Extrapolation Techniques for Asymmetry Measurements

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Overview

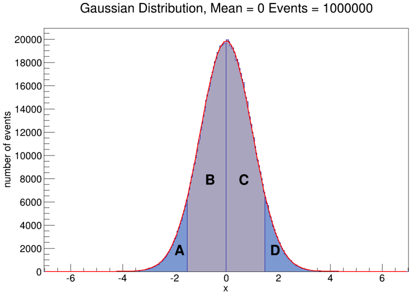
Background, Motivation, and Goals

- Common in particle physics to measure asymmetries – in particular in collider experiments
- Often data can only be measured for a finite portion of the detector, must extrapolate to the total asymmetry

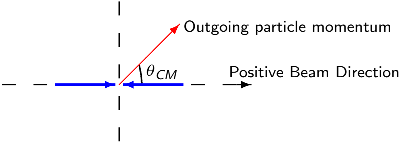
$$A^{\text{total}} = \frac{(C+D)-(A+B)}{A+B+C+D}$$

$$A^{\text{finite}} = \frac{C-B}{B+C}$$

- Can we use a simple *constant* multiplicative factor $A^{\text{total}} = R \cdot A^{\text{finite}}$?
- If so, how much statistics needed to get reliable results, especially in the limit of small asymmetries ?



Classic Example: forward-backward asymmetry (A_{FB}) measured in collider detectors:

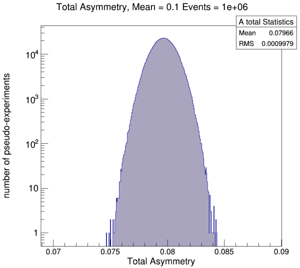


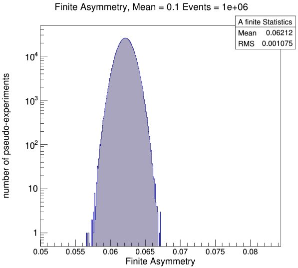
Background, Motivation, and Goals

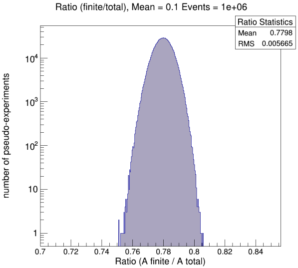
- We start with a single Gaussian with a mean of μ as a good working model to build a foundation and give good insights into more complicated distribution models
- Examples from collider physics have shown that this approximation sometimes works
- It is not obvious if a linear extrapolation technique *should* work
- Since we typically use MC methods to estimate such values, we need to understand whether we *can* confidently use a constant R to linearly extrapolate, and understand the amount of statistics needed to get a reasonable measurement of it

Study 1: Monte Carlo Simulation

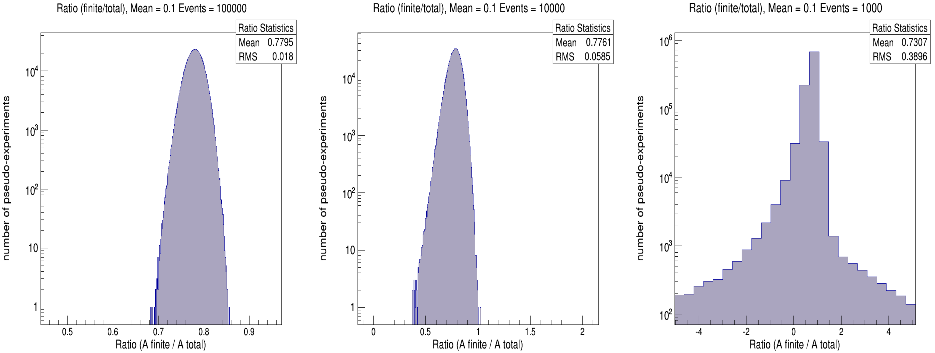
- In our simple Gaussian model, A is linearly proportional to μ (the mean of the distribution)
- Example: $\mu = 0.1$ corresponds to $A^{\text{total}} \approx 8\%$ which is what we typically see in forward-backward asymmetry top quark measurements at the Tevatron
- Run many MC pseudo-experiments each with a large number of events, get distributions for A^{total}, A^{finite}, and R:







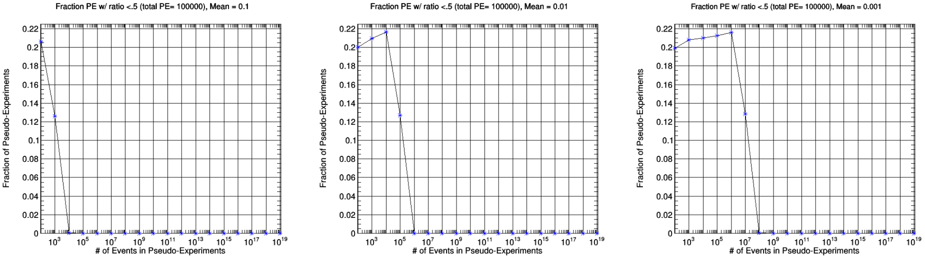
Study 1: Monte Carlo Simulation



- With enough statistics (i.e. large N), measurements of R are very accurate
- As N decreases, measurement of R becomes unreliable, and can no longer correctly reproduce A^{total} from A^{finite}
- This is observed for all values of μ

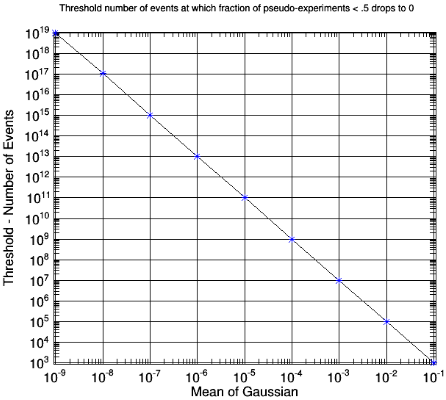
Study 1: Monte Carlo Simulation

- With this understanding, we now aim to quantify this behavior to properly understand how many MC events in the original distribution, N, are needed to give reliable measurements of R
- We define f as the fraction of pseudo-experiments with $R < 0.5$ (very far from expected value)



Study 1: Monte Carlo Simulation

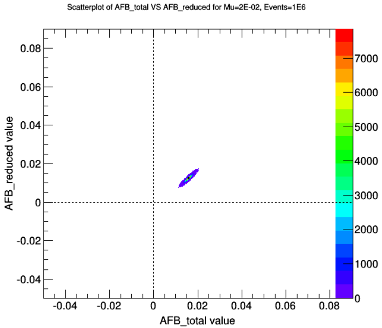
- Want $f \approx 0$, define a threshold value and observe the relationship between the number of events needed for reliable measurements and μ
- N falls as $\frac{1}{\mu^2}$

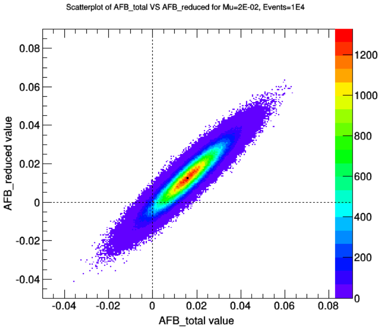


- Measurements of R for all values of μ with enough statistics give the same value
- Conclusion is that R *is* indeed constant for all μ for this simple Gaussian model, and a huge amount of MC statistics are needed to accurately measure the actual value for small μ (or equivalently small A)

Study 2: Closed Form Statistical Solution

- Let's take a closer look at *why* the MC methods break down





- Require A^{total} (denominator of R) to be greater than *at least* 1σ away from 0 – to avoid the potential divide by 0 problem (math jargon: this is where the distribution transitions to a Cauchy regime)

Study 2: Closed Form Statistical Solution

- The statistical question becomes: how many events, N, are required for the mean of ${A_{FB}}^{\text{total}}$ to be some number $(k \cdot \sigma)$ away from 0, thus giving reliable measurements

$$\sigma_{{A_{FB}}^{\text{total}}} = \frac{A^{\text{total}}}{k}$$

- Using statistics (see backup slides), we are able to find N as a function of μ for our single Gaussian model:

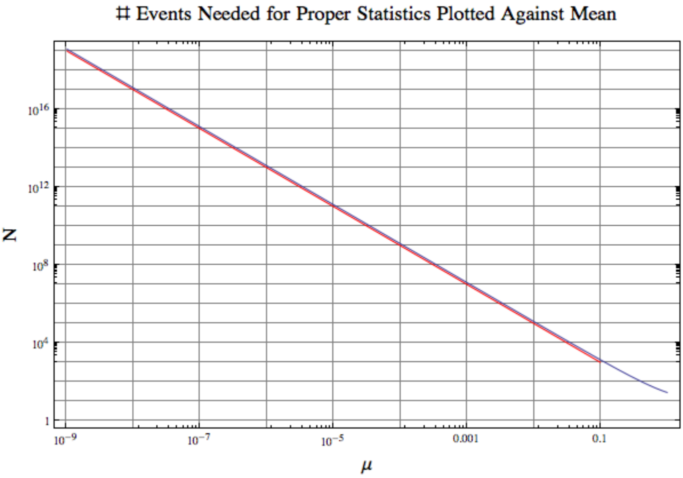
$$N = 2k^2 \cdot \frac{\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)}{\operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)^2}$$

- Some limiting cases:
  - As $\mu \to 0$, $N \to \infty$
  - Using the approximation $\operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right) \approx \sqrt{\frac{2}{\pi}}\,\mu$ for small μ, we find that $N \propto \frac{1}{\mu^2}$ which is precisely what we just saw from our MC study

Study 2: Closed Form Statistical Solution

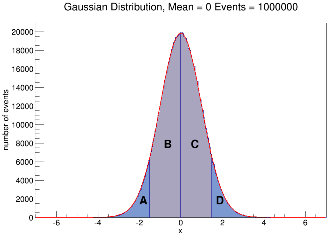
- Closed form solution: blue (for $k = 2$)
- MC data: red

- Excellent agreement!



Study 3: Closed Form Numerical Solution

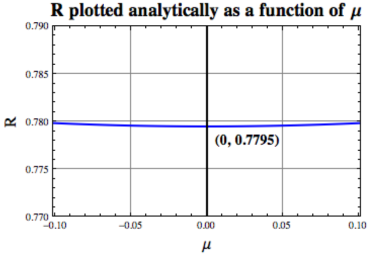
- We calculate R as a function of μ using *Mathematica*
- Set $\sigma = 1.0$
- Plot R in the limit $\mu \to 0$
- For large values of μ, R only rises by 0.04% relative to $\mu = 0$



$$A^{total} = \frac{(C + D) - (A + B)}{A + B + C + D}$$

$$A^{finite} = \frac{C - B}{B + C}$$

$$R = \frac{A^{finite}}{A^{total}}$$



Conclusions

- We have used three methods to study the linear extrapolation of A^{finite} to an inclusive A^{total}
- While we have only studied the simple Gaussian model, we observed that a linear extrapolation **can** be used, and while MC methods work reliably (even for small A) they can require much more significant statistics than expected
- Our results have the potential to be applied for many different asymmetry measurements in collider experiments, and have already been useful at the Tevatron for the $t\bar{t}$ forward-backward asymmetry

Thank You For Listening!
Any Questions?

We need enough statistics such that A_{FB}^{total}, the denominator of R, is more than 1 sigma away from 0 (we will set it to be k, where k will be determined later). In other words, we want to know how many events it takes in a pseudo-experiment to ensure the mean of the full asymmetry will be k standard-deviations away from zero.
To do this we start with the equation

$$\sigma_{A_{FB}^{total}} = \frac{A_{FB}^{total}}{k} \tag{1}$$

where σ_{AFB}^{total} is the variation (or uncertainty) of the measured value of A_{FB}^{total}. We will find both $\sigma_{A_{FB}^{total}}$ and A_{FB}^{total} as functions of N and μ and substitute them into Eq. 1 to get the functional relation between N and μ for "good statistics".

Backup Slides: The Statistical Solution Calculation

We begin with our definition of asymmetry,

$$A_{FB}^{total} = \frac{N_+ - N_-}{N_+ + N_-} \tag{2}$$

where $N_+ = C + D$ and $N_- = A + B$ as on Slide 2. Next we define $N = N_+ + N_-$ as the total number of events in the original Gaussian distribution, and rewrite this as:

$$A_{FB}^{total} = \frac{2N_+ - N}{N}. \tag{3}$$

We note that since our distributions are Gaussian, we can write N_+ in terms of N and μ, with the relation given by

$$\begin{aligned} N_+ &= \frac{N}{\sqrt{2\pi}} \int_0^\infty \mathrm{dx}\ e^{-(x-\mu)^2/2} \\ &= \frac{N}{2}\left(\mathrm{erf}\left(\frac{\mu}{\sqrt{2}}\right) + 1\right) \end{aligned} \tag{4}$$

Backup Slides: The Statistical Solution Calculation

Plugging this in to Eq. 3 and reducing, we get

$$A_{FB}^{total} = \frac{\cancel{2}\frac{\cancel{N}}{\cancel{2}}\left(\text{erf}\left(\frac{\mu}{\sqrt{2}}\right) + \cancel{1}\right) - \cancel{N}}{\cancel{N}}$$
$$= \text{erf}\left(\frac{\mu}{\sqrt{2}}\right) \tag{5}$$

We next find $\sigma_{A_{FB}^{total}}$ by beginning with the definition given in Bevington (92) applied to our problem,

$$\sigma_{A_{FB}^{total}} = \left(\frac{\partial A_{FB}^{total}}{\partial N_+}\right)\sigma_{N_+} + \left(\frac{\partial A_{FB}^{total}}{\partial N}\right)\sigma_N. \tag{6}$$

Taking a simple derivative of A_{FB}^{total} from Eq. 3 gives us

$$\left(\frac{\partial A_{FB}^{total}}{\partial N_+}\right) = \frac{2}{N} \tag{7}$$

Backup Slides: The Statistical Solution Calculation

To be consistent with the previous study, we fix N and allow N_+ to vary. This means that $\sigma_N = 0$, and from simple statistics

$$\sigma_{N_+} = \sqrt{N_+} \tag{8}$$

Plugging Eqs. 7 and 8 into Eq. 6, we get

$$\sigma_{A_{FB}^{total}} = \frac{2}{N} \cdot \sqrt{N_+}. \tag{9}$$

Plugging Eq. 4 into this, we get

$$\begin{aligned} \sigma_{A_{FB}^{total}} &= \frac{2}{N} \cdot \sqrt{\frac{N}{2}\left(\operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right) + 1\right)} \\ &= \sqrt{\frac{2}{N}} \cdot \sqrt{\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)} \end{aligned} \tag{10}$$

Backup Slides: The Statistical Solution Calculation

Finally, plugging Eqs. 5 and 10 back into Eq. 1 gives us

$$\sqrt{\frac{2}{N}} \cdot \sqrt{\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)} = \frac{\operatorname{erf}(\frac{\mu}{\sqrt{2}})}{k}, \tag{11}$$

and solving for N, we get

$$N = \frac{2k^2\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)}{\operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)^2} \tag{12}$$

This is, as we set out to solve for, the number of events it takes per pseudo-experiment to ensure the mean of the full asymmetry will be k standard-deviations away from zero, and thus give good statistics. Discussion of the implication of this result is included in the main slides.

Study 3: Closed Form Numerical Solution

$$A_{FB}^{total} = \frac{\frac{1}{\sqrt{2\pi}\sigma}\int_0^{\infty} dx\left[\exp(-\frac{(x-\mu)^2}{2\sigma^2}) - \exp(-\frac{(-x-\mu)^2}{2\sigma^2})\right]}{\frac{1}{\sqrt{2\pi}\sigma}\int_0^{\infty} dx\left[\exp(-\frac{(x-\mu)^2}{2\sigma^2}) + \exp(-\frac{(-x-\mu)^2}{2\sigma^2})\right]}$$

$$A_{FB}^{finite} = \frac{\frac{1}{\sqrt{2\pi}\sigma}\int_0^{1.5} dx\left[\exp(-\frac{(x-\mu)^2}{2\sigma^2}) - \exp(-\frac{(-x-\mu)^2}{2\sigma^2})\right]}{\frac{1}{\sqrt{2\pi}\sigma}\int_0^{1.5} dx\left[\exp(-\frac{(x-\mu)^2}{2\sigma^2}) + \exp(-\frac{(-x-\mu)^2}{2\sigma^2})\right]}$$

$$R = \frac{A_{FB}^{finite}}{A_{FB}^{total}}$$