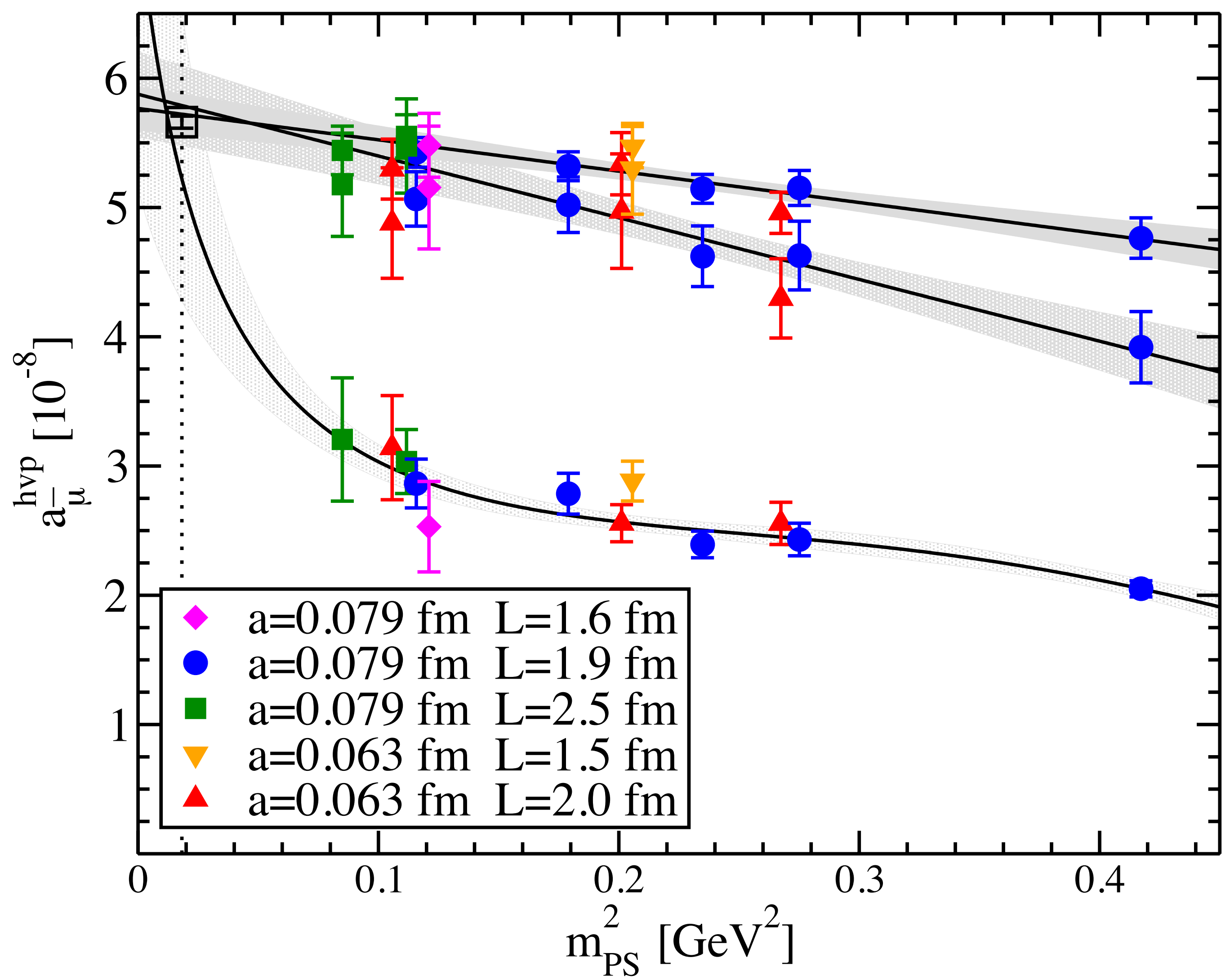
(Statistical errors only)

Modified observables for HVP

[Feng et al., Phys.Rev.Lett. 107 (2011) 081802]

- European Twisted Mass Collaboration focusing on reducing the uncertainty in a_μ^{HVP} due to the extrapolation to the physical light-quark mass

  ➡ Introduce a change of variables that modifies the observable $\Pi(Q^2) \rightarrow \widetilde{\Pi}(Q^2 \times H^2_{lat} / H^2_{phys})$

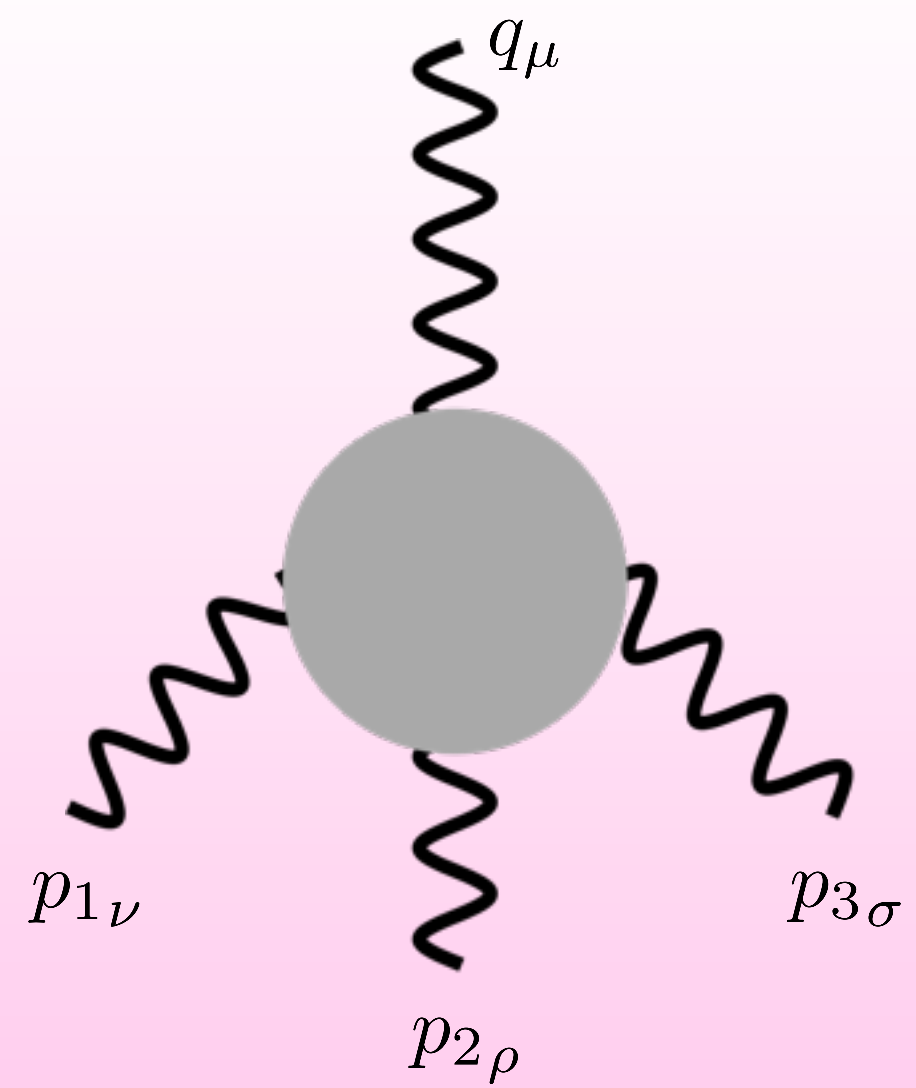


- Modified observable $\widetilde{\Pi}(Q^2)$ equals desired vacuum polarization function at the physical light-quark mass, but has a milder quark-mass dependence and better controlled chiral extrapolation

- *E.g.* choice $H=m_\rho$ removes dominant curvature

"Conventional" approach for HLbL

- Can follow a similar approach to that used for HVP
- Calculate the correlation function of four electromagnetic currents and insert into a continuum two-loop QED integral
- Computationally costly because one must compute the four-index tensor for all possible combinations of loop momenta (p_1,p_2) and several values of the external momentum q
- Exploratory calculations under way [Rakow, Lattice 2008], but viability of this method has yet to be demonstrated



"Disconnected" contributions to HLbL

- Preliminary calculations use the **quenched approximation of QED**, *i.e.* the photon field is not included in the Monte Carlo evolution of the gauge fields

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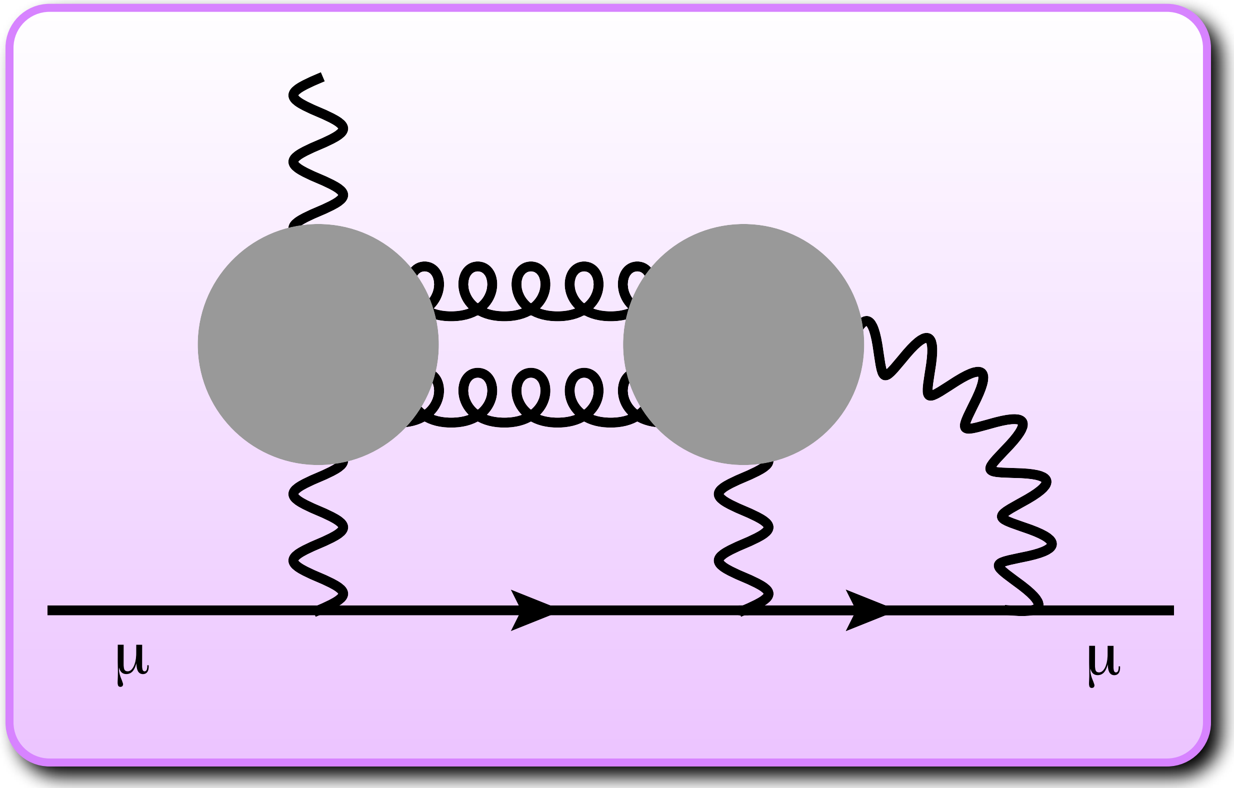
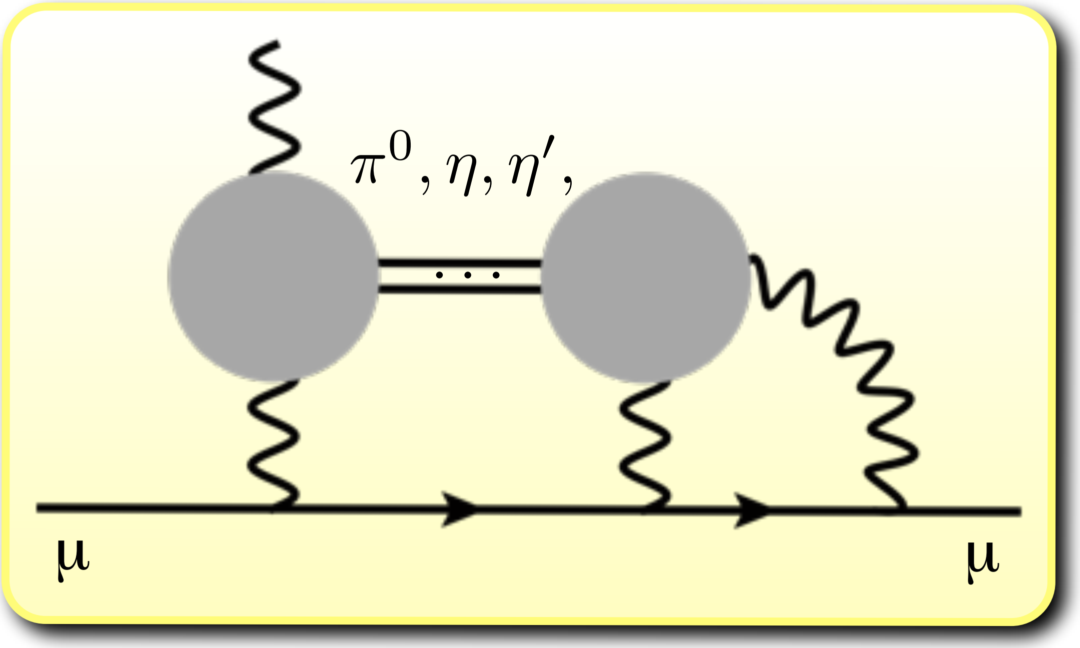


- Currently studying various approaches to address this such as:
  - Brute-force calculation of the disconnected diagrams (computationally costly)
  - Adding the contractions "by-hand"
  - QED reweighting

- Disconnected contributions may be similar in size to the connected ones, so they are essential for a complete calculation with controlled errors

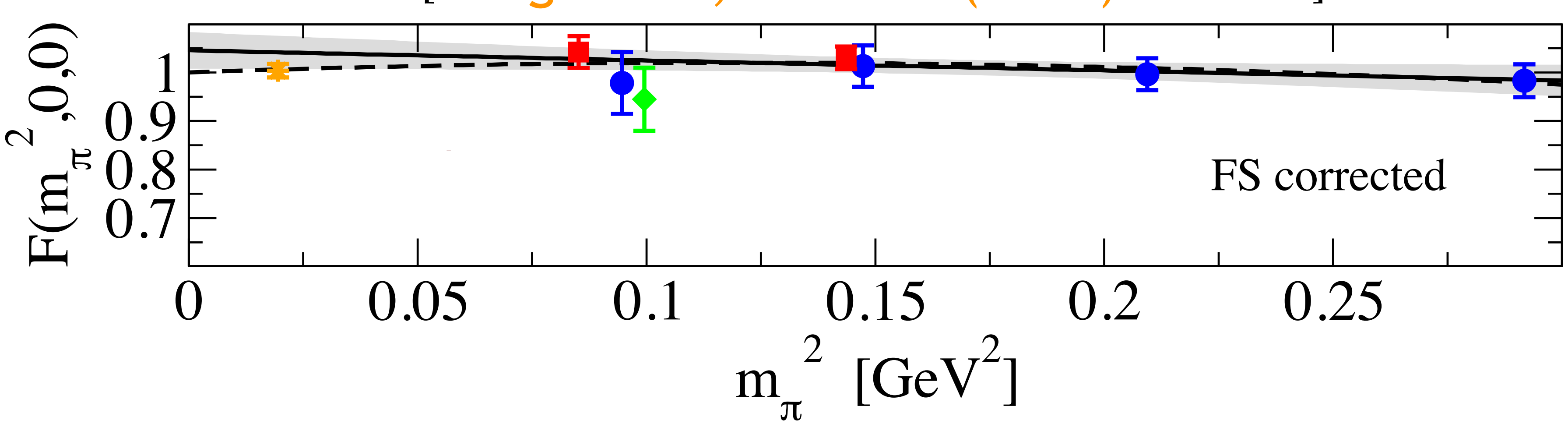
$\pi^0\gamma^*\gamma^*$ form factor

- Dominant contribution to a_μ^{HLbL} from π^0 exchange
- Theoretical estimates of incorporate π^0 exchange contribution modulated by the $\pi^0\gamma^*\gamma^*$ form factor and normalized to the $\pi^0\rightarrow\gamma\gamma$ decay width
  - As a simpler intermediate step, lattice calculations of $F_{\pi 0\gamma\gamma}(k_1,k_1)$ and $\Gamma_{\pi 0\gamma\gamma}$ can check these inputs to model calculations



- JLQCD recently published the first lattice-QCD result for on-shell $\pi^0\rightarrow\gamma\gamma$ decay with controlled errors $\Gamma_{\pi 0\gamma\gamma} = 7.83(31)(49)\mathrm{eV}$

[Feng *et al.*, PRL 109 (2012) 182001]



- Consistent with PrimEx [PRL 106 (2011) 162303], but errors not yet competitive

Quantum Chromodynamics

- QCD Lagrangian contains 1 + n_f + 1 parameters:

$$\mathcal{L}_{\rm QCD} = \frac{1}{2g^2}\mathrm{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] - \sum_{f=1}^{n_f}\bar\psi_f\left(\not{D} + m_f\right)\psi_f + \underbrace{\frac{i\bar\theta}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]}_{\text{violates } CP}$$

  - Gauge coupling g^2 — r_1, m_Ω, Υ(2S-1S), or f_π
  - n_f quark masses m_f — m_π, m_K, $m_{J/\psi}$, m_Υ, …
  - Experimental bound on $|\theta|<10^{-10}$ from neutron EDM — $\theta = 0$

- **Once the parameters of the QCD Lagrangian are fixed, everything else is a prediction of the theory**

Lattice calculations

- Compute operator expectation values on an ensemble of gauge fields $[\mathcal{U}]$ with a distribution exp[-S_{QCD}]:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \underbrace{\mathcal{D}\mathcal{U}}_{\text{MC}} \underbrace{\mathcal{D}\psi_{\text{sea}} \mathcal{D}\bar{\psi}_{\text{sea}}}_{\text{by hand}} e^{-S_{\text{QCD}}[\mathcal{U}, \psi_{\text{sea}}, \bar{\psi}_{\text{sea}}]} \mathcal{O}[\mathcal{U}, \psi_{\text{val}}, \bar{\psi}_{\text{val}}]$$

$$\downarrow$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\mathcal{U} \prod_{f=1}^{n_f} \det \left(\not{D} + m_f \right)_{\text{sea}} e^{-S_{\text{gauge}}[\mathcal{U}]} \mathcal{O}[\mathcal{U}, \psi_{\text{val}}, \bar{\psi}_{\text{val}}]$$

- **Quenched**: replace det→1 (uncontrolled "approximation" ⇒ not used in modern calculations)
- **Partially-quenched**: let $m_{val} \neq m_{sea}$ (recover QCD when $m_{val} = m_{sea} = m_{phys}$)
- **Mixed-action**: let $D_{val} \neq D_{sea}$ (recover QCD when lattice spacing a→0)
- **n_f=2+1**: strange sea quark + degenerate up/down quarks as light as possible (standard)
- **n_f=2+1+1**: add charmed sea quark (some results available)

Systematics in lattice calculations

(1) Monte carlo statistics & fitting

(2) Tuning lattice spacing and quark masses

- Require that lattice results for a few quantities (*e.g.* m_π, m_K, m_{Ds}, m_{Bs}, f_π) agree with experiment

(3) Matching lattice gauge theory to continuum QCD

- Use fixed-order lattice perturbation theory, step-scaling, or other partly- or fully-nonperturbative methods

(4) Chiral extrapolation to physical up, down quark masses

(5) Continuum extrapolation

- Simulate at a sequence of quark masses & lattice spacings and extrapolate to $m_{lat} \rightarrow m_{phys}$ & $a \rightarrow 0$ using functional forms derived in chiral perturbation theory

"GOLD-PLATED" lattice processes

- Easiest quantities to compute **with controlled systematic errors and high precision** have only hadron in initial state and at most one hadron in final state, where the hadrons are stable under QCD (or narrow and far from threshold)
  - Includes meson masses, decay constants, semileptonic and rare decay form factors, and neutral meson mixing parameters
  - Enable determinations of all CKM matrix elements except $|V_{tb}|$
  - Excludes ρ, K^* mesons and other resonances, fully hadronic decays such as $K\rightarrow\pi\pi$ and $B\rightarrow DK$, and long-distance dominated quantities such as D^0-mixing
- Although many nucleon matrix elements are gold plated, calculations are generally more challenging than for mesons
  - Computationally demanding because statistical noise in correlation functions grows rapidly with Euclidean time
  - Extrapolation to physical light-quark masses difficult because baryon chiral perturbation theory converges less rapidly