Multi-Differential Jet Cross-sections

Andrew Larkoski, Ian Moult, Duff Neill

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Motivation

- Amazing angular and energy resolutions of modern detectors.
- With high luminosity one can probe increasingly detailed questions about the structure/substructure of QCD radiation.
- Such detailed questions often require take the form of multi-differential cross-sections, often on a single jet.
- Powerful discriminators for QCD v. W/Z jets or quark v. gluon are often ratios of IRC safe observables.
- Such rations are not IRC safe, but may be "Sudakov" safe when calculated by marginalizing resummed multi-differential cross-sections.

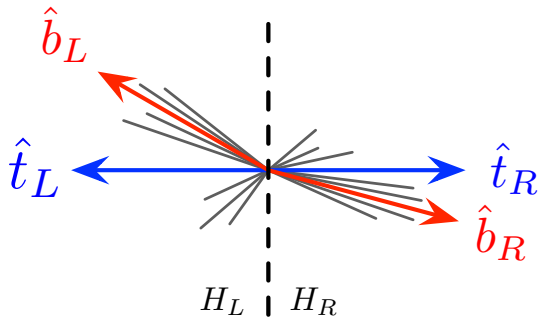
arXiv:1307.1699-Andrew Larkoski, Jesse Thaler

Towards Multi-Differential Phase Space: Jet Angularities

Consider $e^+e^- \to hadrons$, with event shape:

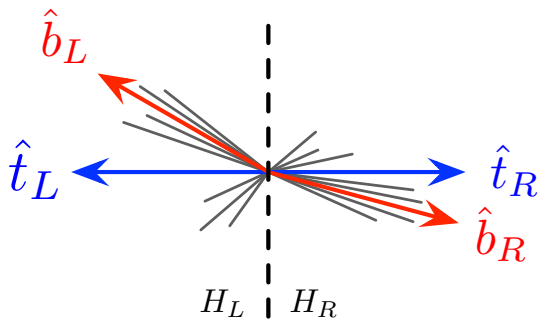
$$e_\beta = \sum_i \frac{E_i}{Q}\left(\sin\left(\frac{\theta_{\hat{n}i}}{2}\right)\right)^\beta \sim \sum_i z_i \theta_{\hat{n}i}^\beta$$

- $\hat{n}$ "recoil-free" jet axes, e.g., axis min. e_1
- $e_\beta \ll 1$ selects jet like structure.



1401.2158: Andrew Larkoski, D.N., Jesse Thaler

Factorization Theorem for $e^+e^- \to h$ in dijet limit



Minimize axis for e_1. Then measure angularity e_β

$$\frac{d\sigma}{de_\beta} = H \int de_n de_{\bar{n}} de_s \delta(e_\beta - e_n - e_{\bar{n}} - e_s) J_n(e_n)\, J_{\bar{n}}(e_{\bar{n}})\, S(e_s)$$

No recoil convolution for all β.

Modes Found in Factorization Theorem

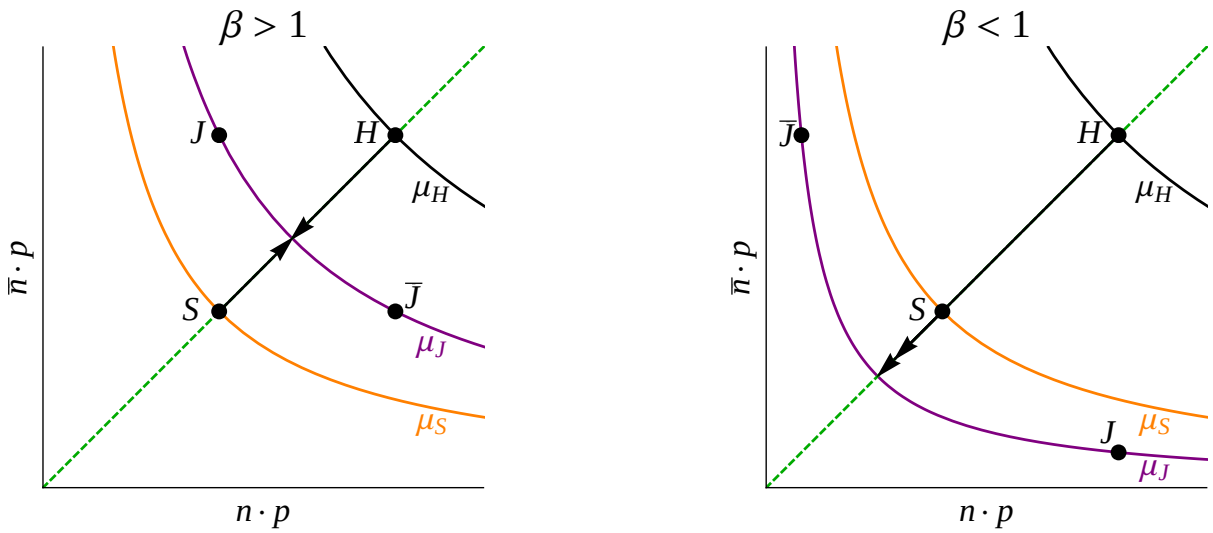
$$\frac{d\sigma}{de_\beta} = H \int de_n de_{\bar{n}} de_s \delta(e_\beta - e_n - e_{\bar{n}} - e_s) J_n(e_n)\, J_{\bar{n}}(e_{\bar{n}})\, S(e_s)$$

Soft-Collinear Effective Field Theory Modes:

$$\lambda \sim e_\beta \text{ and } p = (\bar{n}\cdot p, n\cdot p, p_\perp)$$

$$p_n \sim Q(1, \lambda^{\frac{2}{\beta}}, \lambda^{\frac{1}{\beta}})$$

$$p_s \sim Q(\lambda, \lambda, \lambda)$$



Resummation

In Laplace space ($e_\beta \to s_\beta$):

$$\mu \frac{d}{d\mu} \ln\, J_n\left(s_\beta, \frac{\mu}{Q}\right) = -\frac{\Gamma[\alpha_s(\mu)]}{1-\beta} \ln\, \left(s_\beta \frac{\mu^\beta}{Q^\beta}\right) + \gamma_J$$

$$\mu \frac{d}{d\mu} \ln\, S\left(s_\beta, \frac{\mu}{Q}\right) = 2\frac{\Gamma[\alpha_s(\mu)]}{1-\beta} \ln\, \left(s_\beta \frac{\mu}{Q}\right) + \gamma_S$$

Canonical Scales:

$$\mu_J^2 \sim Q^2 \lambda^{\frac{2}{\beta}}$$

$$\mu_S^2 \sim Q^2 \lambda^2$$

Resummation

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$$\mu \frac{d}{d\mu} \ln S\left(s_\beta, \frac{\mu}{Q}\right) = 2\frac{\Gamma[\alpha_s(\mu)]}{1-\beta} \ln\left(s_\beta \frac{\mu}{Q}\right) + \gamma_S$$

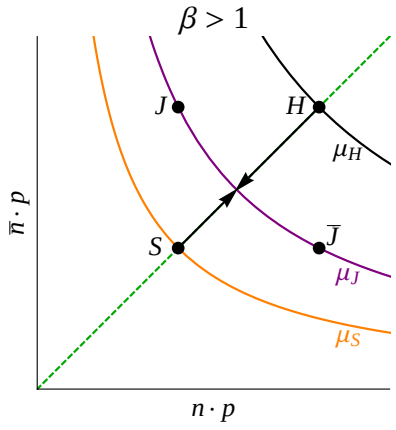
Neighborhood of Canonical Scales:

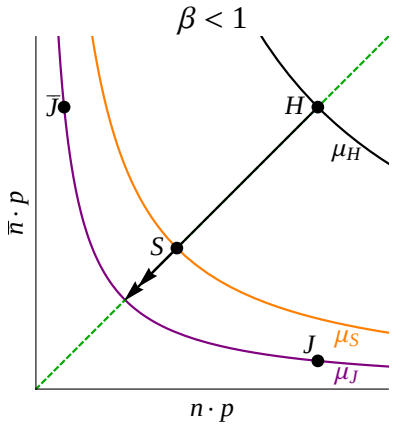
$$\mu_J = Q\left(\frac{\mu_S}{Q}\right)^{\frac{1}{\beta}}$$

Resummation Solved:

Resummed Soft Function:

$$\ln\left(\frac{S\left(s_\beta, \frac{\mu_J}{Q}\right)}{S\left(s_\beta, \frac{\mu_S}{Q}\right)}\right) = \frac{2}{1-\beta}\int_{\mu_s}^{\mu_J}\frac{d\mu}{\mu}\left\{\Gamma[\alpha_s(\mu)]\ \ln\left(s_\beta\frac{\mu}{Q}\right) + \gamma_s[\alpha_s(\mu)]\right\}$$





Measuring two Angularities

- Measure two angularities e_α, e_β, $\alpha > \beta$.
- What is the phase space?

Single emission contributes as:

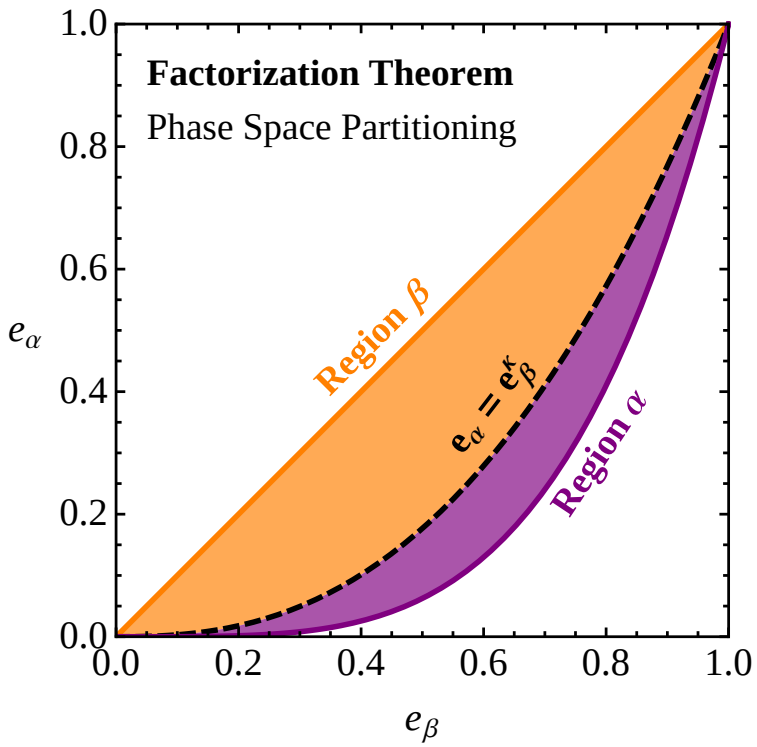
$$e_\alpha \sim z\,\theta^\alpha$$
$$e_\beta \sim z\,\theta^\beta$$

Soft: z sets $e_\alpha \sim e_\beta \ll 1$, and $\theta \sim 1$

Collinear: θ sets $\left(e_\alpha\right)^{\frac{1}{\alpha}} \sim \left(e_\beta\right)^{\frac{1}{\beta}} \ll 1$, and $z \sim 1$

Double Differential Angularities Phase-Space

Two Boundaries of phase space, two factorization regions:



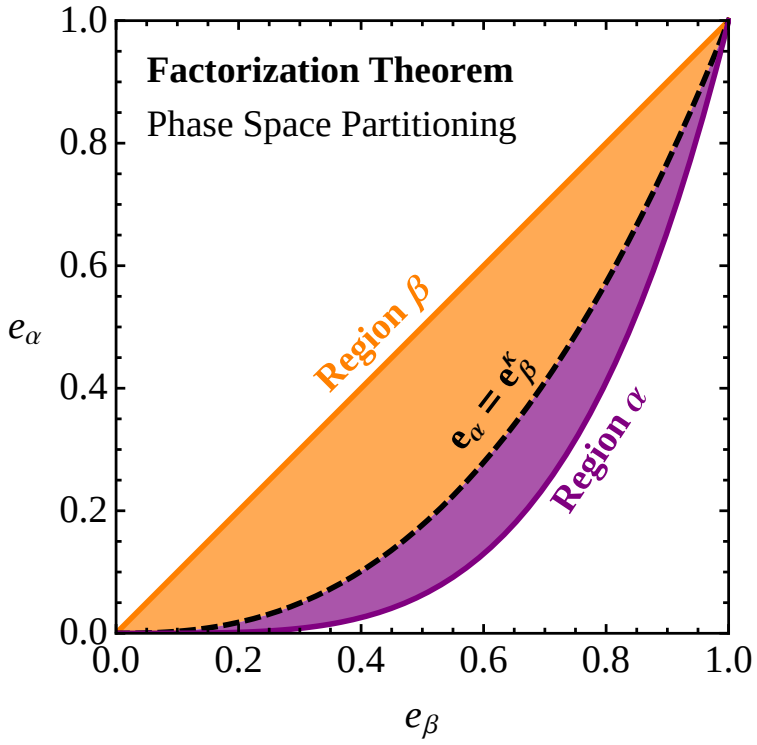
Region β: $e_\alpha \sim e_\beta$.

Region α: $e_\alpha \sim e_\beta^{\frac{\alpha}{\beta}}$.

Boundaries of Phase-Space

Region α: $e_\alpha \sim e_\beta^{\alpha/\beta}$.

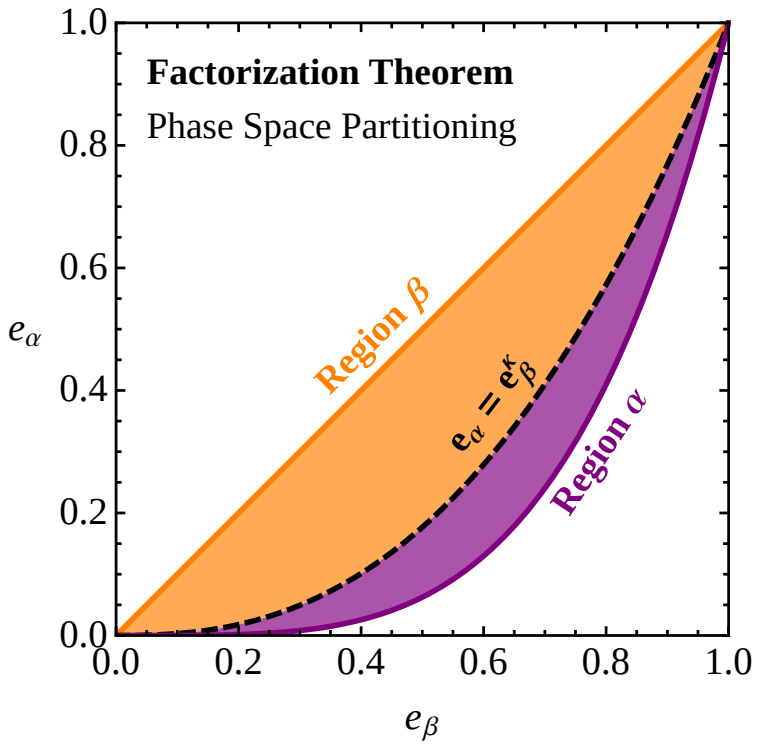
- Set by conservation of energy: $\frac{e_\alpha^{1/\alpha}}{e_\beta^{1/\beta}} \sim z^{1/\alpha - 1/\beta}$, $0 < z < 1$



Boundaries of Phase-Space

Region β: $e_\alpha \sim e_\beta$.

- Set by angular size of jets: $\frac{e_\alpha}{e_\beta} \sim \theta^{\alpha-\beta} < \frac{\pi}{2}, R$



Region β: $e_\alpha \sim e_\beta$

Soft and Collinear contributions:

$$e_\alpha \sim z_s \theta_s^\alpha + z_c \theta_c^\alpha$$
$$e_\beta \sim z_s \theta_s^\beta + z_c \theta_c^\beta$$

Since $e_\alpha \sim e_\beta$ and $\theta_c^\alpha \ll \theta_c^\beta$,
Collinears power suppressed in e_α:

$$\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\beta) \otimes_\beta S(e_\alpha, e_\beta)$$

Region β: $e_\alpha \sim e_\beta$

Soft and Collinear contributions:

$$e_\alpha \sim z_s \theta_s^\alpha + O\left(z_c \theta_c^\alpha\right)$$
$$e_\beta \sim z_s \theta_s^\beta + z_c \theta_c^\beta$$

Since $e_\alpha \sim e_\beta$ and $\theta_c^\alpha \ll \theta_c^\beta$,
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Region α: $e_\alpha \sim e_\beta^{\frac{\alpha}{\beta}}$

Soft and Collinear contributions:

$$e_\alpha \sim z_s \theta_s^\alpha + z_c \theta_c^\alpha$$
$$e_\beta \sim z_s \theta_s^\beta + z_c \theta_c^\beta$$

Since $e_\alpha \sim e_\beta^{\frac{\alpha}{\beta}} \rightarrow e_\alpha \ll e_\beta$,
Softs power suppressed in e_β:

$$\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\alpha, e_\beta) \otimes_\alpha S(e_\alpha)$$

Region α: $e_\alpha \sim e_\beta^{\frac{\alpha}{\beta}}$

Soft and Collinear contributions:

$$e_\alpha \sim z_s \theta_s^\alpha + z_c \theta_c^\alpha$$
$$e_\beta \sim z_c \theta_c^\beta + O\left(z_s \theta_s^\beta\right)$$

Since $e_\alpha \sim e_\beta^{\frac{\alpha}{\beta}} \rightarrow e_\alpha \ll e_\beta$,
Softs power suppressed in e_β:

$$\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\alpha, e_\beta) \otimes_\alpha S(e_\alpha)$$

Jet and soft functions

Single differential functions:

$$J(e_\beta) = \frac{(2\pi)^3}{N_c}\langle 0|\bar{\chi}_{\bar{n}}\,\delta(n\cdot\hat{P}-Q)\delta(\hat{e}_\beta - e_\beta)\delta^{(2)}(\hat{P}_\perp)\frac{\not{n}}{2}\chi_{\bar{n}}|0\rangle$$

$$S(e_\alpha) = \frac{1}{N_c}\mathrm{tr}\langle 0|T\Big\{S^\dagger_{\bar{n}}S_n\Big\}\delta(\hat{e}_\alpha - e_\alpha)\bar{T}\Big\{S^\dagger_n S_{\bar{n}}\Big\}|0\rangle$$

Double Differential functions:

$$J(e_\alpha, e_\beta) = \frac{(2\pi)^3}{N_c}\langle 0|\bar{\chi}_{\bar{n}}\delta(n\cdot\hat{P}-Q)\delta(\hat{e}_\alpha - e_\alpha)\delta(\hat{e}_\beta - e_\beta)\delta^{(2)}(\hat{P}_\perp)\,\frac{\not{n}}{2}\chi_{\bar{n}}|0\rangle$$

$$S(e_\alpha, e_\beta) = \frac{1}{N_c}\mathrm{tr}\,\langle 0|\,T\Big\{S^\dagger_{\bar{n}}S_n\Big\}\delta(\hat{e}_\alpha - e_\alpha)\delta(\hat{e}_\beta - e_\beta)\bar{T}\Big\{S^\dagger_n S_{\bar{n}}\Big\}|0\rangle$$

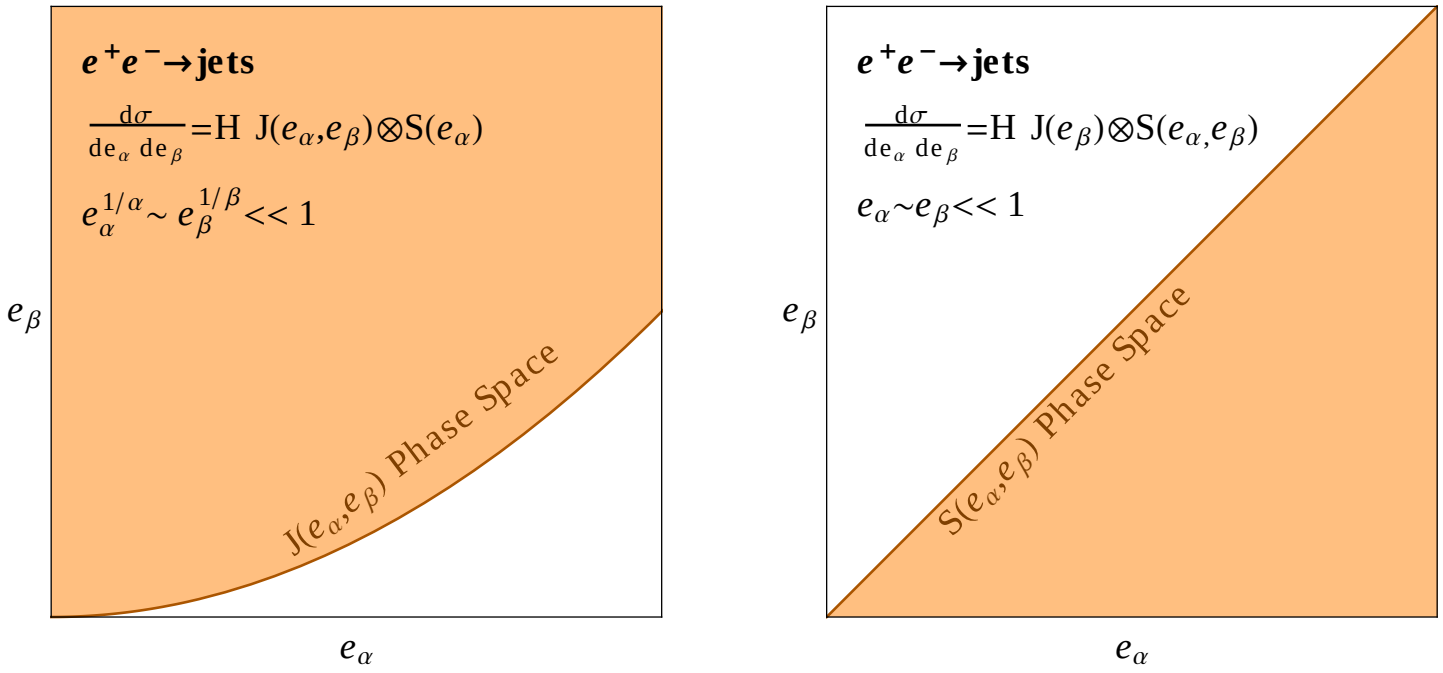
Jet and soft functions

Double differential functions renormalize exactly the same as single differential:

$$\mu \frac{d}{d\mu} \ln J(e_\alpha, e_\beta) = \delta(e_\beta) \mu \frac{d}{d\mu} \ln J(e_\alpha)$$
$$\mu \frac{d}{d\mu} \ln S(e_\alpha, e_\beta) = \delta(e_\alpha) \mu \frac{d}{d\mu} \ln S(e_\beta)$$

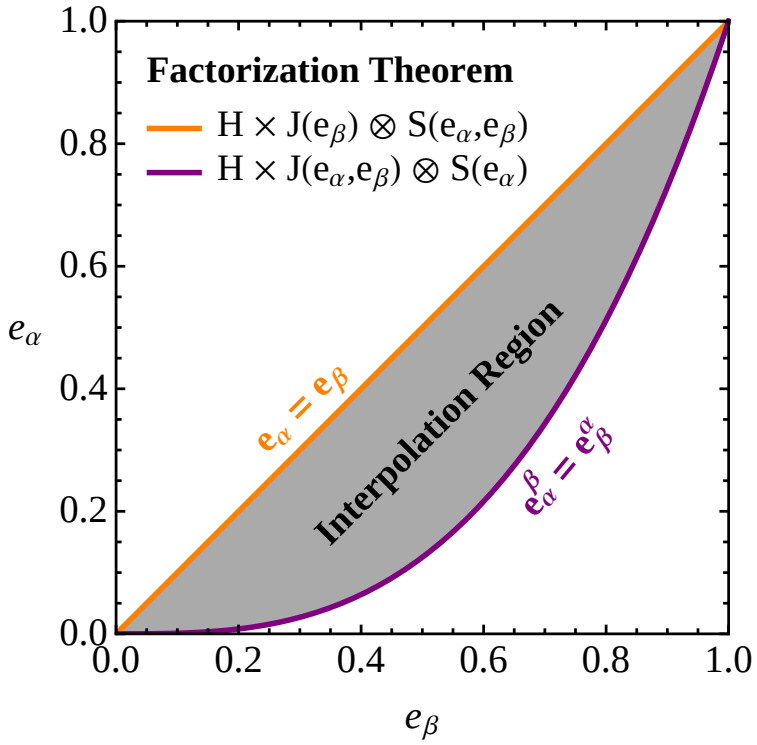
$$\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\alpha, e_\beta) \otimes_\alpha S(e_\alpha)$$
$$\frac{d\sigma}{de_\alpha de_\beta} = HJ(e_\beta) \otimes_\beta S(e_\alpha, e_\beta)$$

Double Differential Angularities Phase-Space



Region α: Fact. Theorem calculates correct lower boundary
Region β: Fact. Theorem calculates correct upper boundary.

Double Differential Angularities Resummation



Region α: Fact. Theorem resums $\alpha_s^n \ln^m(e_\alpha)$ like $\frac{d\sigma}{de_\alpha}$.
Region β: Fact. Theorem resums $\alpha_s^n \ln^m(e_\beta)$ like $\frac{d\sigma}{de_\beta}$.

Double Differential Angularities: Interpolating Resummation

- No naive factorization description covering all phase space.
- How do we move from one boundary to the other smoothly, while resumming all large logarithms?*
- How do we move between the fixed order descriptions?

*Assuming one can suitably define a large log.

Basis for Writing an Interpolating Resummation

Use Two Facts:

- Boundary conditions on the double cumulative distribution.
- The structure of single differential x-sec resummation.

Double Differential Angularities: Interpolating Resummation

Consider cumulative distribution:

$$\Sigma(e_\alpha, e_\beta) = \int^{e_\alpha} de'_\alpha \int^{e_\beta} de'_\beta \frac{d\sigma}{de'_\alpha de'_\beta}$$

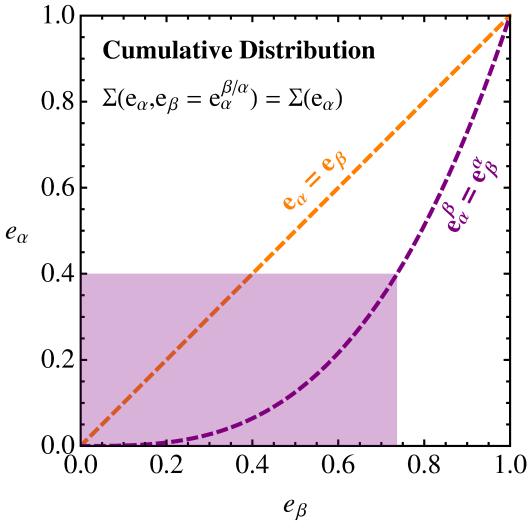
Double Differential Angularities: Interpolating Resummation

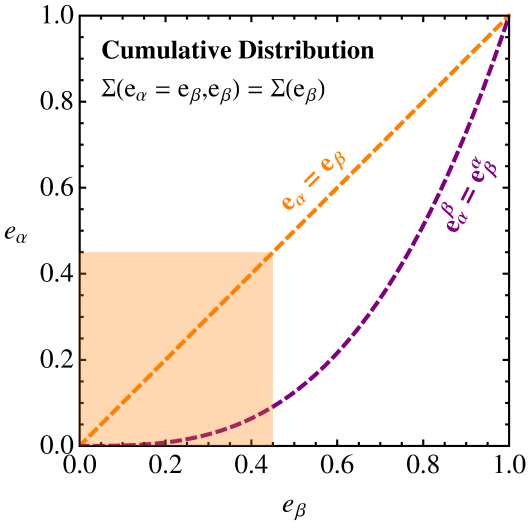
Boundary conditions on cumulative distribution:

$$\Sigma(e_\alpha, e_\beta = e_\alpha^{\beta/\alpha}) = \Sigma(e_\alpha), \qquad \Sigma(e_\alpha = e_\beta, e_\beta) = \Sigma(e_\beta)$$

$$\frac{\partial}{\partial e_\alpha}\Sigma(e_\alpha, e_\beta)\Big|_{e_\beta = e_\alpha^{\beta/\alpha}} = \frac{d\sigma}{de_\alpha}(e_\alpha), \qquad \frac{\partial}{\partial e_\beta}\Sigma(e_\alpha, e_\beta)\Big|_{e_\alpha = e_\beta} = \frac{d\sigma}{de_\beta}(e_\beta)$$

$$\frac{\partial}{\partial e_\alpha}\Sigma(e_\alpha, e_\beta)\Big|_{e_\beta = e_\alpha} = 0, \qquad \frac{\partial}{\partial e_\beta}\Sigma(e_\alpha, e_\beta)\Big|_{e_\beta = e_\beta^{\alpha/\beta}} = 0$$





Structure of Resummation of Single Differential

Using RG equations of factorized cross-section:

$$\Sigma(e_\beta) = \int_0^{e_\beta} de'_\beta \frac{d\sigma}{de'_\beta} = \frac{e^{-\gamma_E R'(e_\beta)}}{\Gamma(1 + R'(e_\beta))} e^{-R(e_\beta) - T(e_\beta)}$$

- $R(e_\beta)$ is the radiator.
- $R(e_\beta)$ has an exact (Laplace space) description in terms of the integrals over cusp parts of hard, jet, soft anomalous dimensions.
- Fact. Theorems predict $\Sigma(e_\alpha, e_\beta)$ to have exact same form near a boundary.

Ansatz for Resummation of Double Differential (At Least to NLL)

Using single differential x-sec form, make ansatz:

$$\Sigma(e_\alpha, e_\beta) = \frac{e^{-\gamma_E \tilde{R}(e_\alpha, e_\beta)}}{\Gamma(1 + \tilde{R}(e_\alpha, e_\beta))} e^{-R(e_\alpha, e_\beta) - T(e_\alpha, e_\beta)}$$

- Demand for satisfy boundary conditions.
- Thus reduces exactly to single differential resummation on boundary.

Ansatz for Resummation of Double Differential (At Least to NLL)

Deriving expressions for $\tilde{R}(e_\alpha, e_\beta)$ and $R(e_\alpha, e_\beta)$:

- Start from $R(e_\beta)$ as sum of cusp pieces of hard, jet and soft anom. dim.
- Re-arrange jet and soft parts as linear combinations with $O(1)$ changes of canonical RG destinations.
- e.g., $\mu_J \to \mu_J \left(\frac{e_\alpha^\beta}{e_\beta^\alpha}\right)^c$ in region-α
- Apply boundary conditions.

One-Loop Result for Double Differential Radiator Ansatz

$$R^{(1)}(e_\alpha, e_\beta) = \frac{C_i}{2\pi\alpha_s\beta_0^2}\left[\frac{1}{\alpha-1}U\left(2\alpha_s\beta_0\log e_\alpha\right) - \frac{\beta}{\beta-1}U\left(2\alpha_s\beta_0\frac{\log e_\beta}{\beta}\right)\right.$$
$$\left. + \frac{\alpha-\beta}{(\alpha-1)(\beta-1)}U\left(2\alpha_s\beta_0\frac{\log e_\alpha^{1-\beta}e_\beta^{\alpha-1}}{\alpha-\beta}\right)\right],$$

$$U(x) = x\ln x$$

$K_T - \log$

New logarithmic structure not appearing in either factorization theorem:

$$\frac{\log e_\alpha^{1-\beta} e_\beta^{\alpha-1}}{\alpha-\beta} \sim \log(z\theta) \sim \log\left(\frac{k_T}{Q}\right)$$

Interpolates between double differential soft and jet functions:

$$\begin{tikzcd}
\underline{e_\beta \sim e_\alpha^{\beta/\alpha}} & & \underline{e_\alpha \sim e_\beta} \\
J(e_\alpha, e_\beta) \arrow[rr, leftrightarrow, "\log e_\beta^{1/\beta}"] \arrow[dr, leftrightarrow, "\log e_\alpha^{\frac{1-\beta}{\alpha-\beta}} e_\beta^{\frac{\alpha-1}{\alpha-\beta}}"'] & & J(e_\beta) \\
S(e_\alpha) \arrow[r, leftrightarrow, "\log e_\alpha"] & S(e_\alpha, e_\beta) &
\end{tikzcd}$$

$K_T - \log$

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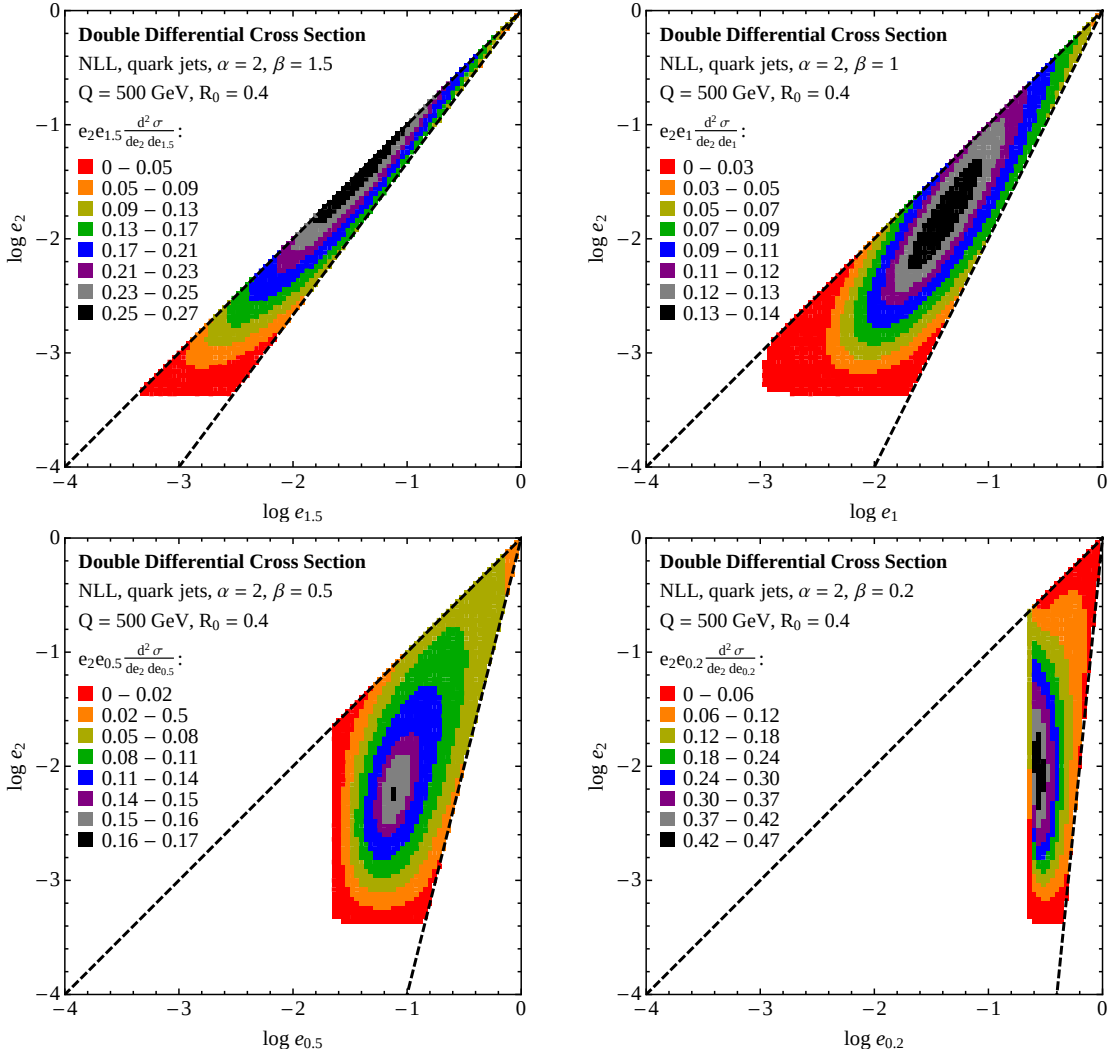
S(e_\alpha) \arrow[r, leftrightarrow, "\log e_\alpha"] & S(e_\alpha, e_\beta) &
\end{tikzcd}$$

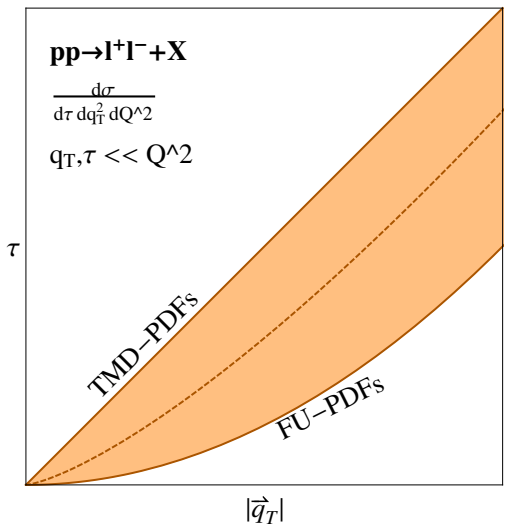
Plots

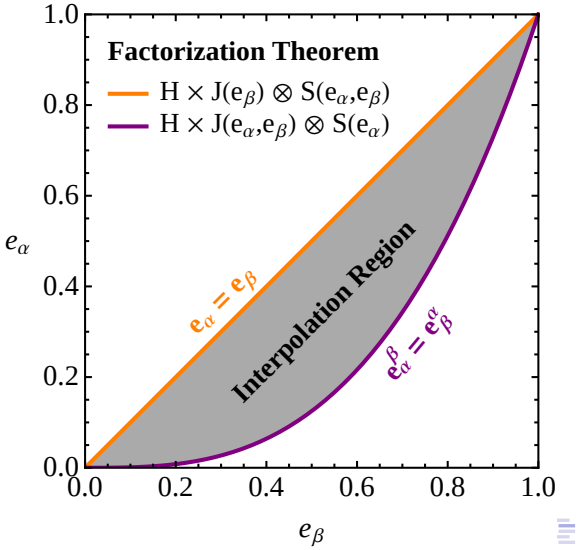


Note: Pure resummation (NLL) not sufficient to reproduce boundary behavior, need low scale matrix elements.

Possible Future Directions

- Justify/Disprove/Modify Ansatz. (Progress in this direction has been made...)
- Include low-scale matrix elements with resummation (NLL').
- Perform similar analysis for TMD-PDF versus FU-PDF factorizations in Drell-Yan.
- Analyze all phase-space factorizations for more complicated multiple differential observables.
- LHC: QCD versus Boosted Z, W: 1 jet versus 2 subjets.

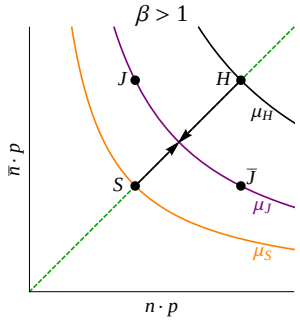
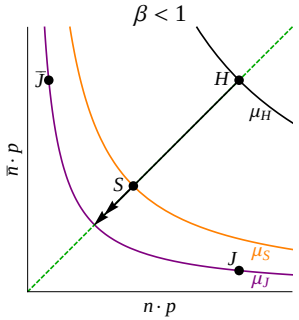




$\beta \to 1$ Limit

Rapidity Resummation from UV-Soft running!

$$\ln\left(\frac{S\left(s_\beta, \frac{\mu_J}{Q}\right)}{S\left(s_\beta, \frac{\mu_S}{Q}\right)}\right) = \frac{2}{1-\beta}\int_{\mu_s}^{\mu_J} \frac{d\mu}{\mu}\Gamma[\alpha_s(\mu)]\,\ln\left(s_\beta\frac{\mu}{Q}\right)$$

$$=_{\beta\to 1} \underbrace{2\,\Gamma[\alpha_s(\mu_S)]\,\ln\left(s_\beta\frac{\mu_S}{Q}\right)}_{\text{CS Kernel}}\ \overbrace{\ln\left(\frac{\nu}{Q}\right)}^{\text{Rapidity Log}}$$

$$\mu_J = Q\left(\frac{\nu}{\mu_S}\right)^{\frac{1-\beta}{\beta}}\left(\frac{\mu_S}{Q}\right)^{\frac{1}{\beta}}$$