NUFAC 07 Institute Lectures

KEK Tokyo
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R. B. Palmer (BNL)

3 Lectures and Tutorial

0. Preface

1. Solenoid Focus

2. Neutrino Factory Introduction

3. Muon Collider Introduction

4. Transverse Ionization Cooling

5. Longitudinal Ionization Cooling

6. Tutorials

0) PREFACE

0.1 Units

When discussing the motion of particles in magnetic fields, I will use MKS units for lengths and magnetic fields and time, so $c \approx 3\ 10^8$ (m/sec). But momentum, energy, and mass will be expressed in "electron Volts". The unit of charge e is plus or minus one.

In these units, the bending radius ρ (m) in a field B (T), of a particle with unit charge is:

$$\rho \ = \ \frac{p}{B\ e\ c} \qquad \text{which for a positive charge gives} \qquad \rho \ = \ \frac{p}{B\ c}$$

In practice, I will often omit the e, meaning that I am referring by default to positive particles

e.g. the radius for a 3 GeV/c particle in 5 Tesla is

$$\rho \ = \ \frac{3\ 10^9}{5\ \times\ 3\ 10^8} \ = \ 2m$$

0.2 Useful Relativistic Relations

$$dE = \beta_v \, dp \tag{1}$$

$$\frac{dE}{E} = \beta_v^2 \, \frac{dp}{p} \tag{2}$$

$$d\beta_v = \frac{dp}{\gamma^2} \tag{3}$$

I use β_v to denote v/c to distinguish it from the Courant-Schneider or Twiss parameters $\beta_\perp$

0.3 Emittance

$$\text{normalized emittance} \;=\; \frac{\text{Phase Space Area}}{\pi\ \text{m c}}$$

The phase space can be transverse:p_x vs x, p_y vs y, or longitudinal Δp_z vs z, where Δp_z and z are with respect to the moving bunch center.

If x and p_x, or y and p_y, are both Gaussian and uncorrelated, then the area is taken to be that of the upright ellipse with radii equal to the rms values in x and x'. When the beams are symmetric in x and y we often use the terms $\sigma_\perp = \sigma_x = \sigma_y$, $\epsilon_\perp = \epsilon_x = \epsilon_y$, $\sigma_\parallel = \sigma_z$, and $\epsilon_\parallel = \epsilon_z$

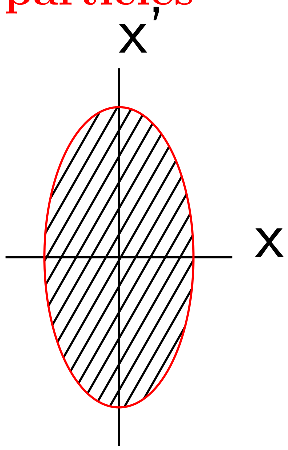
$$\epsilon_\perp = \frac{\pi\ \sigma_{p_\perp}\sigma_\perp}{\pi\ mc} \;=\; (\gamma\beta_v)\sigma_\theta\sigma_\perp \qquad\qquad (\pi\ m\ rad) \tag{4}$$

$$\epsilon_\parallel = \frac{\pi\ \sigma_{p_\parallel}\sigma_z}{\pi\ mc} \;=\; (\gamma\beta_v)\frac{\sigma_p}{p}\ \sigma_z \qquad\qquad (\pi\ m\ ) \tag{5}$$

$$\epsilon_6 = \epsilon_\perp^2\ \ \epsilon_\parallel \qquad\qquad (\pi\ m)^3 \tag{6}$$

Note that the π, added to the dimension, is a reminder that the emittance is phase space/π
The "rad" in the dimensions is in by convention only and has no real meaning

0.4 Beta$_\perp$($beam$) of a beam of many particles



Upright phase ellipse in x' vs x,

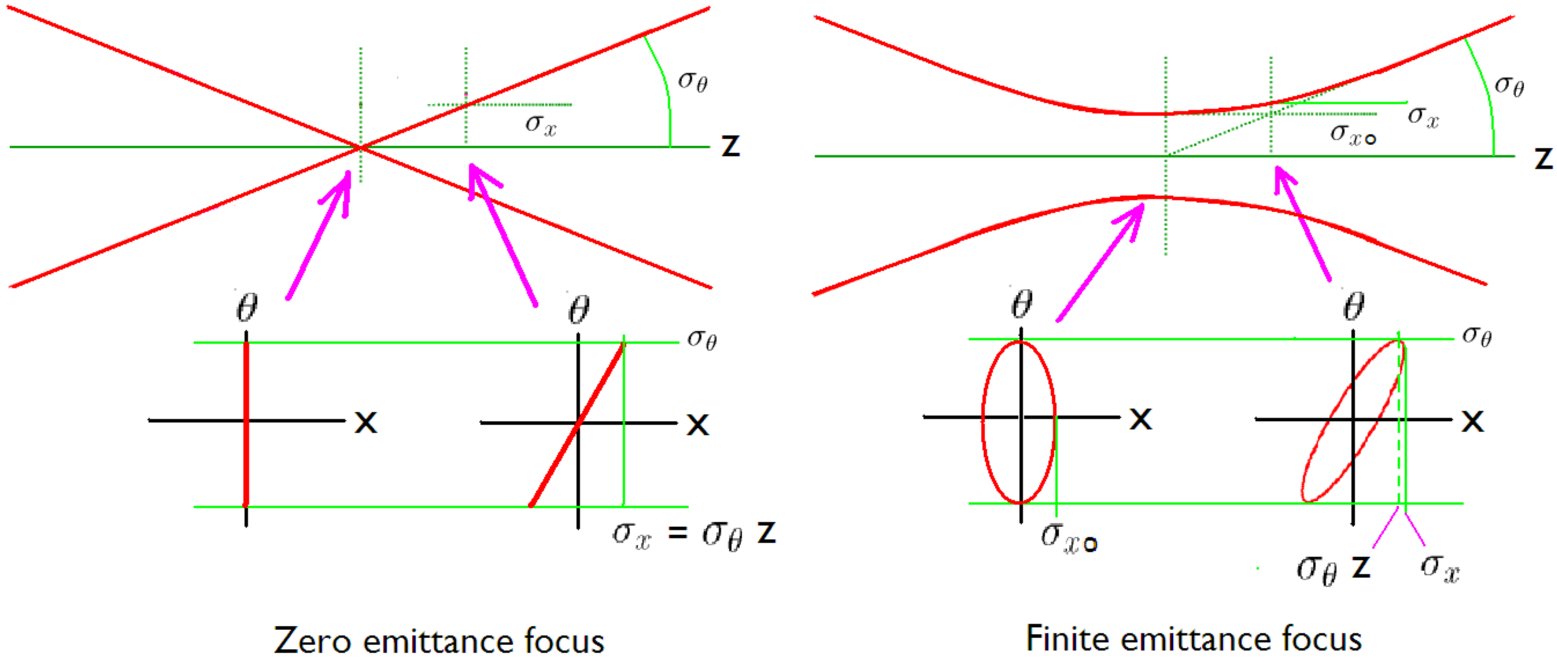
$$\beta_\perp \ = \left(\frac{\text{width}}{\text{height}} \quad \text{of phase ellipse} \right) \ = \ \frac{\sigma_x}{\sigma_\theta} \tag{7}$$

Then, using emittance definition:

$$\sigma_x \ = \sqrt{\epsilon_\perp \ \beta_\perp \ \frac{1}{\beta_v \gamma}} \tag{8}$$

$$\sigma_\theta \ = \sqrt{\frac{\epsilon_\perp}{\beta_\perp} \ \frac{1}{\beta_v \gamma}} \tag{9}$$

0.4.1 Beta$_\perp(beam)$ at a focus



$$\sigma_x(z) \;=\; \sqrt{(\sigma_{xo})^2 \;+\; (\sigma_\theta\ z)^2}$$

since $\sigma_\theta \;=\; \sigma_x(0)/\beta_\perp(o)$

$$\sigma_x(z) = \sigma_x(o)\sqrt{1\;+\;\left(\frac{z}{\beta_\perp}\right)^2}$$

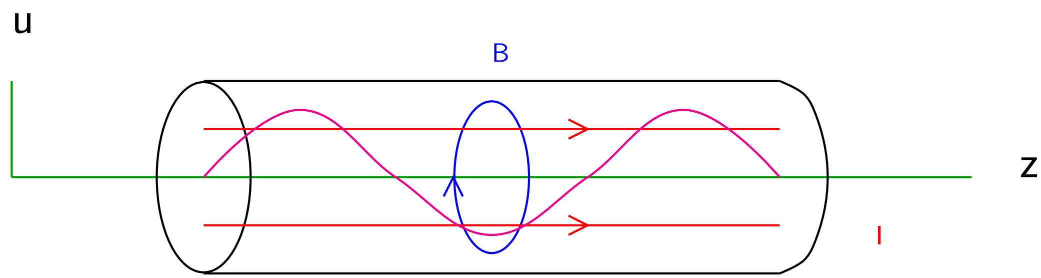
$$\text{and since} \qquad \sigma_x \;=\; \sqrt{\frac{\beta_\perp\ \epsilon_\perp}{\beta_v\gamma}} \;\propto\; \sqrt{\beta_\perp}$$

$$\beta_\perp(z) \;=\; \beta_\perp(o)\ \left(1\;+\;\left(\frac{z}{\beta_\perp(o)}\right)^2\right) \tag{10}$$

0.4.2 Beta$_\perp$(*lattice*) as introduced by Courant & Schneider

$\beta_\perp$(beam) above was defined by the beam, but a lattice can have a $\beta_\perp$(lattice) that may or may not "match" a beam.

e,g. if continuous inward focusing force, as in a current carrying lithium cylinder (lithium lens), then there is a PERIODIC solution:



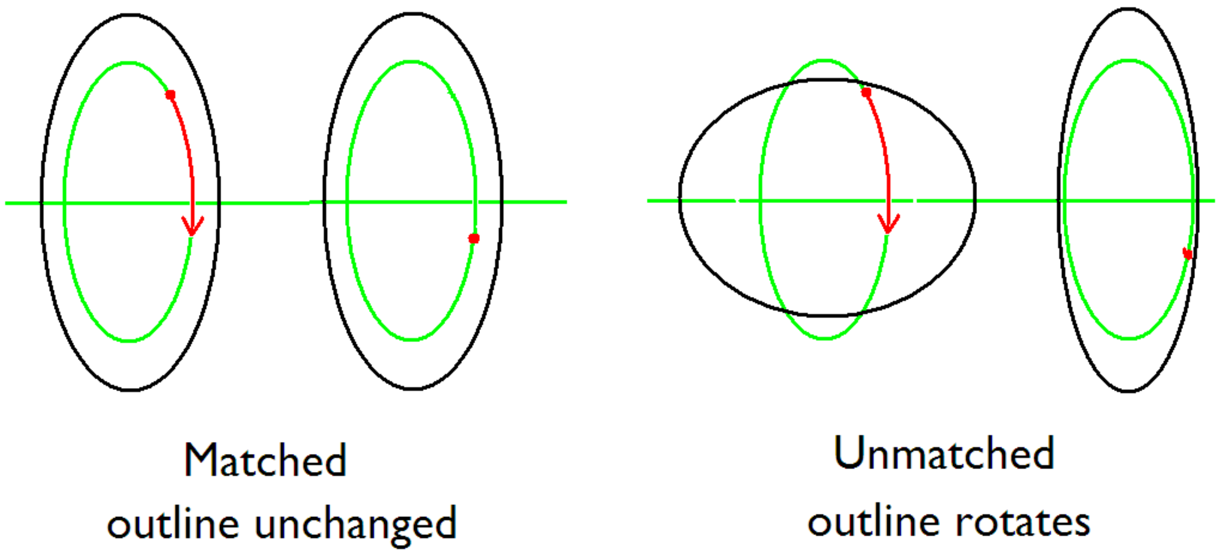
$$\frac{d^2u}{dz^2} = -k\ u \qquad u = A\ \sin\left(\frac{z}{\beta_\perp(\mathrm{lattice})}\right) \qquad u' = \frac{A}{\beta_\perp(\mathrm{lattice})}\ \cos\left(\frac{z}{\beta_\perp(\mathrm{lattice})}\right)$$

where $\quad \beta_\perp(\mathrm{lattice}) = 1/\sqrt{\mathrm{k}} \quad$ and $\quad \lambda = 2\pi\ \beta_\perp(\mathrm{lattice})$

This particle motion is also an ellipse and

$$\frac{\mathrm{width}}{\mathrm{height}}\ \text{of elliptical motion in phase space} = \frac{\hat{u}}{\hat{u'}} = \beta_\perp(\mathrm{lattice})$$

Matching



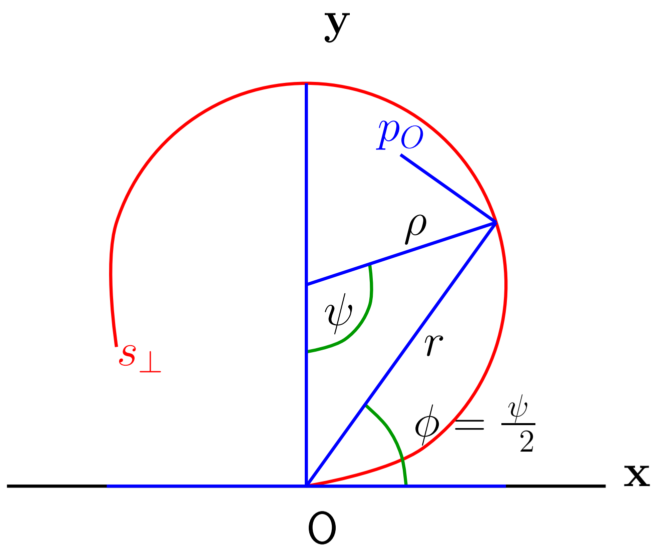
If we have many particles with$\beta_\perp(\mathrm{beam}) = \beta_\perp(\mathrm{lattice})$ then all particles move arround the ellipse, and the shape, and thus $\beta_\perp(\mathrm{beam})$ remains constant, and the beam is "matched" to this lattice.

If the beam's $\beta_\perp(\mathrm{beam}) \neq \beta_\perp$(lattice) of the system then $\beta_\perp(\mathrm{beam})$ of the beam oscillates about $\beta_\perp(\mathrm{Lattice})$: often refered to as a "beta beat".

1) SOLENOID FOCUSING

0.4.3 Motion in Long Solenoid

Consider motion in a fixed axial filed B_z, starting on the axis O with finite transverse momentum $p_\perp$ i.e. with initial angular momentum=0.



$$\rho = \frac{p_\perp}{ec\ B_z} \tag{11}$$

The projected distance along the circumference:

$$s_\perp = \psi\rho = 2\phi\rho$$

The corresponding distance along the axis:

$$z = \frac{p_\parallel}{p_\perp} s_\perp = \frac{p_\parallel}{p_\perp} 2\phi\rho$$

$$\text{so} \quad \phi = \frac{z}{2\rho}\left(\frac{p_\perp}{p_\parallel}\right) = \frac{z}{2}\left(\frac{ecB_z}{2p_\perp}\right)\left(\frac{p_\perp}{p_\parallel}\right) = \frac{z}{2}\left(\frac{ecB_z}{p_\parallel}\right)$$

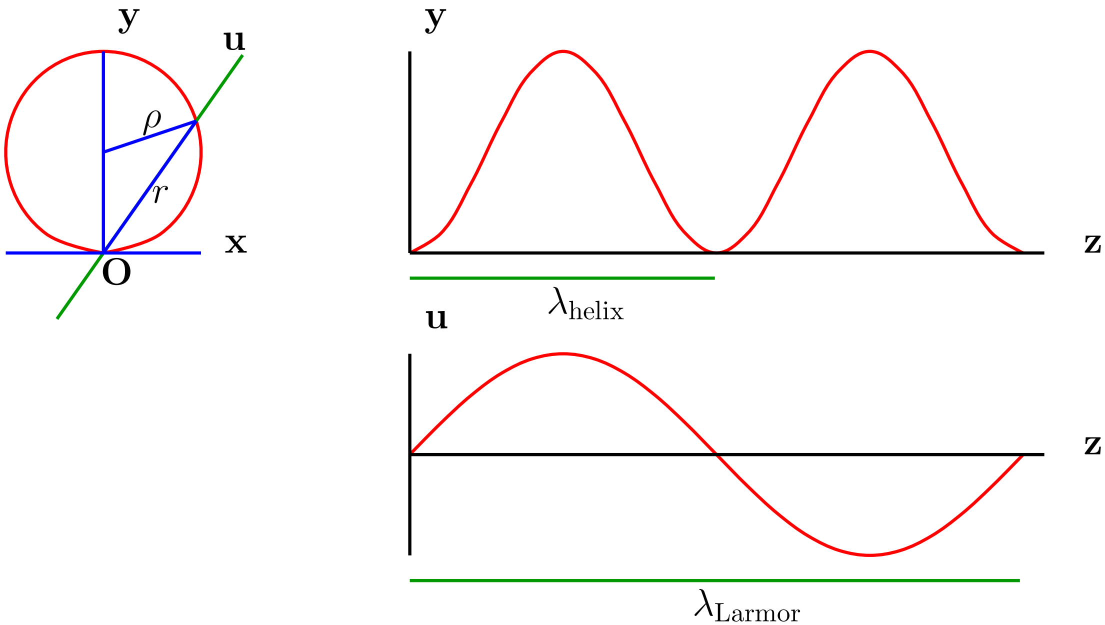
$$\text{For} \quad \psi < 180^o \quad \phi < 90^o : \qquad u = 2\rho\ \sin\left(\frac{\psi}{2}\right) = 2\rho\sin(\phi)$$

$$\text{so} \qquad u = 2\rho\ \sin\left(\frac{z}{\beta_\perp(\text{lattice})}\right) \qquad \text{where} \quad \beta_\perp(\text{lattice}) = \frac{2p_\parallel}{ecB_z} \tag{12}$$

$$\text{i.e. sinusoidal motion about the axis with} \qquad \lambda = 2\pi\beta_\perp(\text{lattice}) \tag{13}$$

0.4.4 Larmor Plane

If The center of the solenoid magnet is at O, then consider a plane that contains this axis and the particle. This, for a particle with initially no angular momentum, is the 'Larmor Plane:



$$y \;=\; 1 \;-\; \cos(\psi)$$

$$u \;=\; \sin(\psi) \tag{14}$$

$$\phi \;=\; \frac{\psi}{2} \qquad \text{so} \qquad \lambda(\text{helix}) \;=\; \frac{\lambda(\text{lattice})}{2} \qquad\qquad \beta_{\perp}(\text{helix}) \;=\; \frac{\beta(\text{lattice})}{2}$$

0.4.5 Focusing Pseudo Force

In this constant B case,

$$u = 2\rho \sin\left(\frac{z}{\beta_\perp(\text{lattice})}\right) \qquad \frac{d^2u}{dz^2} = -\left(\frac{2\rho}{\beta_\perp^2}\right) u$$

$$\text{so} \qquad \frac{d^2u}{dz^2} = -\left(\frac{ecB_z}{2\;p_z}\right)^2 u \tag{15}$$

i.e. The motion in the larmor plane corresponds to the motion of a particle with an inward force proportional to the distance u from the axis. i.e. it is an ideal focusing force.

This result can also be obtained by noting that the momentum p_O perpendicular to the radius r is from equation 12 and 11

$$p_O = p_\perp \sin(\phi) = p_\perp \frac{u}{2\rho} = \frac{uecB_z}{2} \tag{16}$$

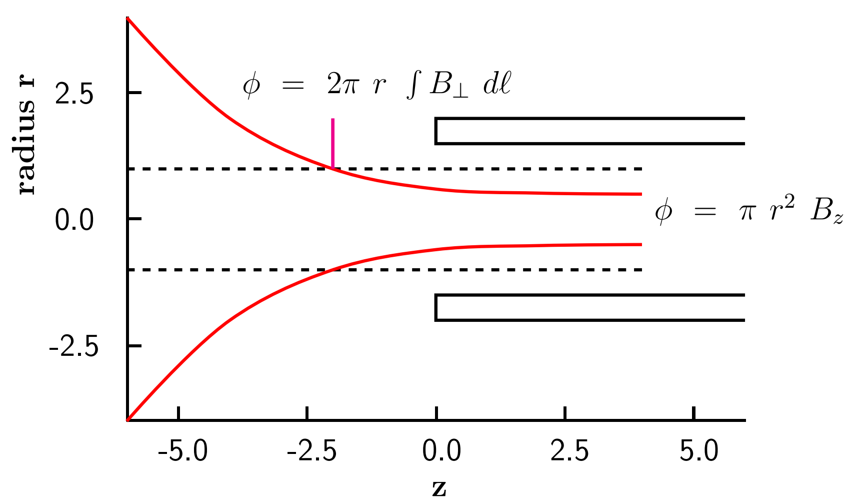
which acting on B_z gives an inward curvature. Though derived here for a constant B_z Equations 15 and 16 are true also for any axi-symmetric fields, constant or varying.

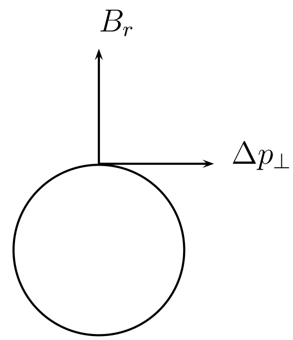
Note: the focusing "Force" $\propto (e\;B_z)^2$ so it works the same for either sign of B, either charge and $\propto 1/p_z^2$. Whereas in a quadrupole the force $\propto 1/p$ So solenoids are not good for high p, but beat quads at low p.

Note however that it depends on p_z, not p. For large amplitude motion $p_z < p$ and there is spherical aberration: large amplitudes focused more strongly than smaller amplitudes.

0.4.6 Entering a solenoid from outside

We will now look at a simple non-uniform B_z case. Let a particle start from the axis with finite transverse momentum, but no angular momentum. After some distance with no field, it reaches a radius u and then enters a solenoid with B_z. As it enters the solenoid it crosses radial field lines and receives some angular momentum.





$$\int B_r\ dz \;=\; \frac{r\ B_z}{2}$$

$$\Delta p_\perp \;=\; c\ \int B_r\ dz \;=\; \frac{B_z\ r\ c}{2} \tag{17}$$

Sof for our case with zero initial transverse momentum,

$$p_\perp \;=\; c\ \int B_r\ dz \;=\; \frac{B_z\ r\ c}{2}$$

Which is the same as eq.16, and will lead to the same inward bending (eq.15), as when the particle started inside the field. In fact equations 15 and 17 are true no matter how the axial field varies

0.4.7 Canonical Angular momentum

In vacuum with axial symmetry, a particle will have a conserved "Canonical Angular Momentum" $\mathcal{M}_o$ equal to the angular momentum outside the axial fields.

$$\mathcal{M}_o \ = \ p_\perp \ r \ \text{(Outside the field)}$$

Inside a varying field $B_z(z)$, the real angular momentum will, from eq. 17, be:

$$\mathcal{M} \ = \ \mathcal{M}_o \ + \ \frac{r^2 \ B_z \ c}{2}$$

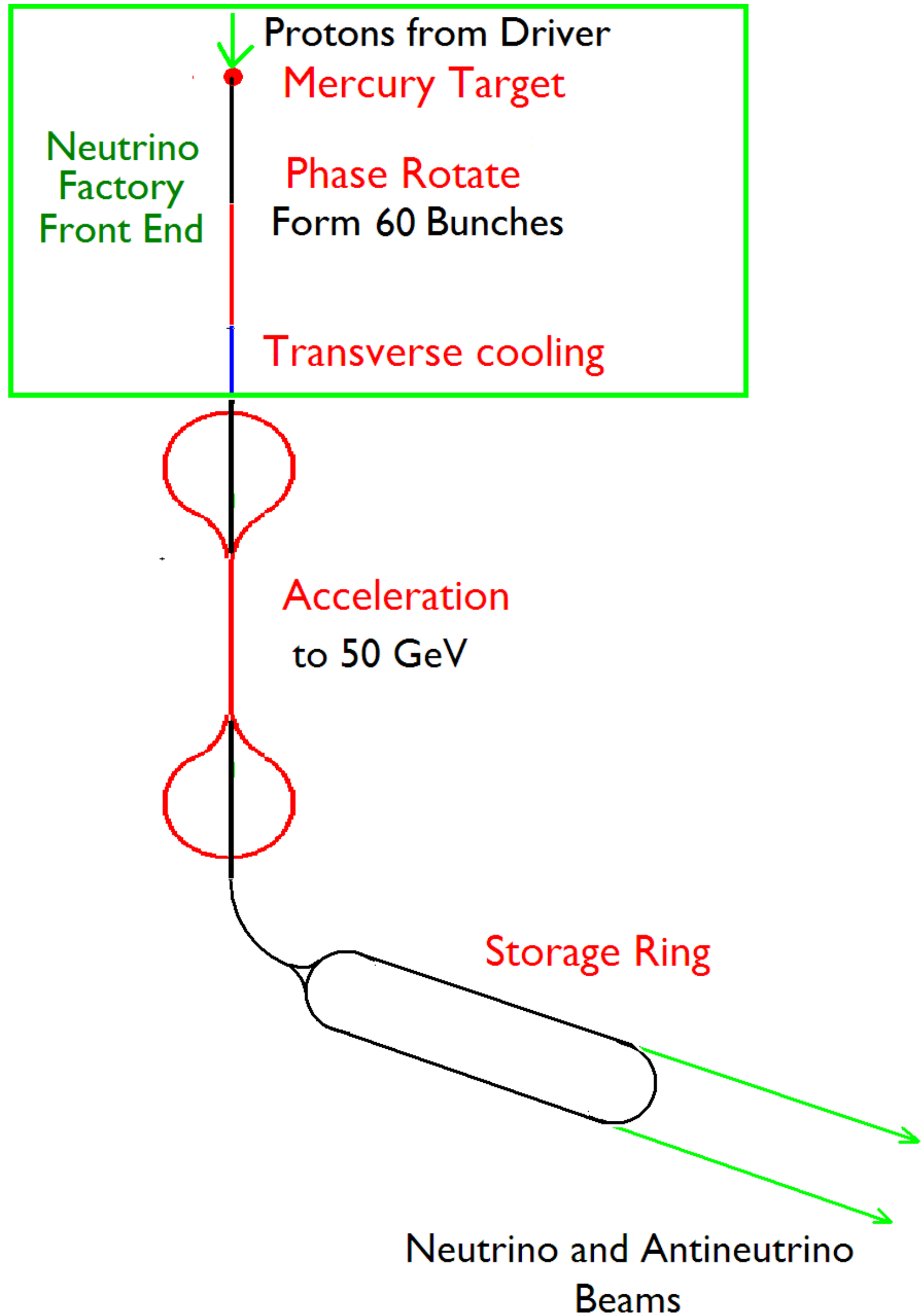
But in the rotating Larmor Frame the angular momentum is always just the Canonical angular momentum, and motion in that frame has only inward focusing forces, with no angular kicks.

2) NEUTRINO FACTORY INTRODUCTION

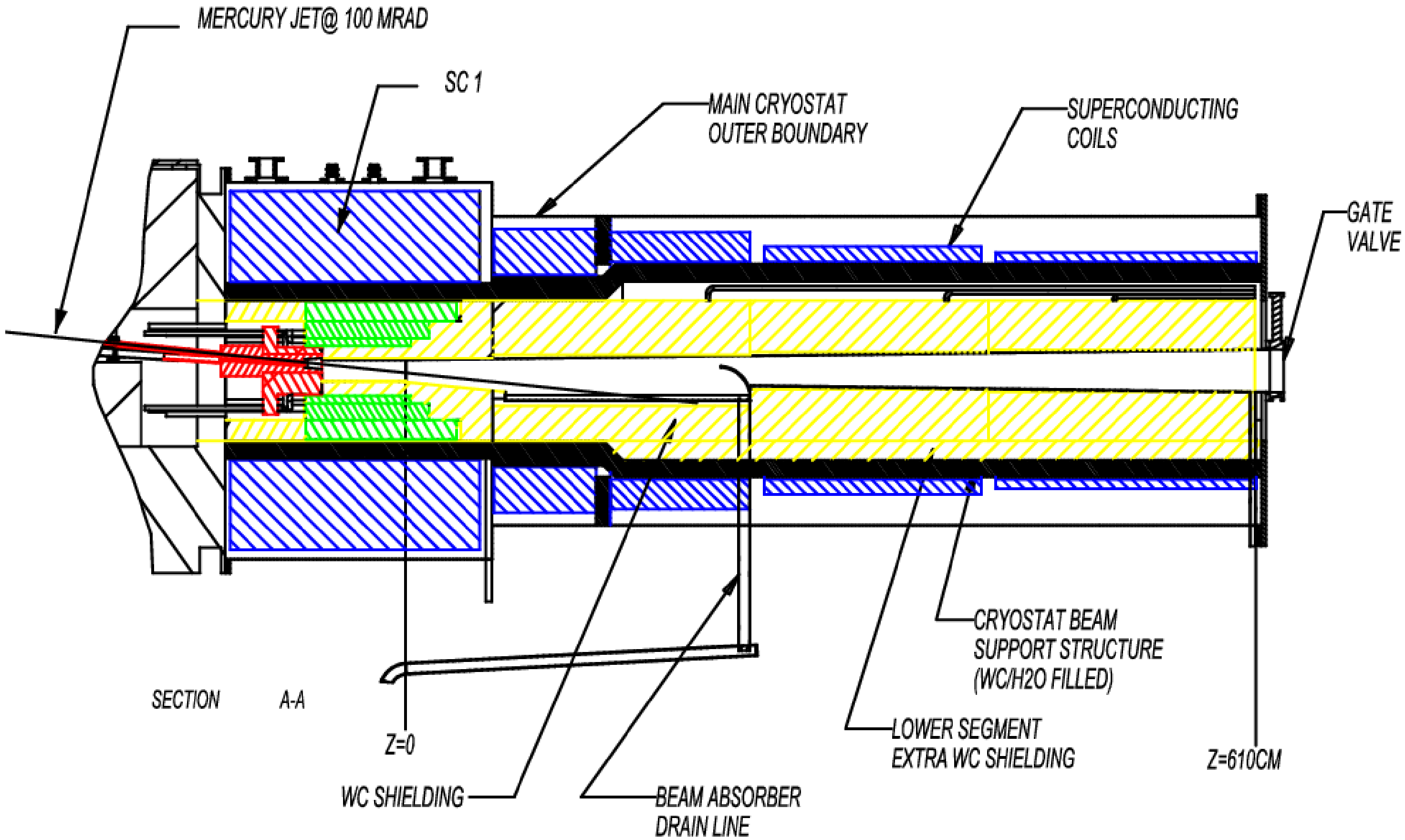
- Requires maximum number of muons
- Can use large acceptance acceleration and storage
- Cooling only if it increases numbers of muons accepted
- Japanese study had used no cooling
- Study 2a and ISS (International Scoping Study) gained 1.7 by cooling
- Neither used longitudinal cooling in part because:
  - phase rotating large dp/p into long or multiple bunches with small dp/p ok

Neutrino Factory Schematic

- Not to scale
  Overall length of order 1 km
- Acceleration in multiple stages not indicated
- μ^+ and μ^- coexist in capture, rotate, & cooling
- They go in opposite directions in "dog bone" RLA
- and in the Storage Ring



#1 Target and Capture and Phase Rotate

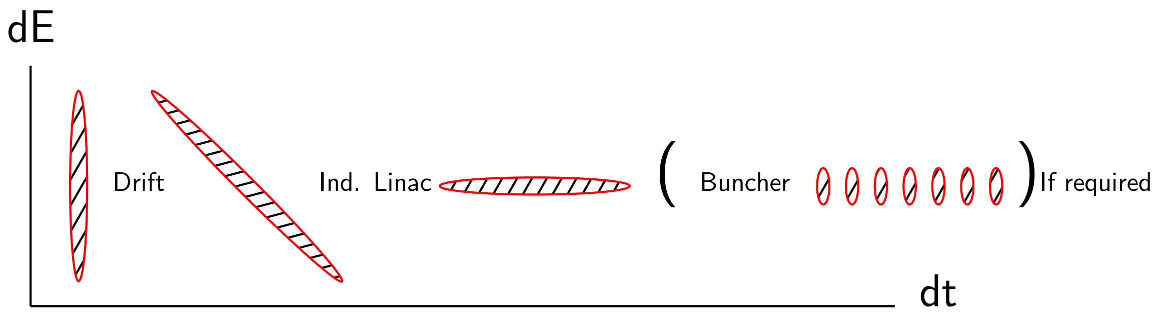


- Liquid mercury Jet 'destroyed' on every pulse
- 20 T Solenoid captures almost all low momentum pions
- Field subsequently tapers down to approx 2 T
- Target tilted to maximize extraction of pions
- MERIT Experiment at CERN will test this concept

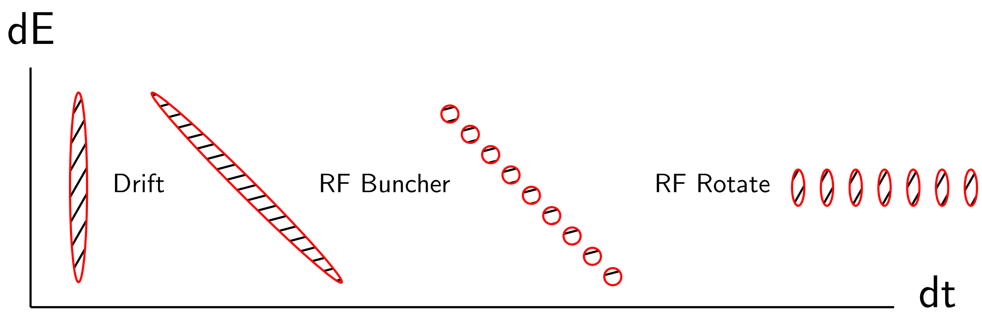
Phase Rotation Schemes

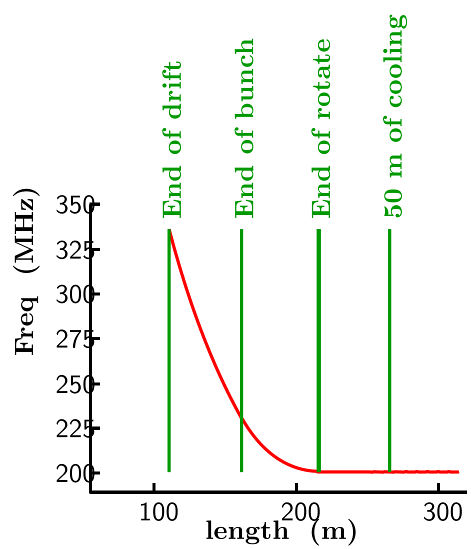
To reduce momentum spread at expence of multiple bunches

Conventional with LF RF or Induction Linacs (Studies 1&2)



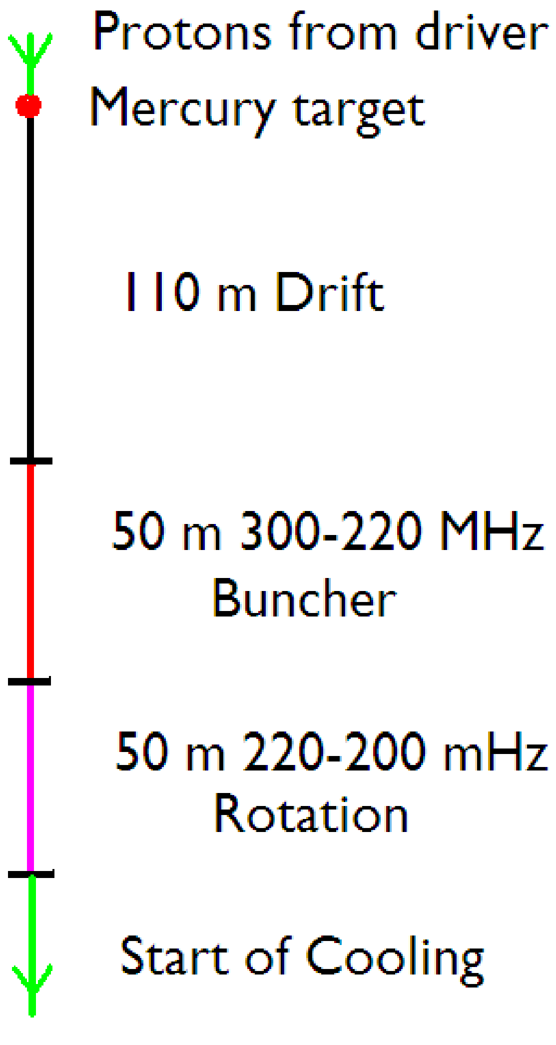
Neuffer Bunched Beam Rotation with 200 MHz RF (Study 2a, ISS)

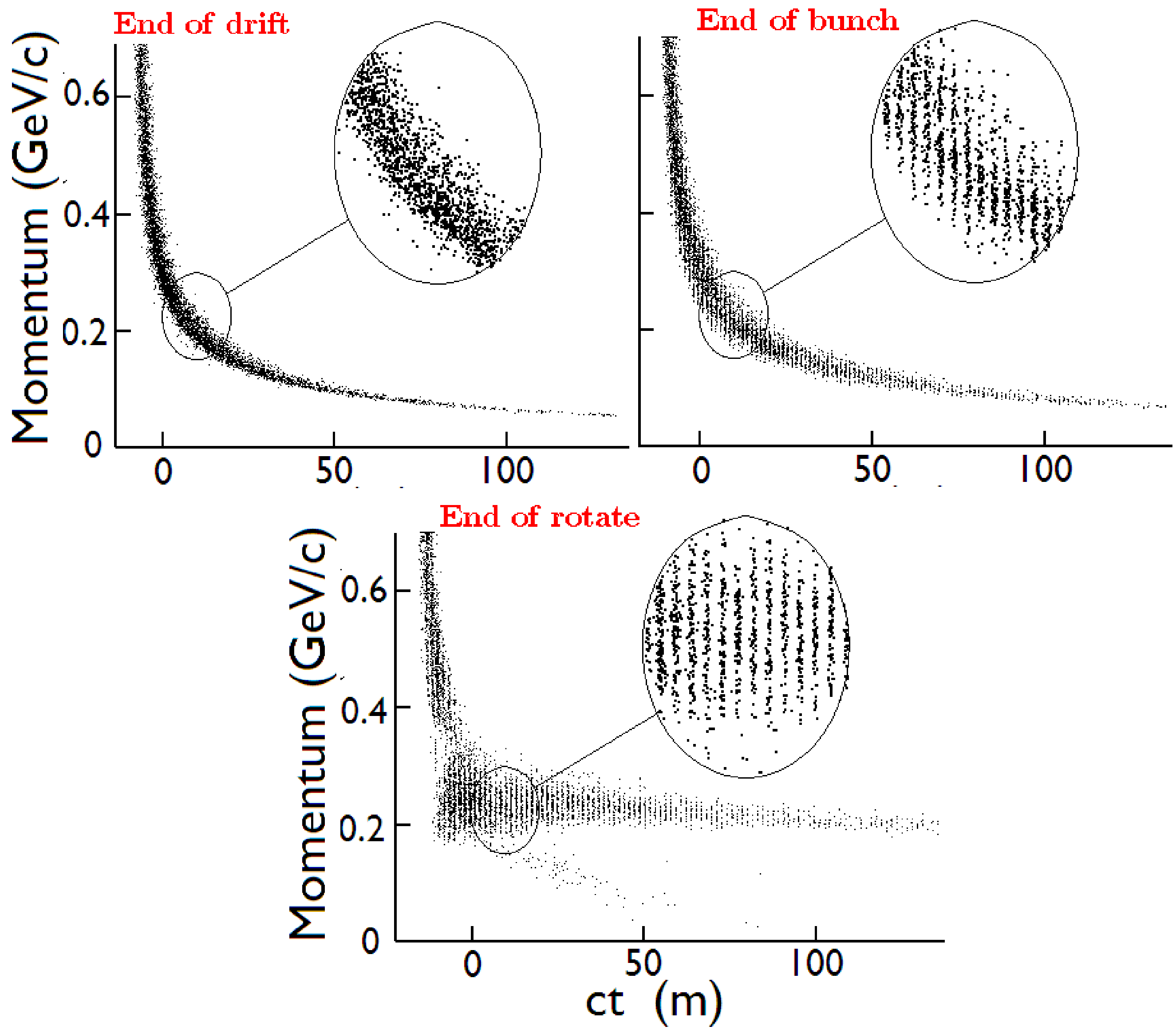




- RF frequency must vary along bunching channel (high mom. bunches move faster than low)
- Higher freq RF is cheaper than Induction Linacs
- Bunched Beam method captures both signs in interleaved bunches

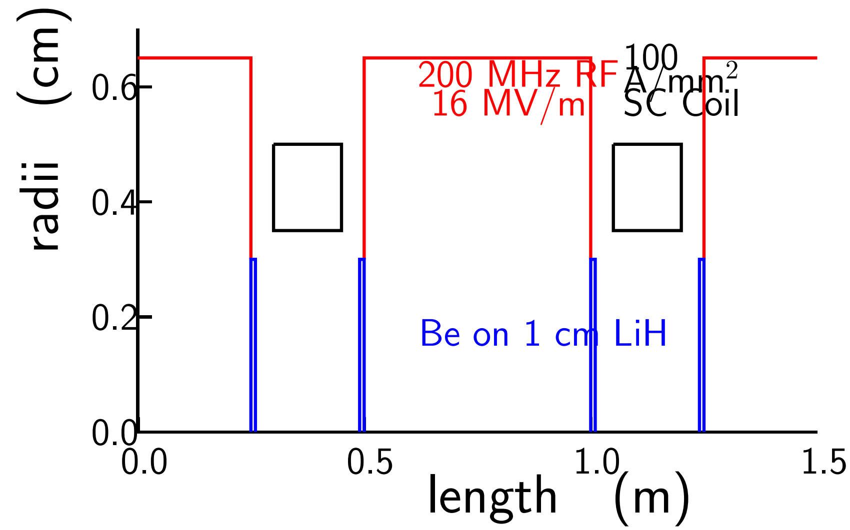
Phase Rotation Simulation

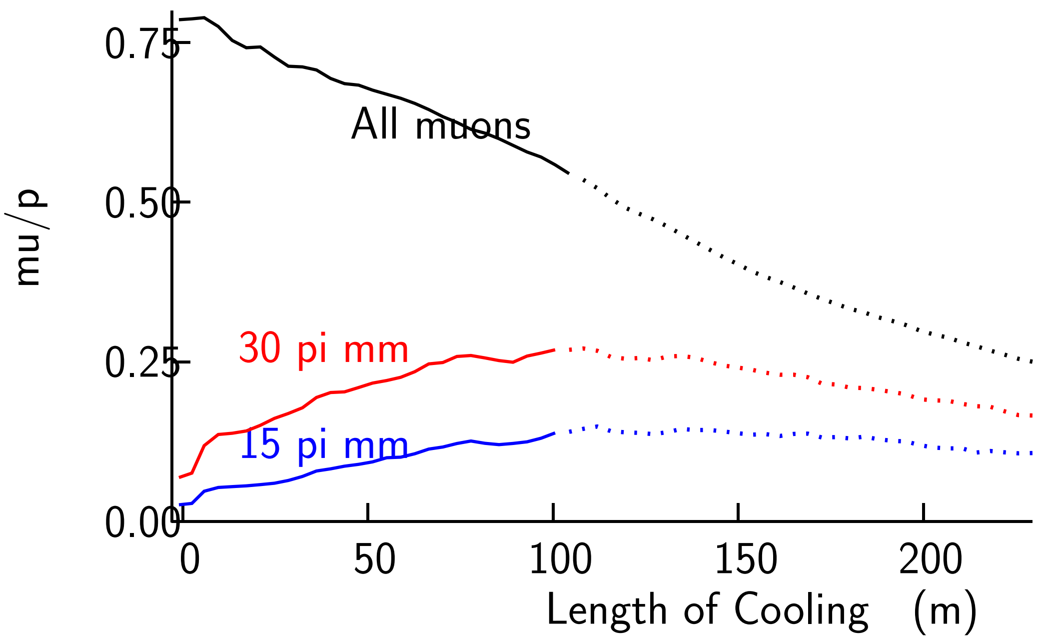


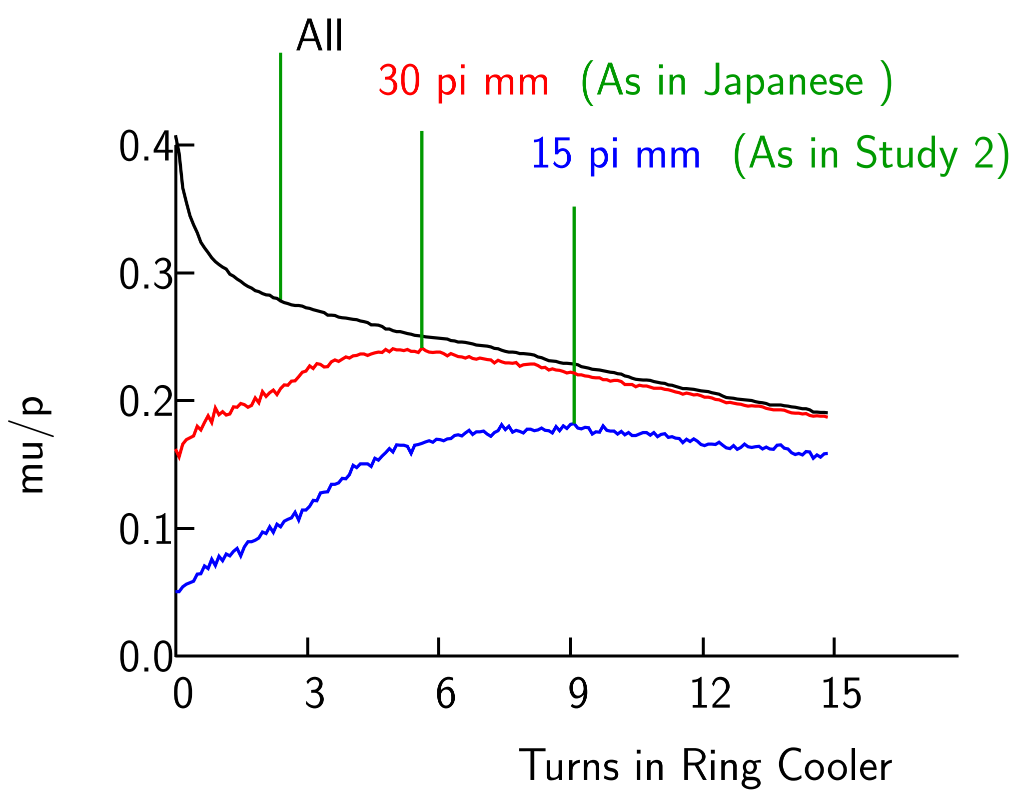


Cooling

- With 30 pi mm acceptance (S2a & ISS), only moderate cooling
- Cooling in FOFO lattice
- Use LiH rather than Hydrogen
- More cooling needed if 15 pi mm (S1 & s2)







- No advantage in longer cooling

No advantage in 6D ring cooling

3) MUON COLLIDER INTRODUCTION

- Requires strong transverse cooling because
  - To get very small emittances
    $\mathcal{L} \propto 1/\sigma_{\rm r}^2 \propto 1/\epsilon_\perp$
- Requires moderate longitudinal cooling because
  - Short bunch
    because $\mathcal{L} \propto 1/\sigma_{\rm r} \propto 1/\beta_\perp$ and req $\sigma_z \leq \beta_\perp$
  - Small dp/p
    because Collider ring with small $\beta_\perp$ is difficult
- Requires few intense muon bunches because
  - Many muons per bunch
    $\mathcal{L} \propto {\rm N}_\mu^2$
  - Relatively few total muons
    because of neutrino radiation

- Very different from Neutrino Factory,
- But they can both use same front end

Luminosity Dependence

A little more compicated than above due to beam-beam tune shift limitations

$$\mathcal{L} \;\propto\; n_{\rm turns}\, f_{\rm bunch}\, \frac{N_\mu^2}{\sigma_\perp^2} \qquad\qquad \Delta\nu \propto \frac{N_\mu}{\epsilon_\perp}$$

$$\mathcal{L} \;\propto\; B_{\rm ring}\, P_{\rm beam}\;\; \Delta\nu\, \frac{1}{\beta^*}$$

- Higher $\mathcal{L}/P_{\rm beam}$ requires lower $\beta_\perp$ or correction of $\Delta\nu$
- Lower emittances do not directly improve Luminosity/Power
- But for maximum plausible N_μ emittances are small

Collider Parameters

	Current	Snowmass	Extrapolation	
C of m Energy	1.5	4	8	TeV
Luminosity	1	4	8	10^{34} cm^{2}sec^{-1}
Beam-beam Tune Shift	0.1	0.1	0.1	
Muons/bunch	2	2	2	10^{12}
Ring <bending field>	5.2	5.18	10.36	T
Ring circumference	3	8.1	8.1	km
Beta at IP $= \sigma_z$	10	3	3	mm
rms momentum spread	0.09	0.12	0.06	%
Muon Beam Power	7.5	9	9	MW
Required depth for ν rad	≈135	135	540	m
Efficiency $N_\mu/N_{\mu o}$	0.07	0.07	0.07	
Repetition Rate	12	6	3	Hz
Proton Driver power	≈4	≈ 1.8	≈ 0.8	MW
Trans Emittance	25	25	25	pi mm mrad
Long Emittance	72,000	72,000	72,000	pi mm mrad

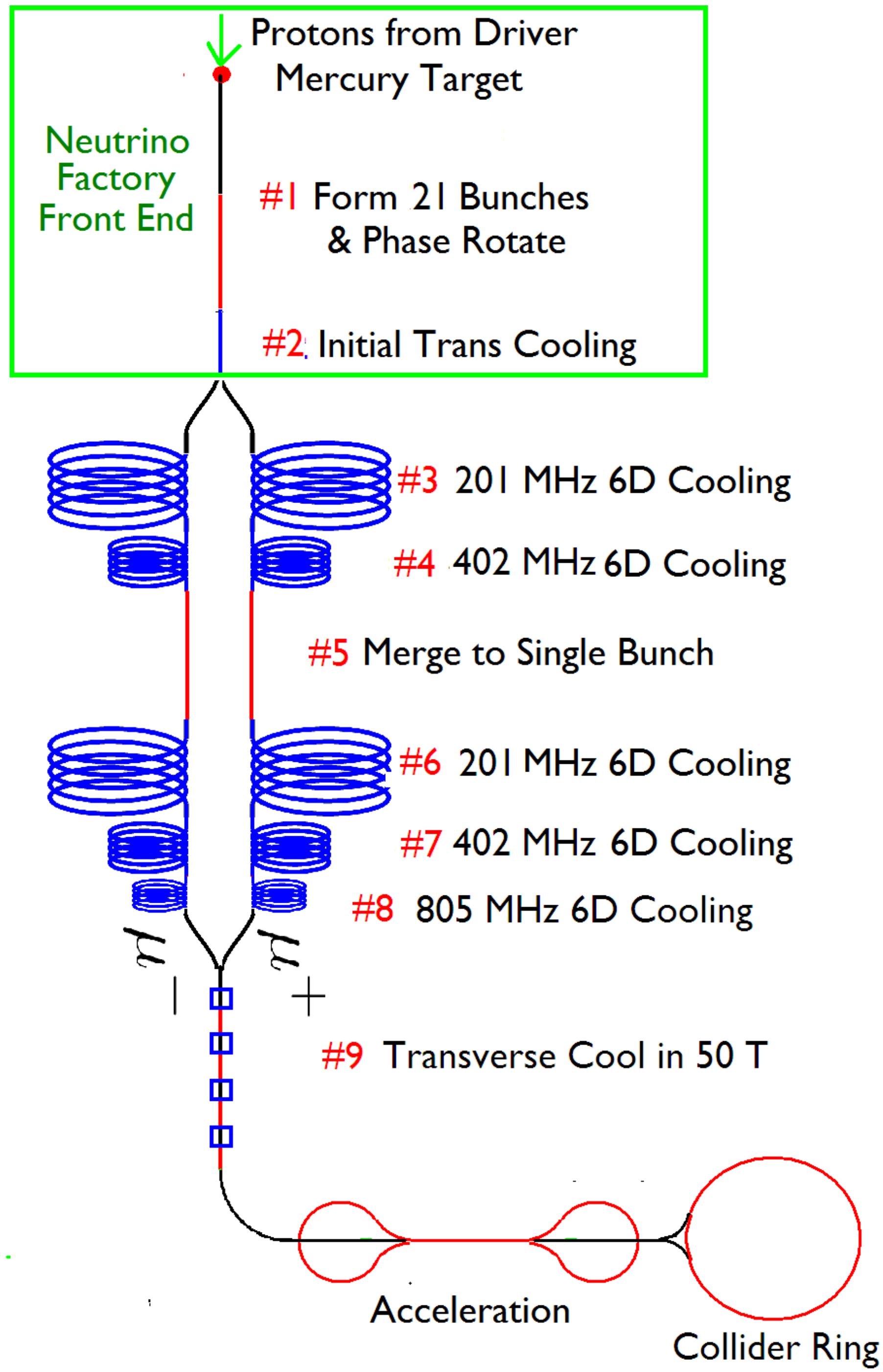
- Emittance and bunch intensity requirement same for all examples
  Because beam-beam tune shift is independent of energy

Capture & Cooling Schematic

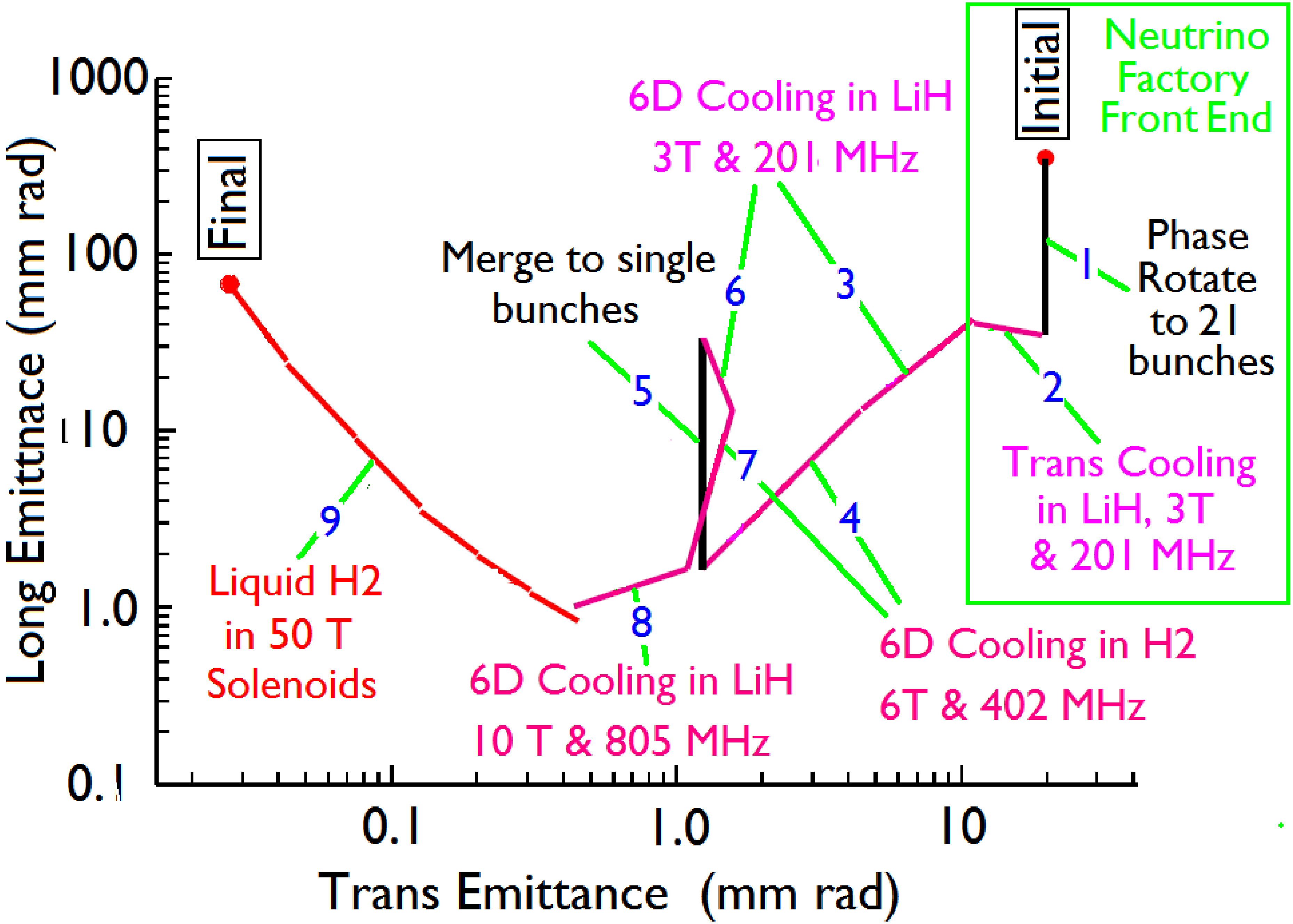
- Not to scale
  Overall length of order 2 km

- Acceleration in multiple stages not indicated

- We will look at numbered components later

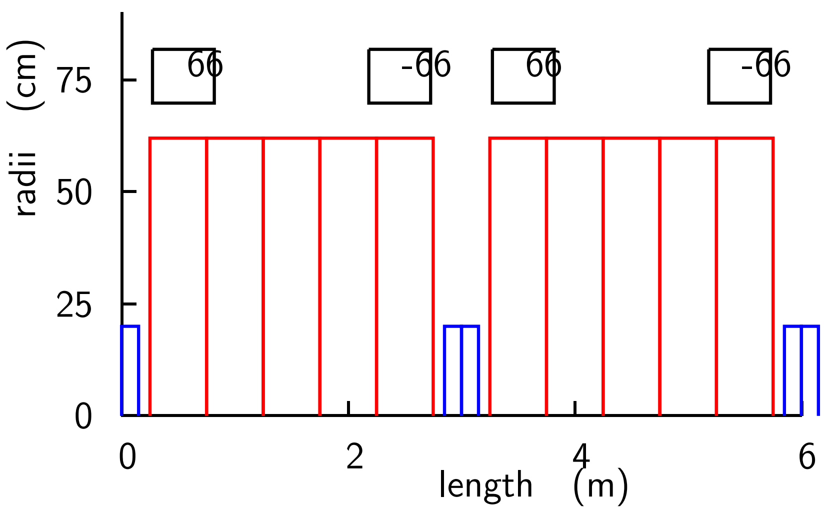


Emittances vs. Stage

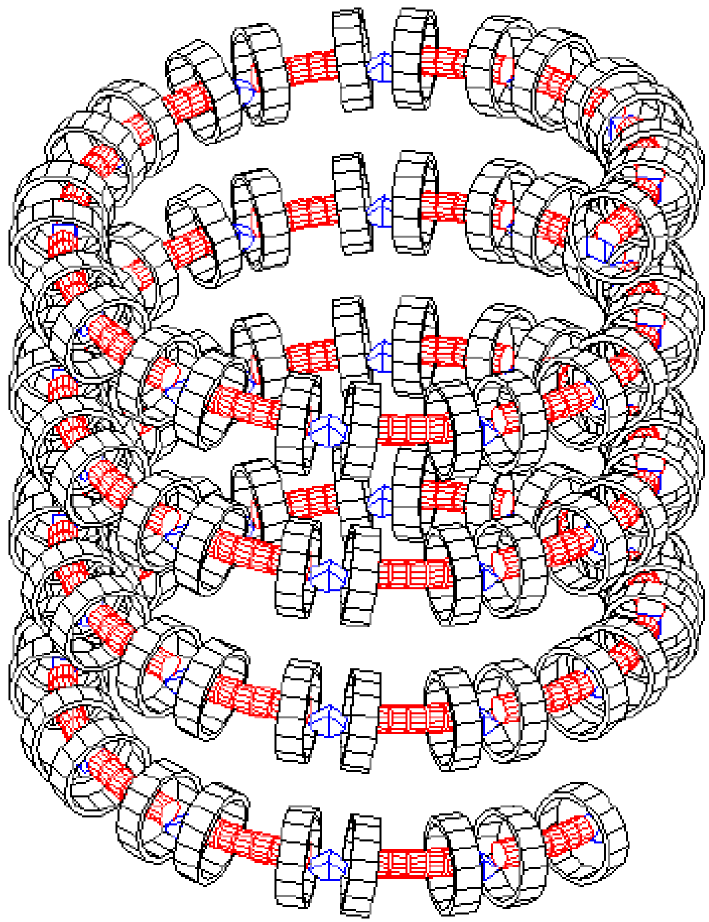


s 3 4 6 7 8: 6D Cooling in "Guggenheim" helices

- RFOFO lattices
- Bending gives dispersion
- Wedge absorbers give emittance exchange → Cooling also in longitudinal
- Use as 'Guggenheim' helix
  - Because bunch train fills ring
  - Avoids difficult kickers
  - Better performance possible by tapering (Not yet assumed)



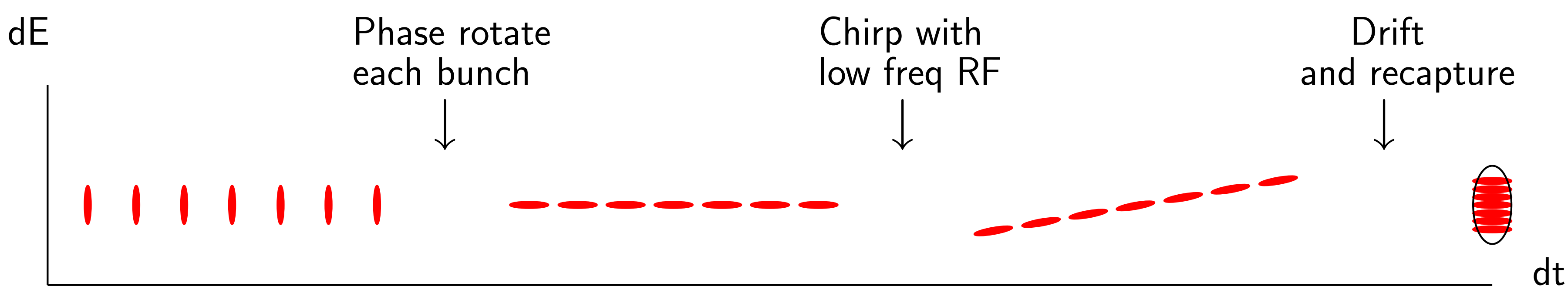
RFOFO Lattice


'Guggenheim'

#5 Bunch Merging

- Luminosity proportional to muons per bunch squared
- Few large bunches required
- Capturing to one large bunch would have required low frequency rf ($\approx$ 30 MHz) with low gradients and inefficiency
- We thus:
  - Capture into multiple bunches at 201 MHz
  - Cool them till small enough to:
  - Merge them and recapture at 201 MHz
  - Re-cool the merged bunches

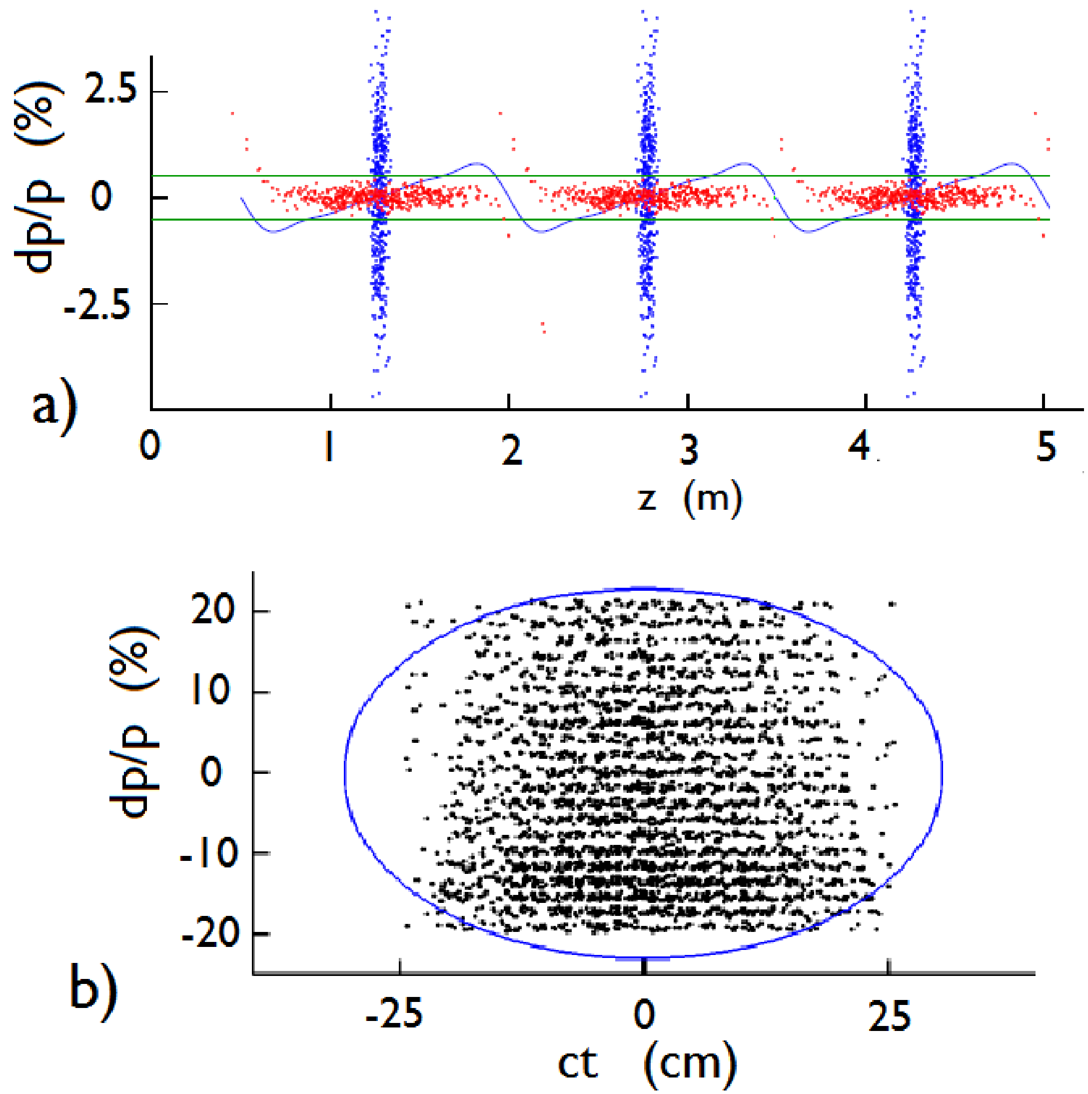
Merging Scheme



One Dimensional Bunch Merging Simulation

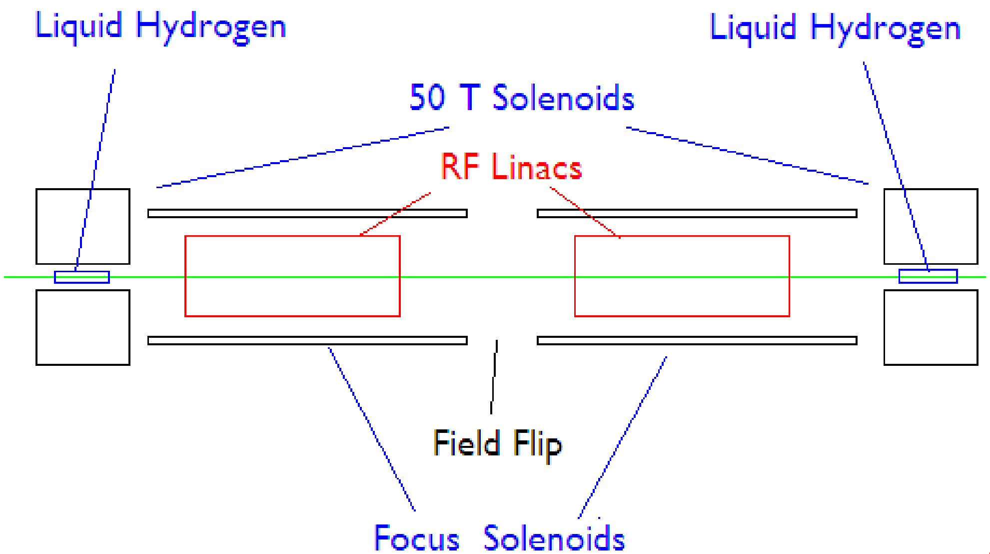
- Drifts in 1 T wigglers,
  simulated in ICOOL
  vs amp and mom

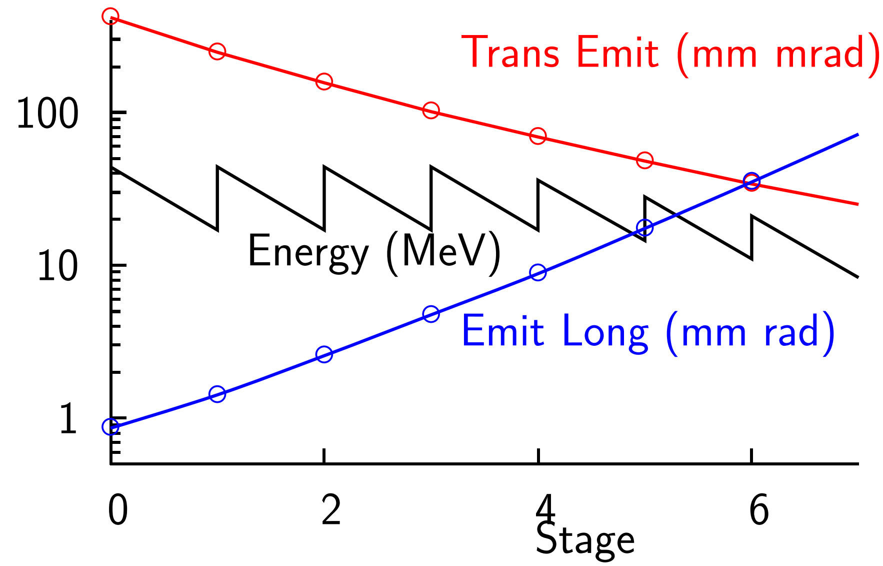
- rf:
  1) at 200 MHz + 2 harmonics
  2) at 5 MHz + 2 harmonics
  Simulated off line
  with parameters from ICOOL



#9 Transverse Cooling in Very High Field Solenoids

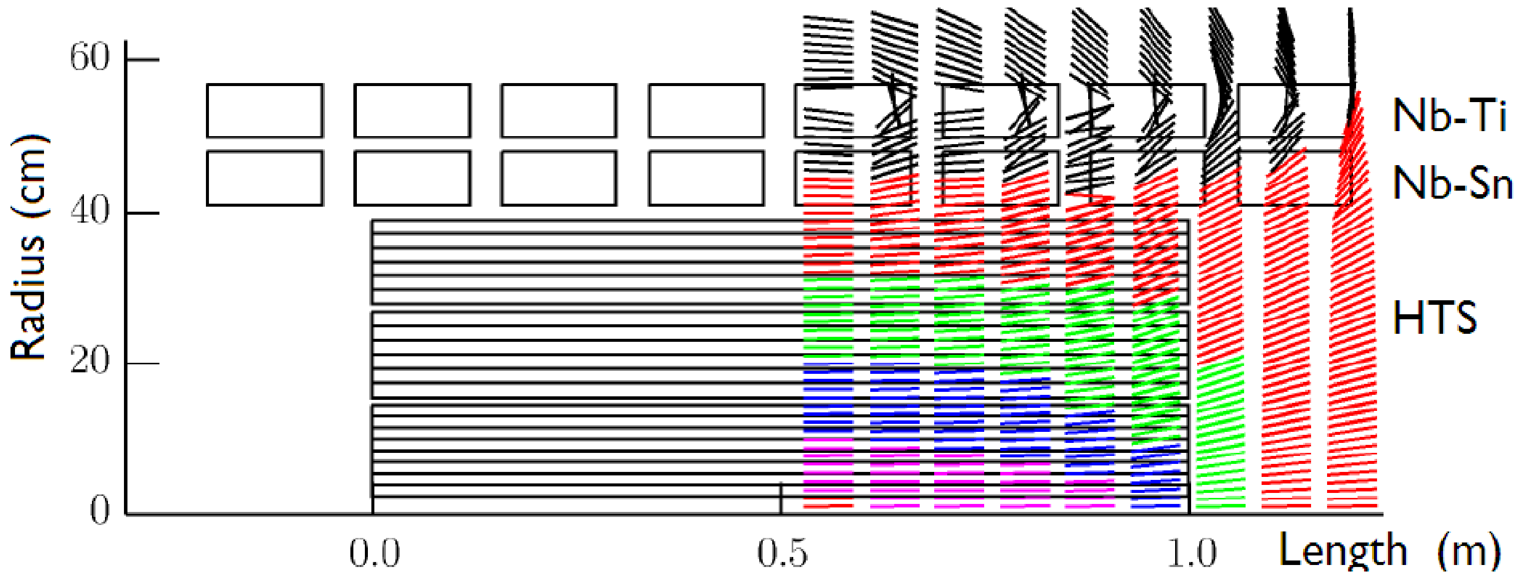
- Lower momenta allow strong transverse cooling, but long emittance rises:
- Effectively reverse emittance exchange





- 50 T HTS Solenoids
  - Layer wound allowing current to vary with radius
  - Vary ss support with radius to keep strain constant
  - Existing HTS tape at 4.2 deg. gave 50 T with rad=57 cm
- 7 solenoids with liquid hydrogen
- ICOOL Simulation (Ideal Matching and reacceleration, Transmission 97%)

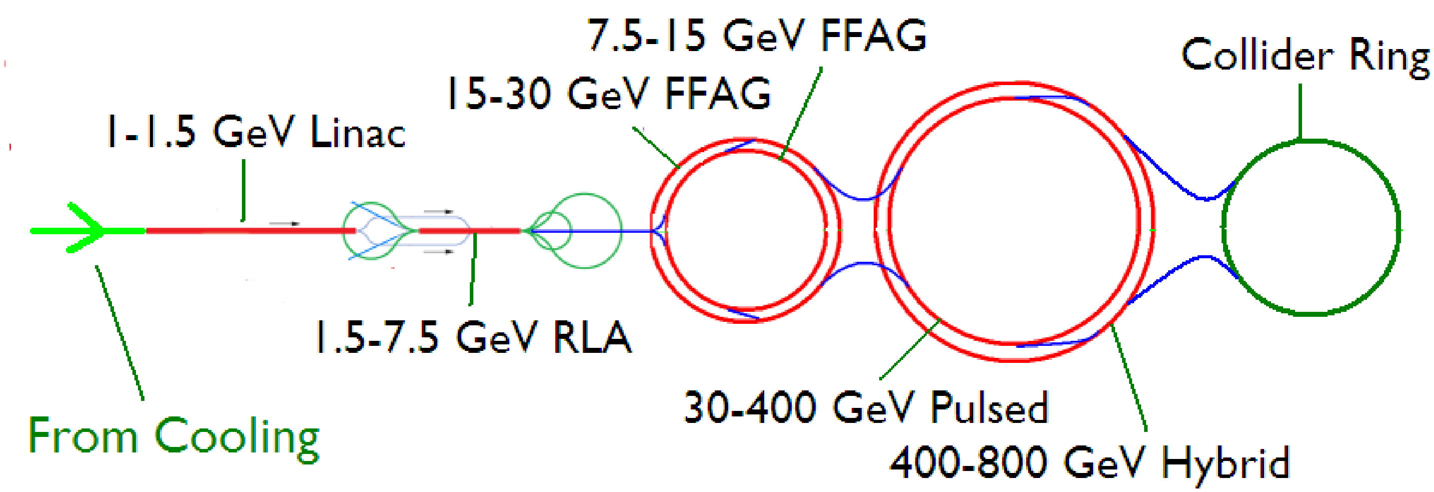
Details of 50 T Solenoid



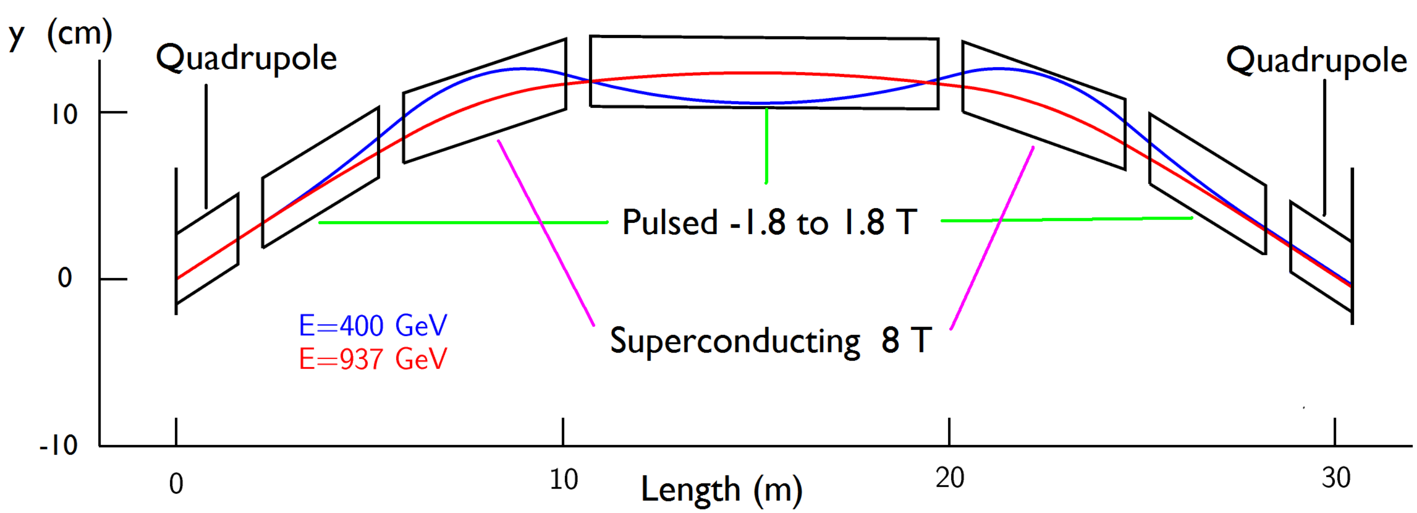
- Design uses BSCCO tape (conductor cost now 2.7 M$, but falling)
- Stored energy 141 MJ (requires multiple local quench protections)
- Questions raised about stress cycling in BSCCO
- YBCO claimed to be much better, but more sensitive to field directions
- Needs characterizations of materials

- Highest field HTS now under construction is only 30 T
- But existing hybrid (Cu and SC) 45 T magnet using 20 MW

Acceleration

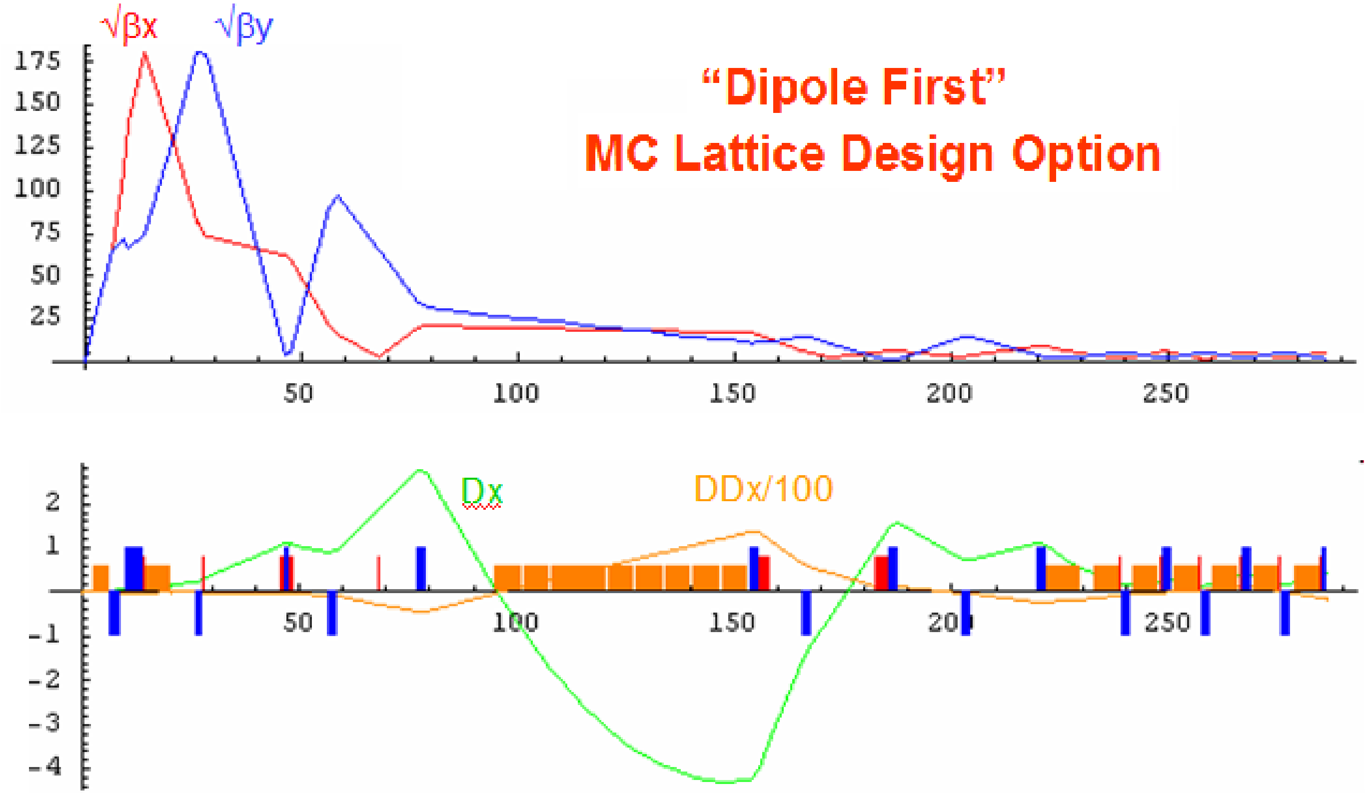


- Hybrid SC and pulsed synchrotron 400-750(930) GeV (in Tevatron tunnel)

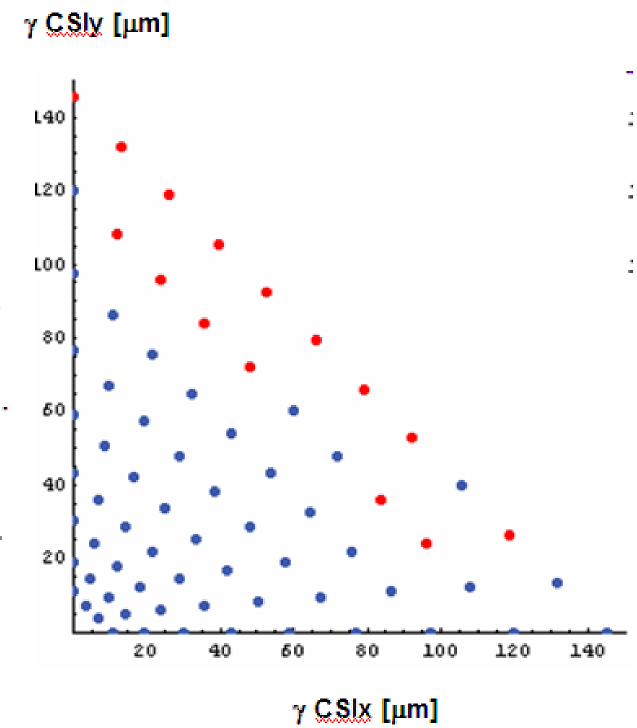


- All RLAs with ILC cavities is an alternative but more expensive

Collider Ring (Y. Alexahin E. Gianfelice-Wendt)



Lattice



Acceptance

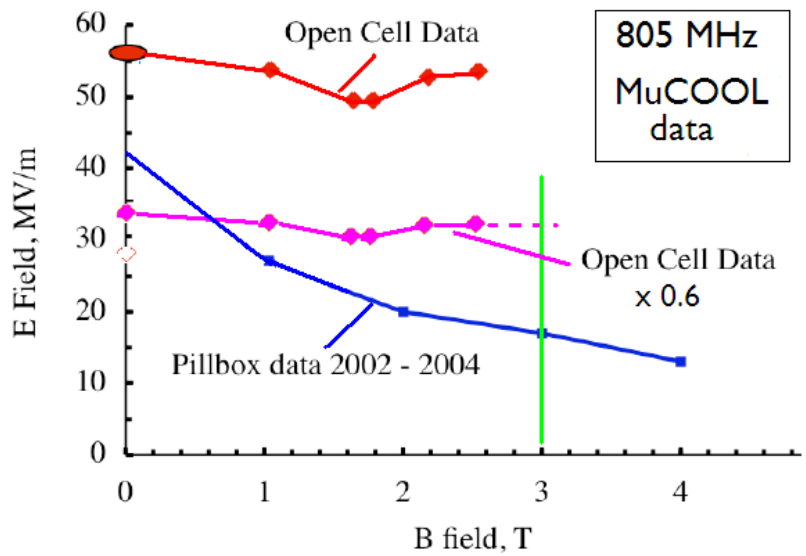
- $\beta^* \;=\; 1\ \mathrm{cm} \quad \Delta p/p \;\approx\; 0.6$ %
  More than adequate for rms dp/p=0.09 %
- $\Delta x, y \;\approx\; 2\sigma$ at 25 mm mrad emittance
  Will require scraping of beam (cut at 1.75 sigma loses only 5% of luminosity)

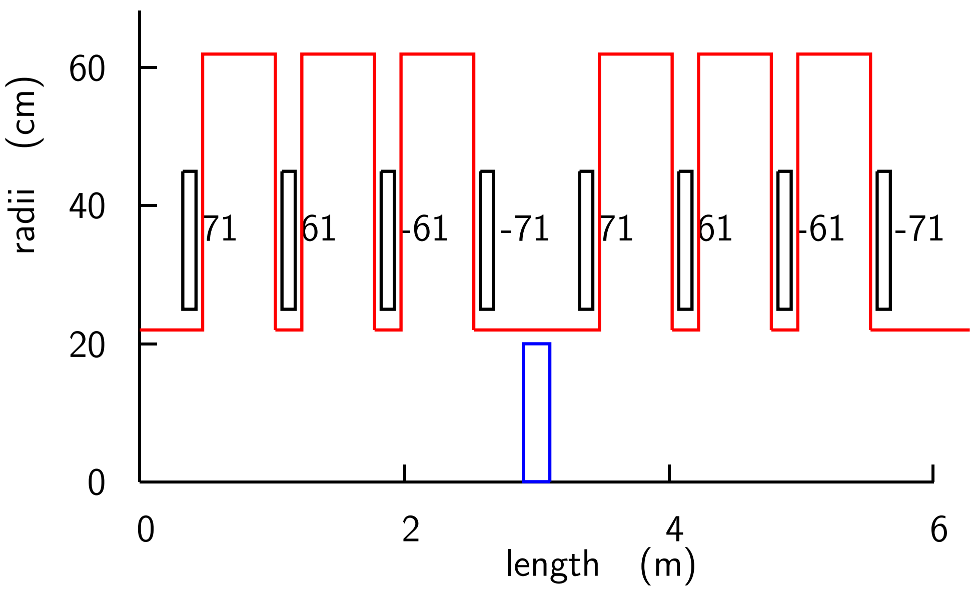
Ongoing Studies

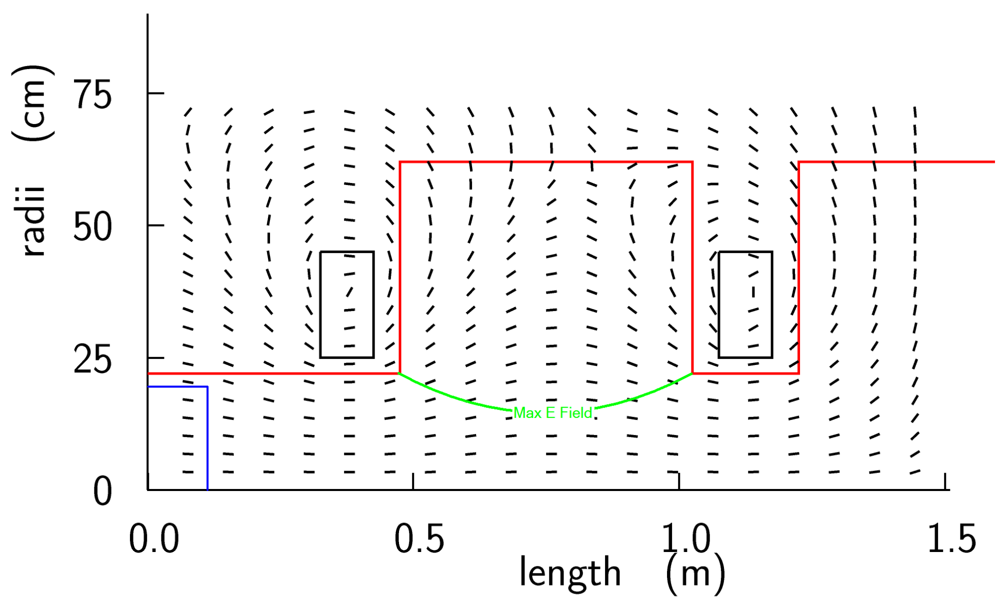
- Fuller simulations
- Space charge tune shifts (moderate, but not in simulations)
- Possible breakdown of vacuum RF in the specified magnetic fields
  - Being studied experimentally by MUCOOL Collaboration
  - Possible solution 1) Gas filled cavities
    works for earlier cooling lattices experiment needed for beam breakdown
  - Possible solution 2) Open Cavities with coils in irises (see next)
    works in simulation experiments needed for breakdown
- Planar wiggler lattice to replace Guggenheims (cools both muon signs)
- Fast Helical cooling in hydrogen gas
  Another alternative to RFOFO Guggenheims being studied by Muons Inc
  but difficult to introduce required rf
- Design of 50 T solenoids
- Use of more, but lower field (e.g. 35 T) final cooling solenoids
- Design detector shielding

Open cell rf with coils in irises

- B field effect on open cavity much less
  average field/surface fields $\approx 1/2$
  but open cavity still better at 3 T
- Should be even better if coils in irises
- Max E field $\perp$ to B



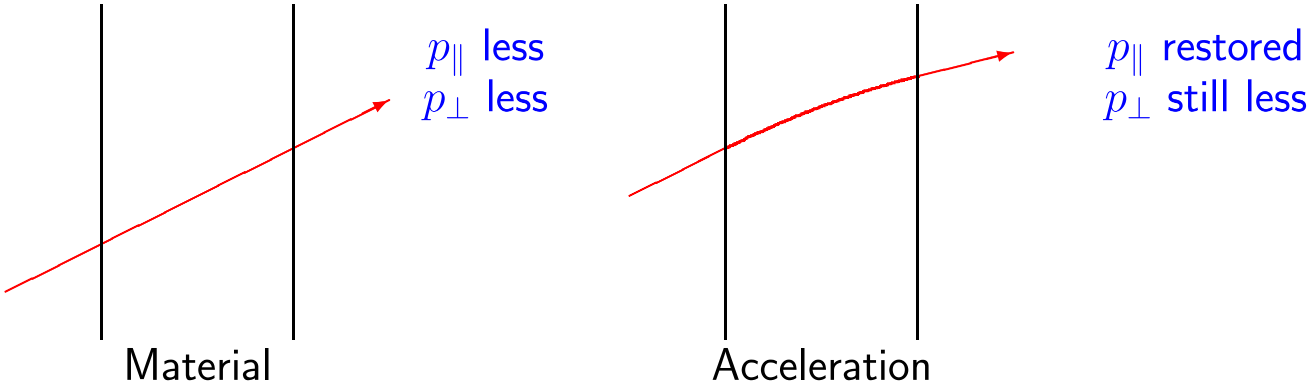




Conclusion

- New 1.5 TeV Collider lattice has more conservative IP parameters
  - Luminosity 1×10^{34} achieved with bunch rep rate ≈12 Hz
  - Collider ring must be deep (eg 135 m of ILC) to control neutrino radiation
  - Proton driver (≈ 4 MW) is challenging
- Complete cooling scheme achieves required muon parameters
  - All components simulated (at some level) with realistic parameters
  - But much work remains
- Possible problem with rf breakdown in specified magnetic fields
  - Solutions with gas ?
  - Open cell rf ?
- Lower cost acceleration possible using pulsed magnets in synchrotons
  - Rings fit in Tevatron tunnel
  - Second ring uses hybrid of fixed and pulsed magnets

1 TRANSVERSE IONIZATION COOLING



1.1 Cooling rate vs. Energy

$$\text{(from eq 4)} \qquad \epsilon_{x,y} = \gamma\beta_v \, \sigma_\theta \, \sigma_\perp$$

If there is no Coulomb scattering, or other sources of emittance heating, then σ_θ and $\sigma_\perp$ are unchanged by energy loss, but p and thus $\beta\gamma$ are reduced. So the fractional cooling $d\epsilon \, / \epsilon$ is (using eq.2):

$$\frac{d\epsilon}{\epsilon} = \frac{dp}{p} = \frac{dE}{E} \frac{1}{\beta_v^2} \tag{18}$$

which, for a given energy change, strongly favors cooling at low energy.

1.2 Heating Terms

$$\epsilon_\perp \;=\; \gamma\beta_v \, \sigma_\theta \, \sigma_\perp$$

Between scatters the drift conserves emittance (Liouiville).
When there is scattering, $\sigma_\perp$ is conserved, but σ_θ is increased.

$$\Delta(\epsilon_\perp)^2 \;=\; \gamma^2\beta_v^2 \, \sigma_\perp^2 \Delta(\sigma_\theta^2)$$

differentiating the LHS and substituting for $\sigma_\perp$ from eq. 8

$$2\epsilon_\perp \, \Delta\epsilon_\perp \;=\; \gamma^2\beta_v^2 \left(\frac{\epsilon_\perp\beta_\perp}{\gamma\beta_v}\right) \, \Delta(\sigma_\theta^2)$$

$$\Delta\epsilon_\perp \;=\; \frac{\beta_\perp\gamma\beta_v}{2} \, \Delta(\sigma_\theta^2)$$

e.g. from Particle data booklet

$$\Delta(\sigma_\theta^2) \approx \left(\frac{14.1 \; 10^6}{p\beta_v}\right)^2 \, \frac{\Delta s}{L_R}$$

$$\Delta\epsilon_\perp \;=\; \frac{\beta_\perp}{\gamma\beta_v^3} \, \Delta E \quad \left(\left(\frac{14.1 \;\; 10^6}{2_\mu}\right)^2 \, \frac{1}{L_R dE/ds}\right)$$

Defining

$$C(mat, E) \;=\; \frac{1}{2} \, \left(\frac{14.1 \;\, 10^6}{\mu)}\right)^2 \, \frac{1}{L_R \; d\gamma/ds} \tag{19}$$

then
$$\frac{\Delta\epsilon_\perp}{\epsilon_\perp} = \Delta E \ \frac{\beta_\perp}{\epsilon_\perp \gamma \beta_v^3} \ C(mat, E) \tag{20}$$

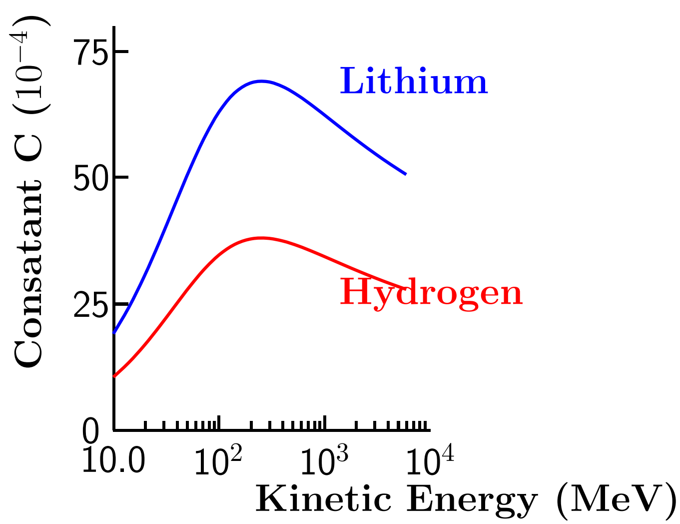
Equating this with equation 18
$$\Delta E \ \frac{1}{\beta_v^2 \ E} = \Delta E \ \frac{\beta_\perp}{\epsilon_\perp \gamma \beta_v^3} \ C(mat, E)$$

gives the equilibrium emittance $\epsilon_{\perp o}$:
$$\epsilon_\perp(min) = \frac{\beta_\perp}{\beta_v} \ C(mat, E) \tag{21}$$

At energies for minimum ionization loss:

material	T ^{o}K	density kg/m^3	dE/dx MeV/m	L_R m	C_o 10^{-4}
Liquid H_2	20	71	28.7	8.65	38
Liquid He	4	125	24.2	7.55	51
LiH	300	820	159	0.971	61
Li	300	530	87.5	1.55	69
Be	300	1850	295	0.353	89
Al	300	2700	436	0.089	248

As a function of energy:



Clearly Liquid Hydrogen is far the best material, but has cryogenic and safety complications, and requires windows which will significantly degrade the performance. At lower energies C is much lower but there is then longitudinal (dp/p) heating.

1.3 Rate of Cooling with finite Coulomb scattering

$$\frac{d\epsilon_\perp}{\epsilon_\perp} = \left(1 - \frac{\epsilon_{\perp \min}}{\epsilon_\perp}\right) \frac{dp}{p} \tag{22}$$

This would suggest that it was always desirable to have $\epsilon_\perp(\min)$ as small as possible in order to maximize the rate of cooling. But having $\epsilon_\perp(\min) \ll \epsilon_\perp$ requires excessive angular acceptance (see following).

1.4 Beam Divergence Angles and required angular acceptance

$$\sigma_\theta = \sqrt{\frac{\epsilon_\perp}{\beta_\perp \ \beta_v \gamma}}$$

from equation 21

$$\epsilon_\perp(min) = \frac{\beta_\perp}{\beta_v} \ C(mat, E) \qquad \mathrm{so} \qquad \beta_\perp = \frac{\epsilon_\perp(\mathrm{min}) \ \beta_v}{C(\mathrm{mat, E})}$$

and substituting this

$$\sigma_\theta = \sqrt{\left(\frac{\epsilon_\perp}{\epsilon_\perp(\mathrm{min})}\right) \left(\frac{C(\mathrm{mat, E})}{\beta_v^2 \gamma}\right)}$$

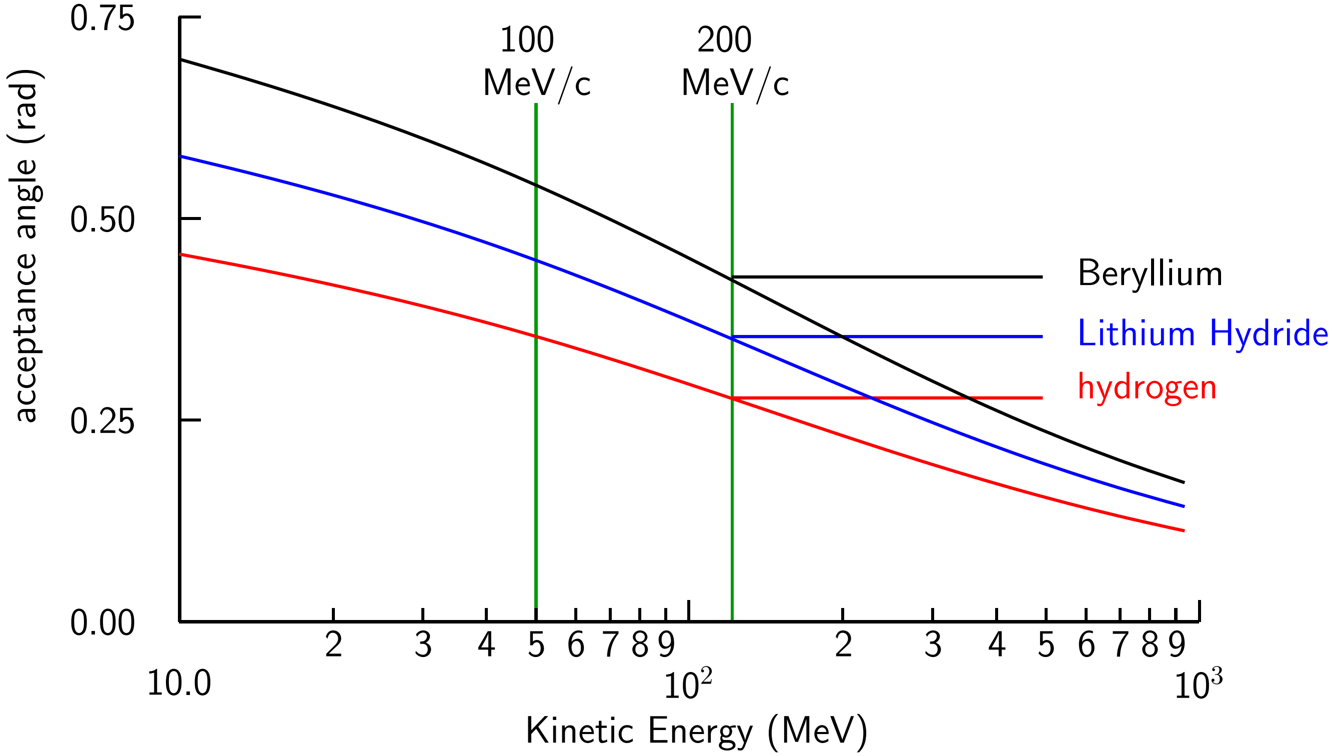
If $\epsilon_\perp(\mathrm{min})/\epsilon_\perp$ is made too small, the rms divergence angles σ_θ becomes too large, but if the fraction is too large, the cooling rate is reduced.

For a cooling rate of 50 % of maximum ($\epsilon_\perp(\mathrm{min})/\epsilon_\perp = 0.5$) and an aperture at 3 σ, the angular aperture $\mathcal{A}$ of the system must be

$$\mathcal{A} = 3\sqrt{2} \sqrt{\frac{C(mat, E)}{\beta_v^2 \gamma}} \tag{23}$$

Note that in order to maintain efficient cooling and avoid excessive acceptance requirement, the ratio $\epsilon_\perp(\mathrm{min})/\epsilon_\perp$ must be held constnat. This requires that the focusing $\beta_\perp$ must be continuously reduced as $\epsilon_\perp$ falls. Thus either frequent changes in the lattice, or a continuously cahnging (tapered) lattice should be used. This is an argument against cooling in many turns in a ring.

Apertures for hydrogen, lithium and berillium are plotted vs. energy below. These are very large angles, and if we limit apertures to less than 0.3, then this requirement sets lower energy limits of about 100 MeV ($\approx$ 170 MeV/c) for Lithium, and about 25 MeV ($\approx$ 75 MeV/c) for hydrogen.



Required acceptances:

- $\theta = 0.3$ at 200 MeV/c in Hydrogen (as in RFOFO lattice) may be as large as is possible
- $\theta = 0.6$ at 100 MeV/c in berylium (as in PIC & REMEX Lattices) looks very hard
- $\theta = 0.5$ at 10 MeV in hydrogen (as in collider final cooling) may be ok in a continuous solenoid

1.5 Focusing Systems

1.5.1 Continuous Solenoid

In a solenoid with axial field B_{sol} (from eq 12)

$$\beta_\perp = \frac{2\ p}{c\ B_{sol}}$$

so from eq. 21

$$\epsilon_\perp(min) = C(mat, E)\ \frac{2\ \gamma\ m_\mu}{B_{sol}\ c} \tag{24}$$

- 45 T hybrid using 20 MW exists at NHFML
- 50 T HTS seems possible

1.5.2 Current Carrying Rod

In a rod carrying a uniform axial current, the azimuthal magnetic field B varies linearly with the radius r. A muon traveling down it is focused:

$$\frac{d^2r}{dr^2} = -\frac{B\ c}{p} = -\left(\frac{c}{p}\ \frac{dB}{dr}\right)\ r$$

$$\text{so orbits oscillate with} \qquad \beta_\perp^2 = \frac{\gamma\beta_v}{dB/dr}\ \frac{m_\mu}{c} \tag{25}$$

If we set the rod radius a to be f_{ap} times the rms beam size $\sigma_\perp$ (from eq.8),

$$\sigma_\perp = \sqrt{\frac{\epsilon_\perp\ \beta_\perp}{\beta_v\gamma}}$$

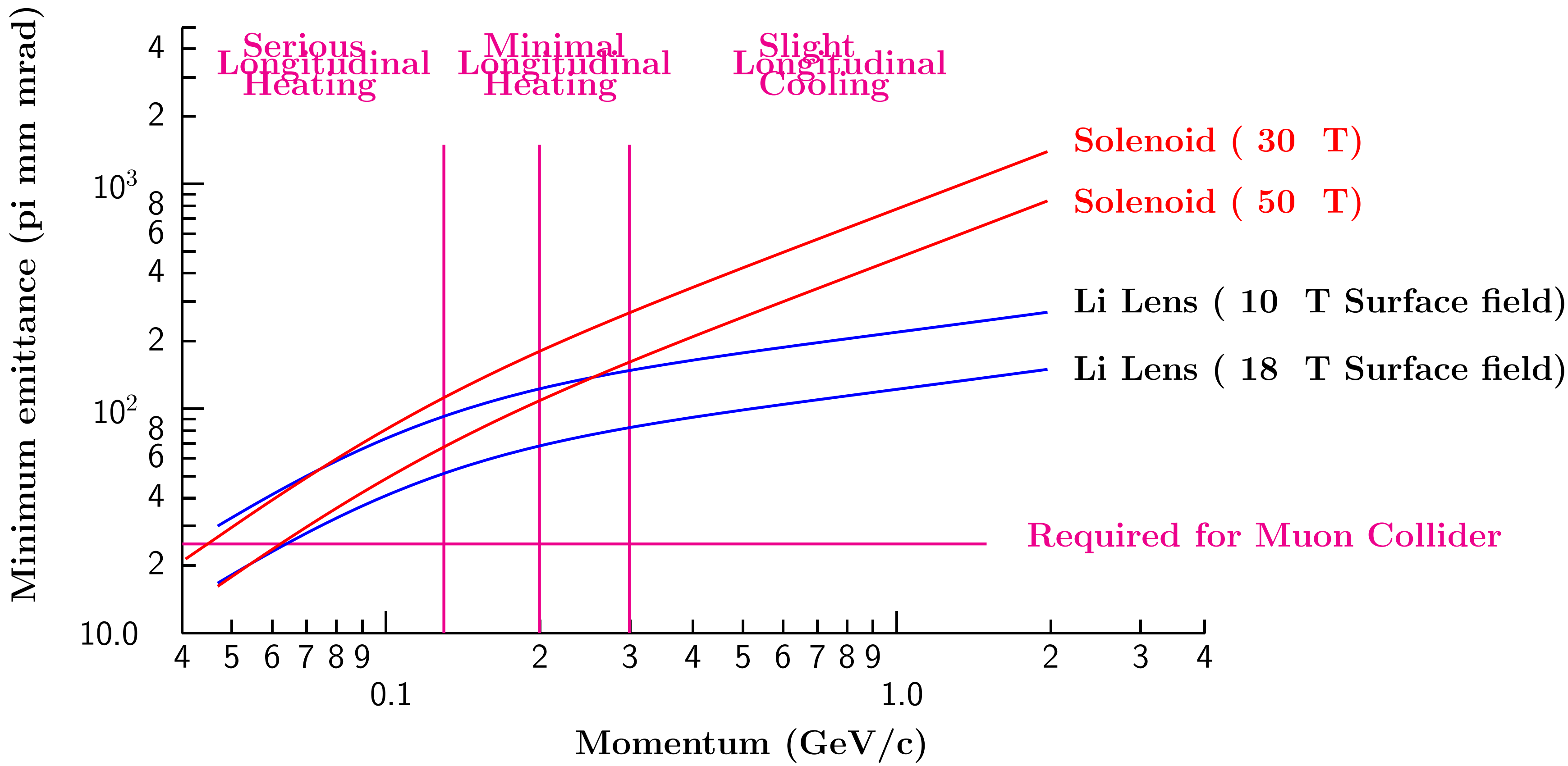
$$\text{if the field at the surface is } B_{max} \qquad \beta_\perp^2 = \frac{\gamma\beta_v m_\mu\ f_{ap}}{B_{max}\ c}\sqrt{\frac{\epsilon_\perp\ \beta}{\gamma\ \beta_v}}$$

$$\text{from which we get :} \qquad \beta_\perp = \left(\frac{f_{ap}\ m_\mu}{B_{max}\ c}\right)^{2/3}\ (\gamma\beta_v\ \epsilon_\perp)^{1/3}$$

$$\text{puting this in equation 21} \qquad \epsilon_\perp(min) = (C(mat,E))^{1.5}\left(\frac{f_{ap}\ m_\mu}{B_{max}\ c\ \beta_v}\right)\ \sqrt{\gamma} \tag{26}$$

A maximum surface field of $\approx$ 10 T is set by breaking of containment pipe

1.5.3 Compare Focusing as a function of the beam momentum



We see that at momenta where longitudinal emittance is not blown up ($\approx$ 200 MeV/c) then current Li Lens technology ($\approx$ 10 T) is comparable to the highest plausible solenoid ($\approx$ 50 T)

But if we allow longitudinal heating and use very low momenta (45-62 MeV/c or 9-17 MeV):

- Even a 30 T solenoid beats the Li Lens
- A 50 T solenoid is comparable with a 16 T surface field Li Lens that may or may not be build able

1.6 Angular Momentum Problem in Solenoid Cases

In the absence of external fields and energy loss in materials, the angular momentum of a particle is conserved.

But a particle entering a solenoidal field will cross radial field components and its angular momentum ($r\ p_\phi$) will change (eq.17).

$$r\ \Delta(p_\phi) \ = \ r\ \Delta\left(\frac{c\ B_z\ r}{2}\right)$$

so if there is no initial angular momentum then in the field B_z, in addition to the random emittance fluctuations, there will be a coherent angular momentum:

$$p_\phi\ r \ = \ \left(\frac{c\ B_z\ r}{2}\right)\ r$$

Material introduced to cool the beam, will reduce all momenta, both longitudinal and transverse, random and average.

Re-acceleration will not change the angular momenta, so the average angular momentum will continuously fall. After any significant transverse cooling of the random emittance phase space, this coherent angular momentum will be largely removed:

$$p_\phi\ r \ \approx \ 0$$

Then when the beam exits the solenoid, this canonical angular momentum becomes a real angular momentum.

$$< p_\phi \ r >_{\text{end}} \ \approx \ -\left(\frac{c \ B_z \ r}{2}\right) \ r$$

This represents an effective emittance

$$\epsilon_\perp(\text{effective}) \ = \ \beta_v \gamma \sigma_\perp \ \left(\frac{< p_\phi >}{p}\right)$$

$$\approx \ \beta_v \gamma \ \sigma_\perp \left(\frac{cB_z \ \sigma_\perp}{2p}\right) \ = \ \beta_v \gamma \ \sigma_\perp^2 \left(\frac{cB_z}{2p}\right)$$

$$= \ \beta_v \gamma \ \left(\frac{\epsilon_\perp \ \beta_\perp}{\beta_v \gamma}\right) \ \left(\frac{cB_z}{2p}\right) \ = \ (\epsilon_\perp \ \beta_\perp(\text{beam})) \ \left(\frac{cB_z}{2p}\right)$$

Since $\beta_\perp$(lattice) in a solenoid is given by

$$\beta_\perp(\text{lattice}) \ = \ \frac{2p}{cB_z}$$

$$\epsilon_\perp(\text{effective}) \ = \ (\epsilon_\perp \ \zeta) \ \left(\frac{2p}{cB_z}\right) \left(\frac{cB_z}{2p}\right) \ = \ (\epsilon_\perp \ \zeta)$$

where

$$\zeta \ = \ \frac{\beta_\perp(\text{beam})}{\beta_\perp(\text{lattice})}$$

There are three different circumstances:

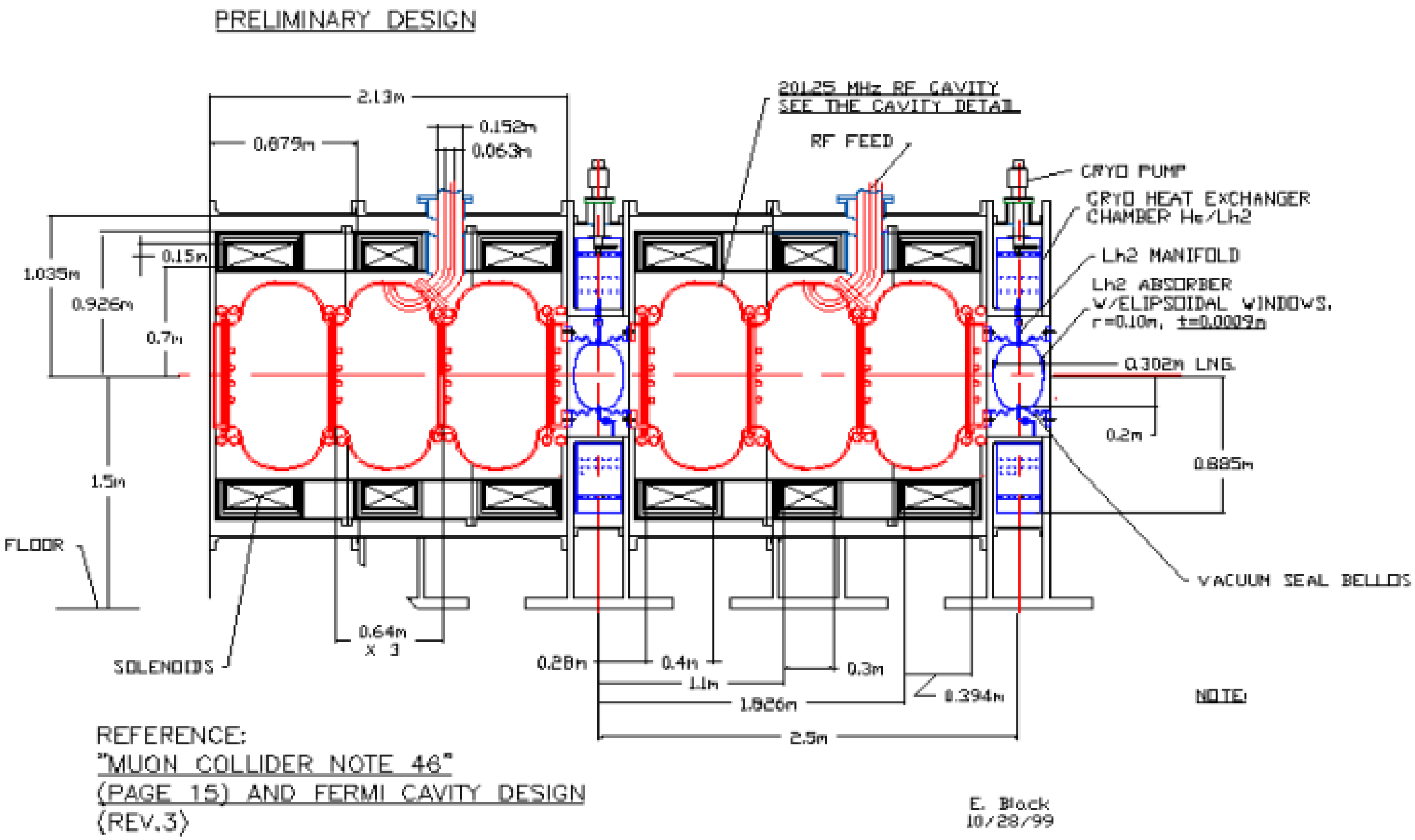
1. In a periodic lattice in which the cooling occurs in a solenoid with the $\beta_\perp$(beam) $= \beta_\perp$(lattice)
   In this case: $\zeta = 1$ so
   $$\epsilon_\perp(\mathrm{effective}) \ = \ \epsilon_\perp$$
   Which effectively increases the final emittance in both x and y by approximatel $\sqrt{2}$

2. Cooling in a non-periodic long solenoid where $\sigma_\perp$ does not change, while only σ_θ is cooled
   In this case $\beta_\perp(\mathrm{beam}) >> \beta_\perp(\mathrm{lattice})$, $\zeta >> 1$ and
   $$\epsilon_\perp(\mathrm{effective}) >> \epsilon_\perp$$
   The effective additional emittance is much more than a factor of $\sqrt{2}$

3. Cooling in a slenoid within a lattice where $\beta_\perp(\mathrm{beam}) << \beta_\perp(\mathrm{lattice})$, $\zeta << 1$ and
   $$\epsilon_\perp(\mathrm{effective}) << \epsilon_\perp$$
   So in this case the problem is negligible

1.7 Continuous Solenoid

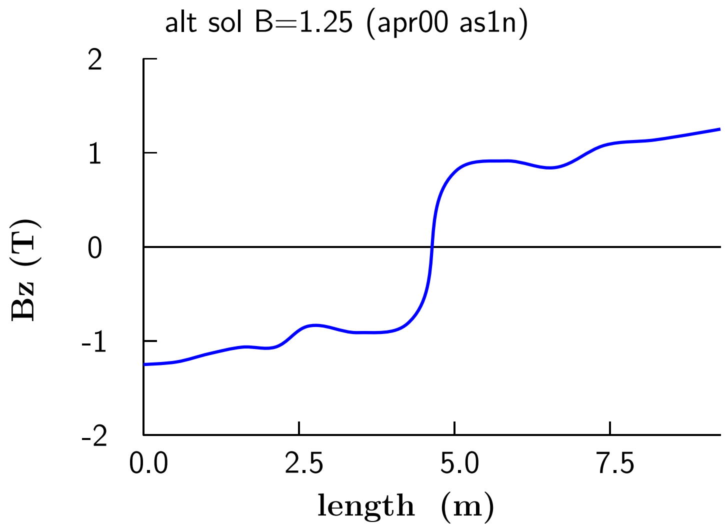
Coils Outside RF: e.g. in Feasiility Study 1, FNAL 1 flip design

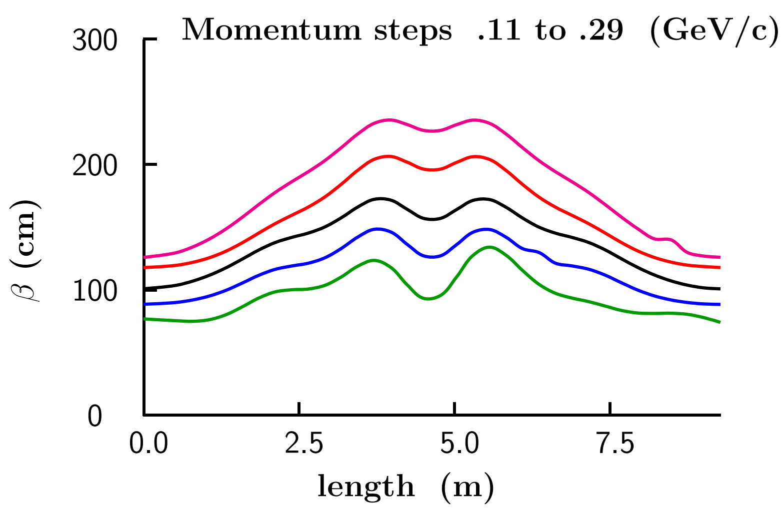


This would be a case with $\zeta >> 1$ so field reversals (flips) are required

One must design the flips to match the betas from one side to the other.

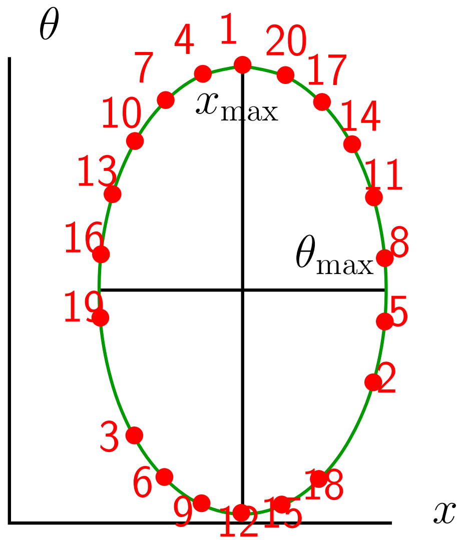
For a computer designed matched flip between uniform solenoidal fields: the following figure shows B_z vs. z and the $\beta_\perp$'s vs. z for different momenta.

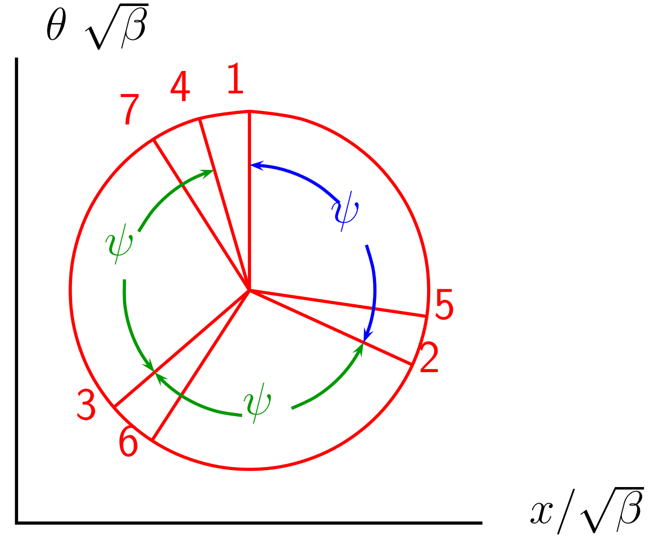




1.8 Periodic lattices

If we observe a slightly off axis tracks angle and position once for each "cell" and plot their angles θ and positions x then they will, if this is a stable lattice, fall on an ellipse, as in the left hand figure below





The $\beta_\perp$ is then given by the ratio of the major and minor axes:

$$\beta_\perp = \frac{x_{\max}}{\theta_{\max}}$$

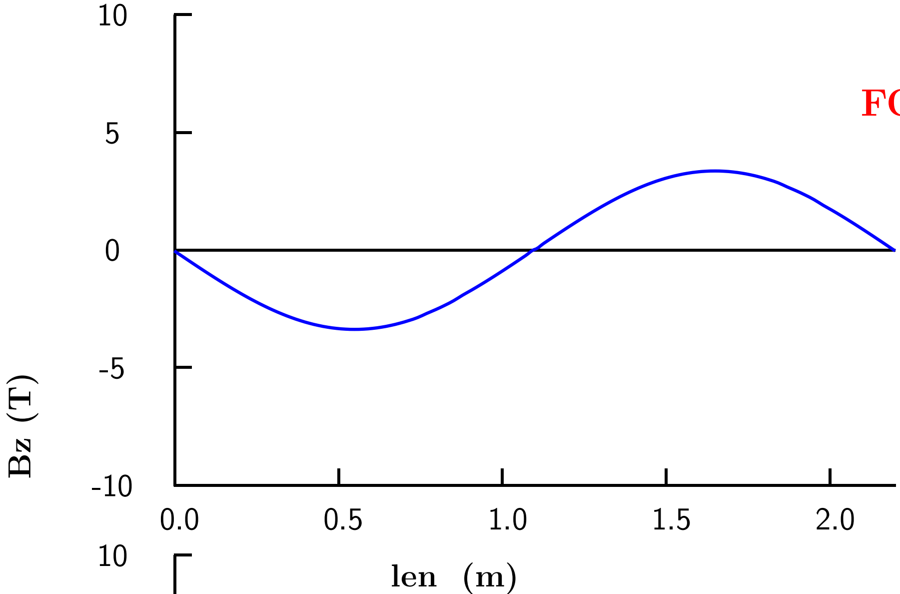
If we replot the points with axes of $x/\sqrt{\beta_\perp}$ and $\theta\sqrt{\beta_\perp}$, then the points fall on a circle.

The angle between successive cells is now found to be a constant and is the phase advance per cell ψ The tune for the cell is

$$\nu = \psi/2\pi$$

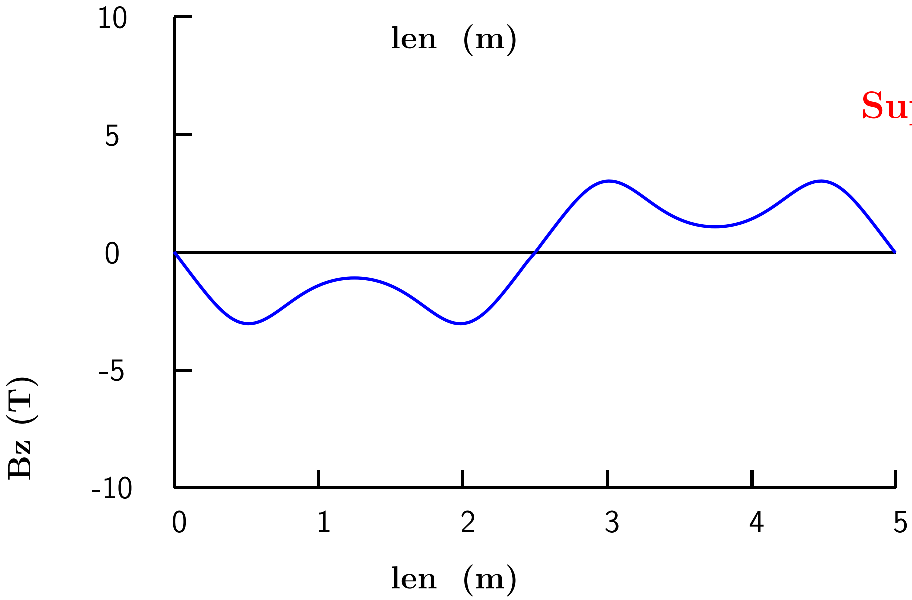
If the tune is an integer or half integer then we have a "reonance" in where the parameter $\beta_\perp$ often goes to zero or infinity

Single and Double Periodicities



FOFO

$B_z(\text{max}) = 3.4$ (T)
$dB_z/dz(\text{max}) = 9.4$ (T/m)

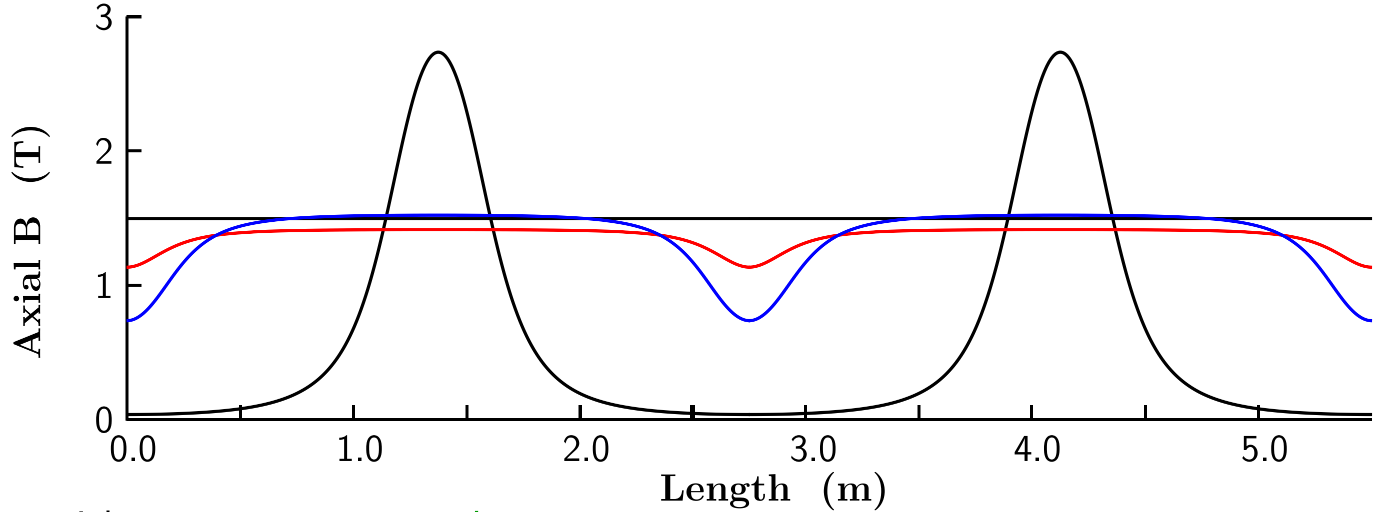


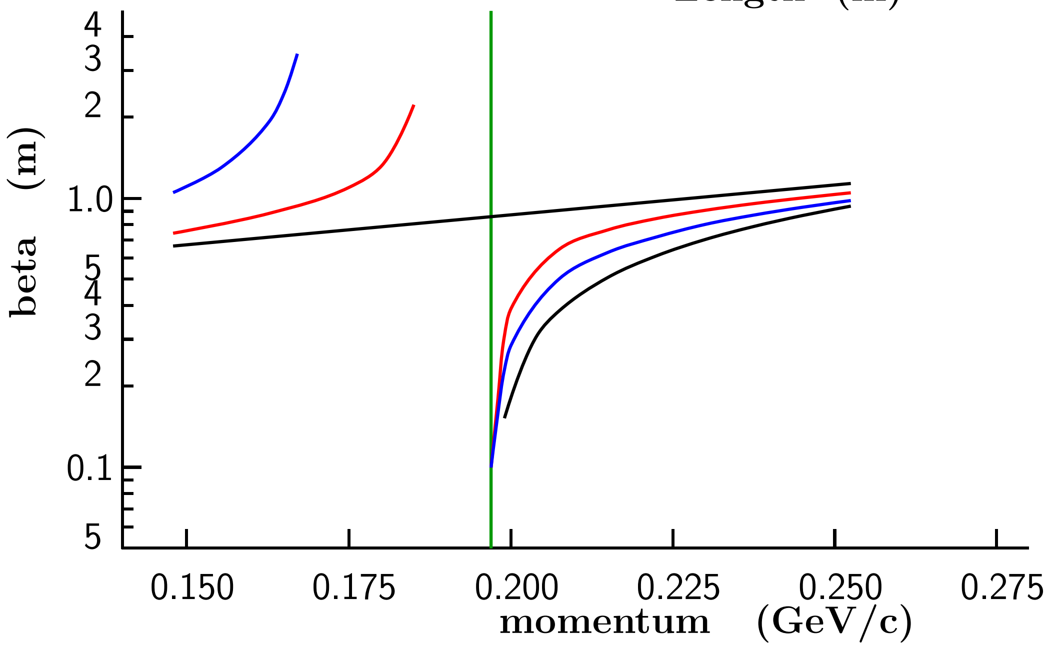
Super FOFO

$B_z(\text{max}) = 2.6$ (T)
$dB_z/dz(\text{max}) = 7$ (T/m)

1.8.1 Single periodicity

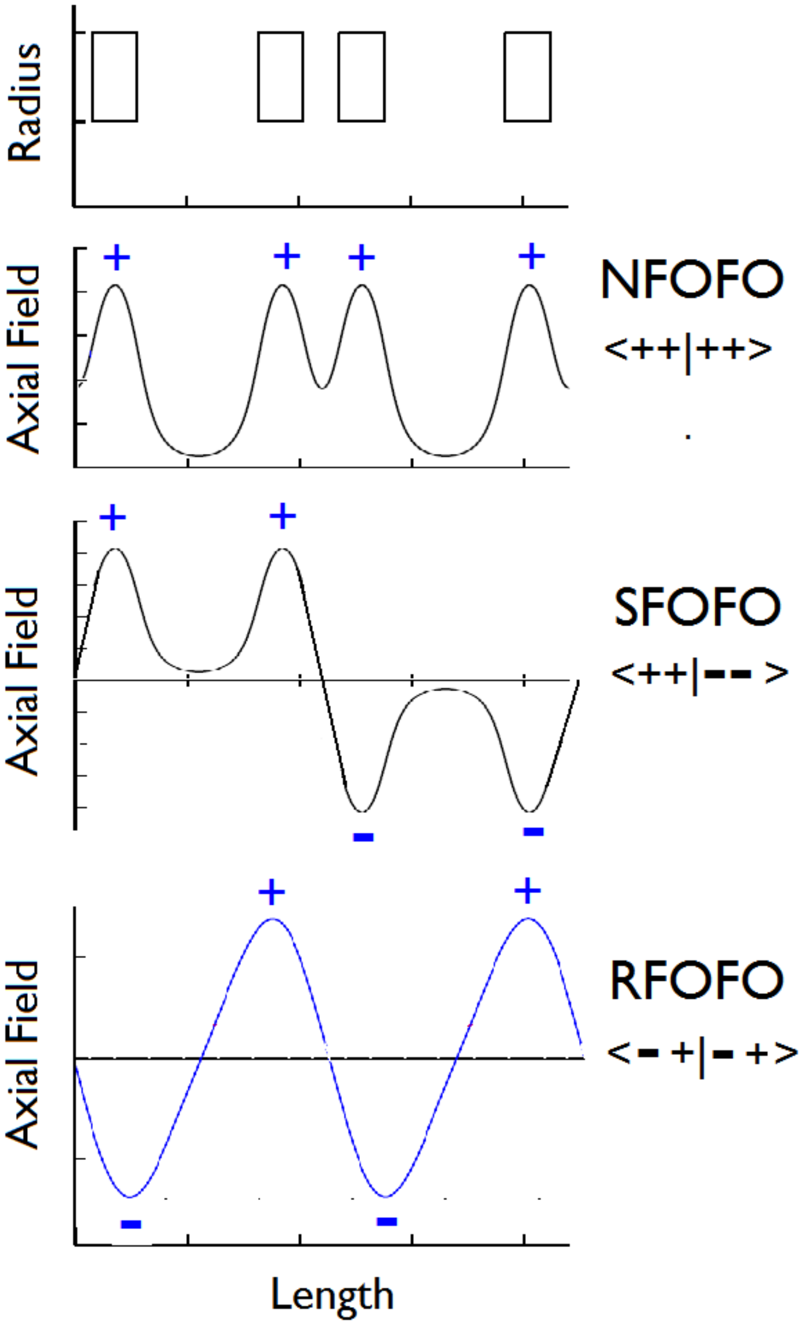
Often refered to as FOFO (focus-focus)



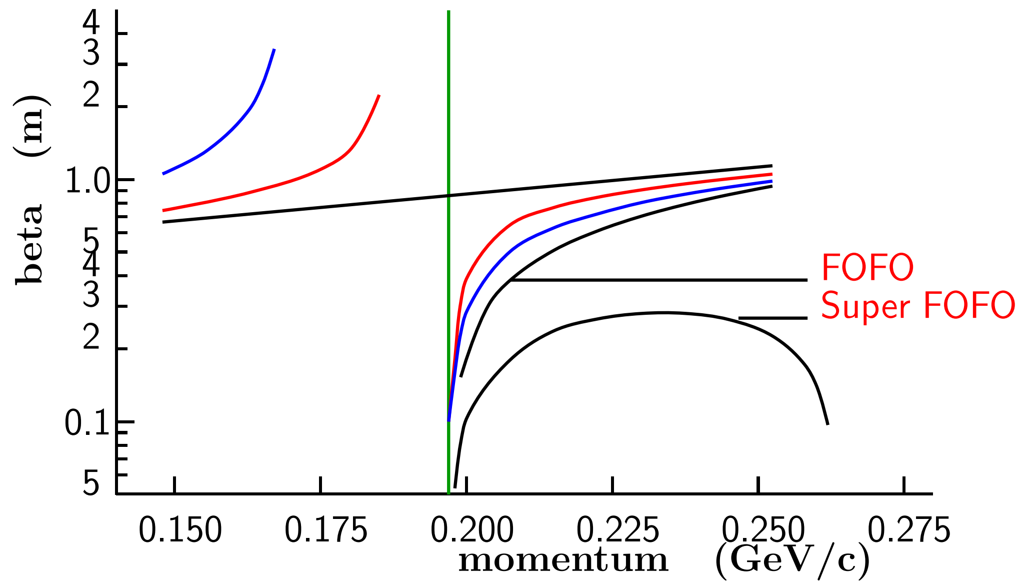


- Resonances are introduced
- Betas are reduced locally
- But momentum acceptance is small
- & smaller if the perturbation is small

1.8.2 Double Periodicity

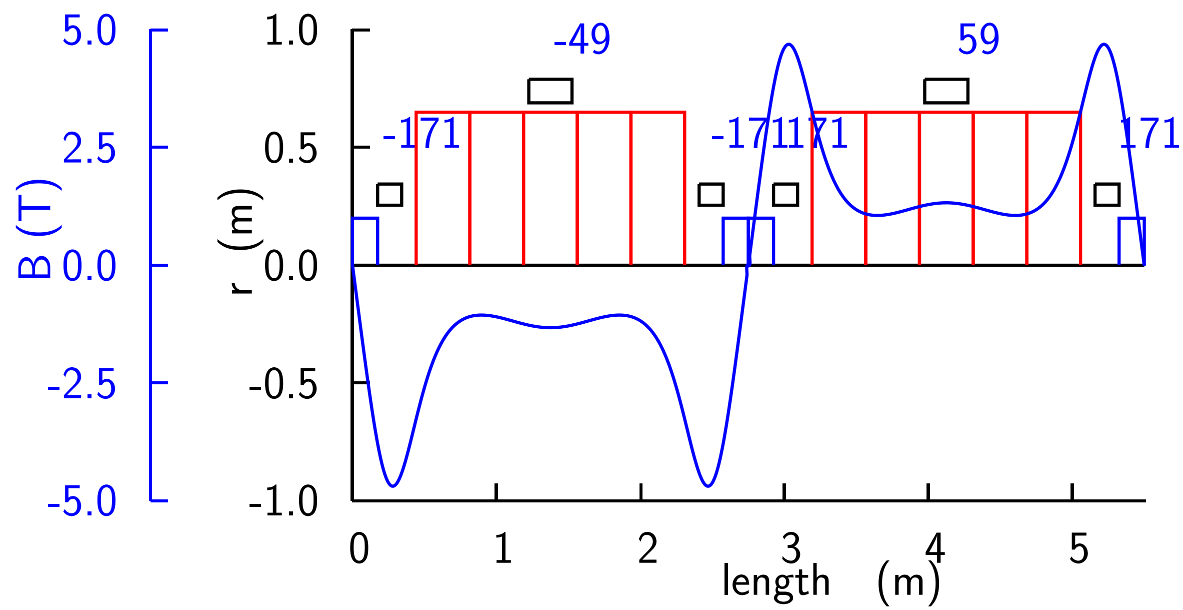


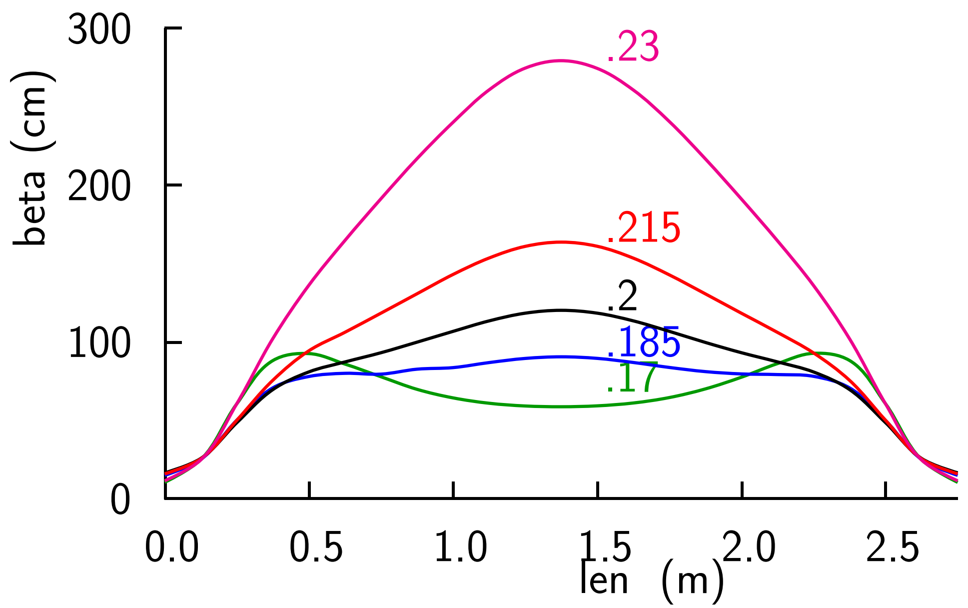
- A second resonance is introduced
- Between the two resonances there are lower betas over a finite momentum range
- Beta lower by about 1/2 solenoid ($\zeta \approx 0.5$)

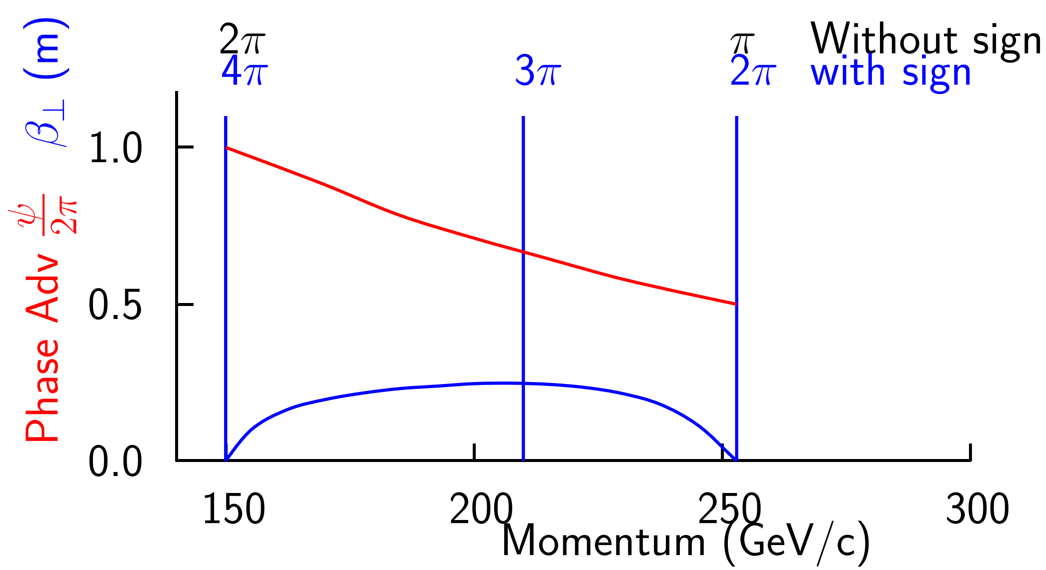


- Because focusing $\propto B^2$ there are several symmetries with similar lattice properties:
  - $< + + | + + >$ minimizes current but induces angular momentum
  - $< + + | - - >$ avoids angular momentum
  - $< - + | - + >$ has all cells truely identical

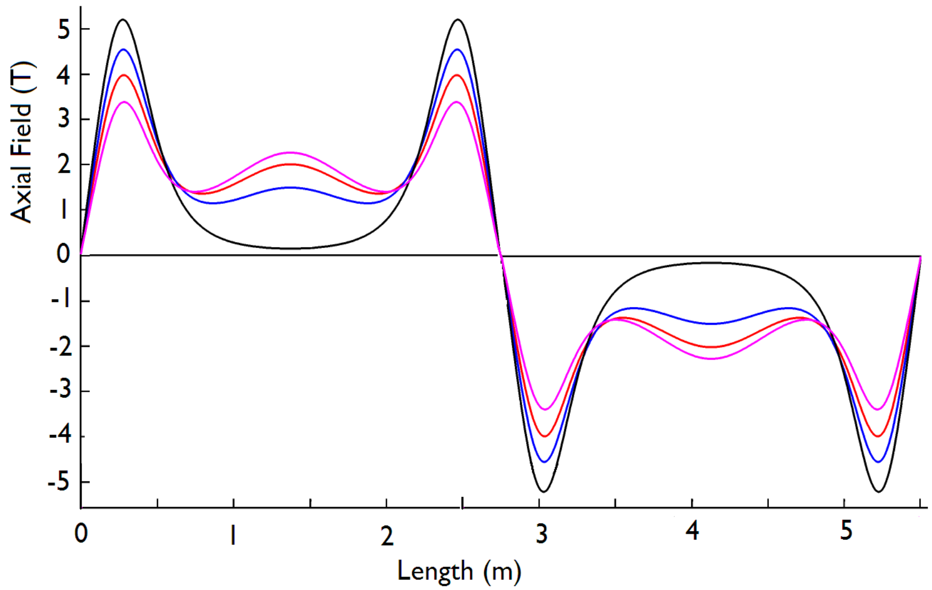
3 coil SFOFO in Study 2 & MICE $<+++|--->_2$



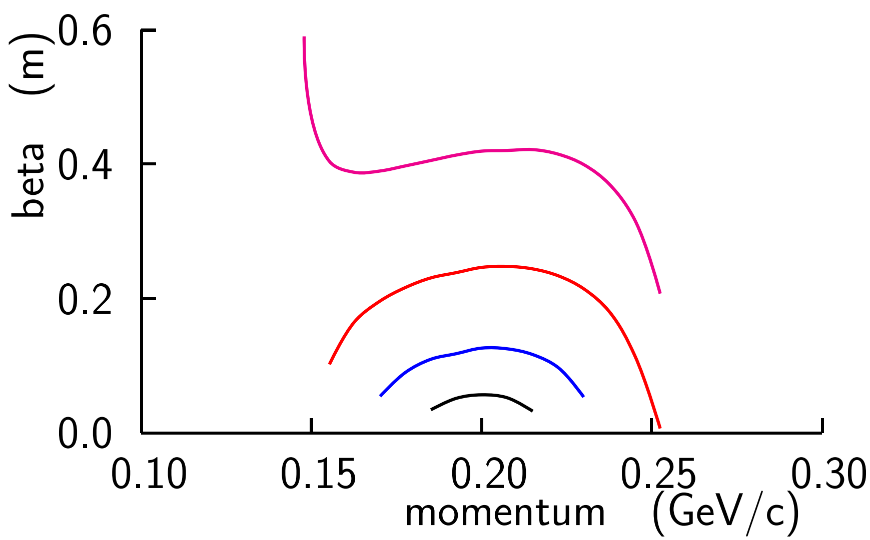




3 coils allows minimum β to be varied



- Reduce "coupling coil" current while raising focus coil current to keep pass band centered

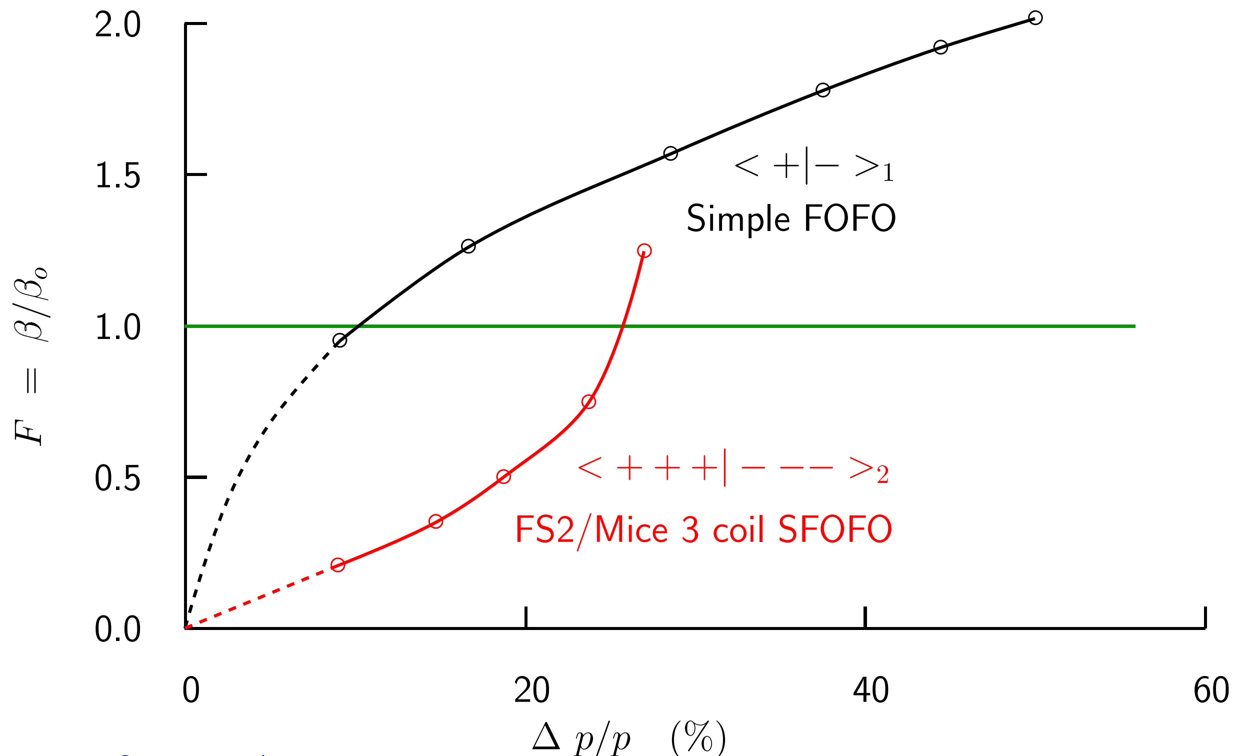


- Wide variation of betas
- Momentum acceptance falls with beta

1.9 Minimum betas vs. Momentum acceptance

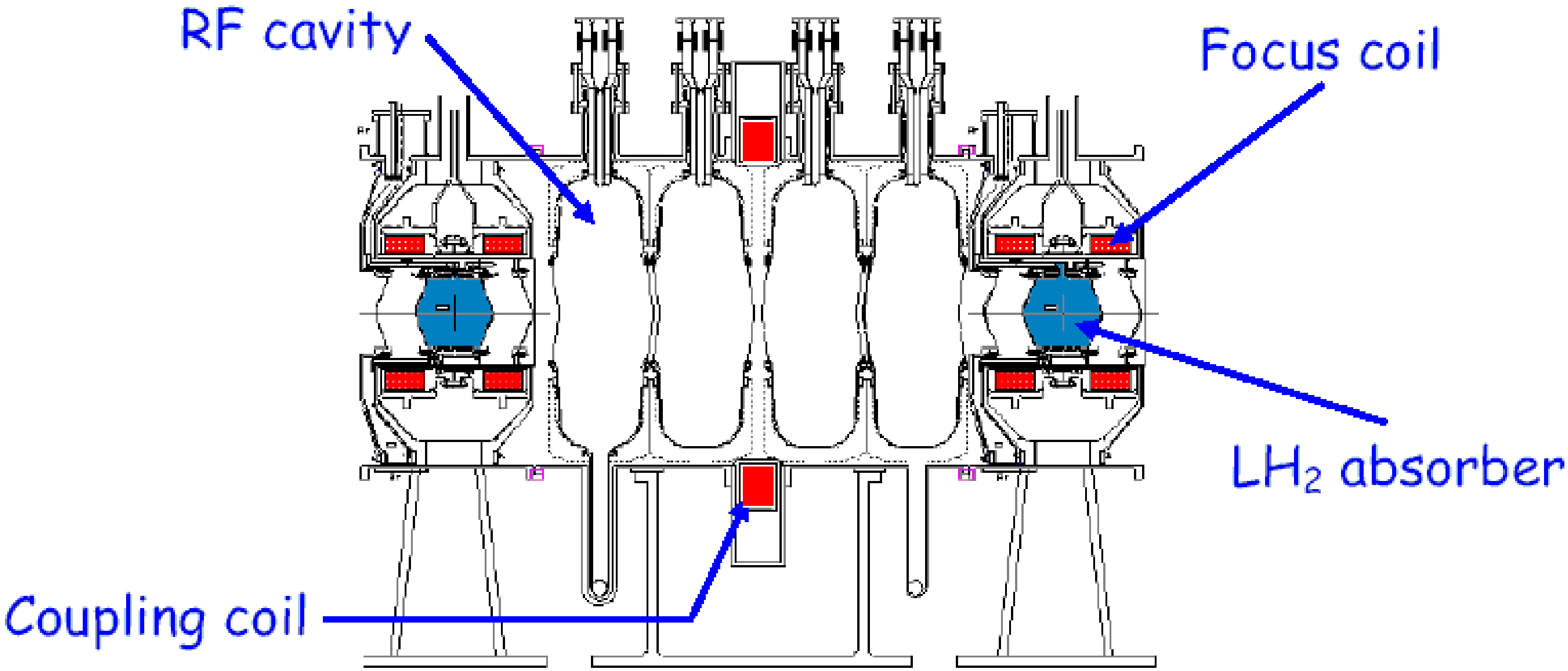
If we normalize the minimum betas to those of long solenoids with fields equal to the maximum axial fields in a set of lattices

$$F = \frac{\beta_\perp}{\frac{p}{B_z(max)\ c}}$$



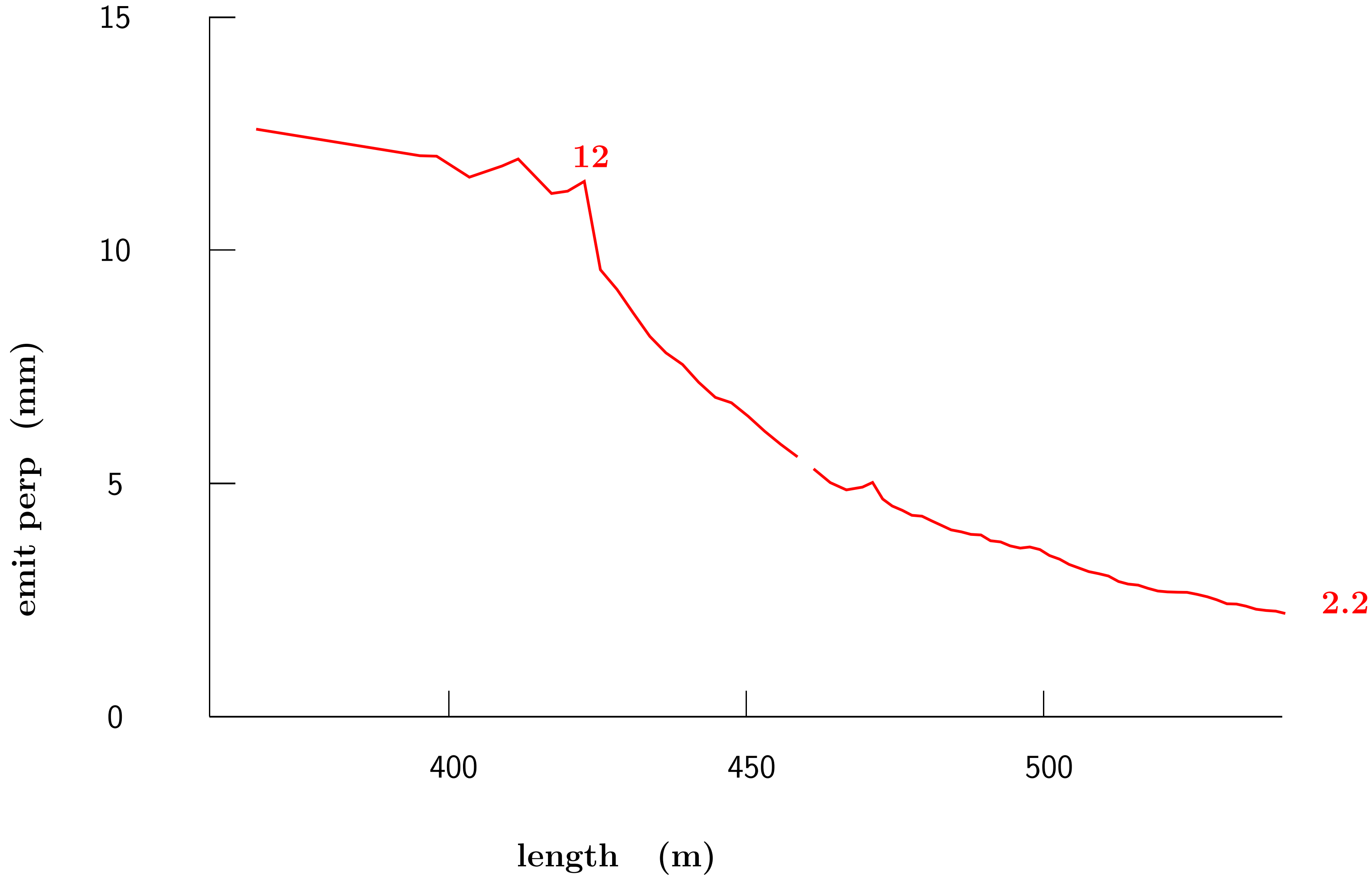
- For SFOFO and small Δp : $\beta \propto \Delta p$
- For FOFO and small Δp : $\beta \propto \sqrt{\Delta p}$
- For very large Δp FOFO or long solenoids required
- But for small Δp SFOFO is superior

SFOFO at start of Study 2 Cooling and in MICE



- This is the lattice to be tested in Muon Ionization Cooling Experiemnt (MICE) at RAL
- In study 2 the lattice is modified vs. length to lower $\beta_\perp$ as ϵ falls
  This keeps σ_θ and ϵ/ϵ_o more or less constant, thus maintains cooling rate

Study 2 Performance



- With RF and Hydrogen Windows, $C_o \approx 45\ 10^{-4}$
- $\beta_\perp$(end)=.18 m,
- β_v(end)=0.85,

5) LONGITUDINAL IONIZATION COOLING

Following the convention for synchrotron cooling we define partition functions:

$$J_{x,y,z} = \frac{\frac{\Delta\ (\epsilon_{x,y,z})}{\epsilon_{x,y,z}}}{\frac{\Delta p}{p}} \tag{27}$$

$$J_6 = J_x + J_y + J_z \tag{28}$$

where the $\Delta\epsilon$'s are those induced directly by the energy loss mechanism (synchrotron radiation for electrons, ionization energy loss in this case). Δp refers to the loss of momentum induced by this energy loss.

In electron synchrotrons, with no gradients fields, $J_x = J_y = 1$, and $J_z = 2$.

In muon ionization cooling, $J_x = J_y = 1$, but J_z is negative or small.

2.1 c.f. Transverse

From last lecture:

$$\epsilon_\perp = \beta_v \gamma \sigma_\perp \sigma_\theta = \beta_v \gamma \sigma_\perp \frac{\sigma_{p\perp}}{p}$$

$$\frac{\Delta \sigma_{p\perp}}{\sigma_{p\perp}} = \frac{\Delta p}{p}$$

and $\sigma_\perp$ does not change, so

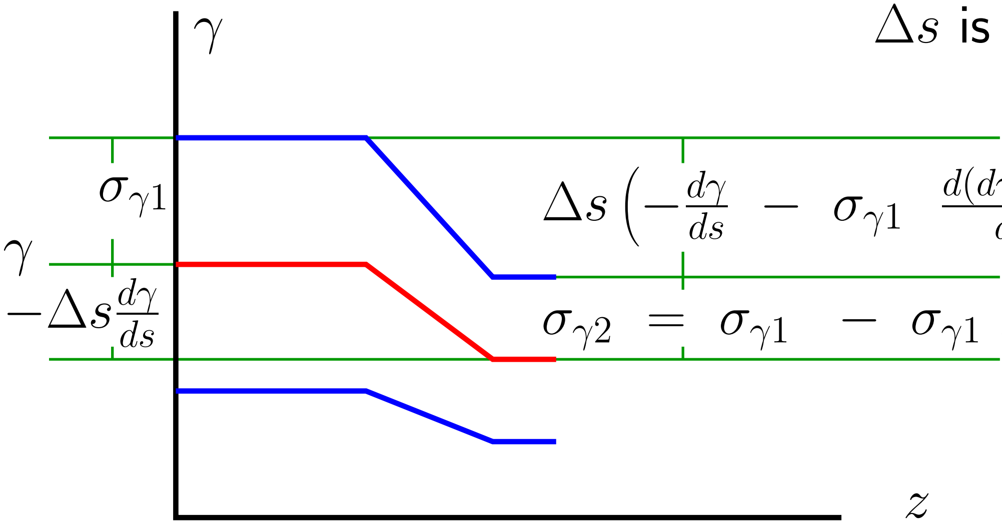
$$\frac{\Delta \epsilon_\perp}{\epsilon_\perp} = \frac{\Delta p}{p} \tag{29}$$

and thus

$$J_x = J_y = 1 \tag{30}$$

2.2 Longitudinal cooling/heating without wedges

Δs is thickness of absorber



$$\sigma_{\gamma 2} = \sigma_{\gamma 1} - \sigma_{\gamma 1} \Delta s \frac{d(d\gamma/ds)}{d\gamma} \qquad \text{i.e.} \quad \Delta\sigma_\gamma = \sigma_{\gamma 1} \Delta s \ d$$

The emittance in the longitudinal direction ϵ_z is (eq.5):

$$\epsilon_z = \gamma \beta_v \frac{\sigma_p}{p} \sigma_z = \frac{1}{m} \sigma_p \sigma_z = \frac{c}{m} \sigma_E \sigma_t = c \ \sigma_\gamma \ \sigma_t \tag{31}$$

where σ_t is the rms bunch length in time, and c is the velocity of light. Drifting between interactions will not change emittance (Louville), and an interaction will not change σ_t, so emittance change is only induced by the energy change in the interactions:

$$\frac{\Delta \epsilon_z}{\epsilon_z} = \frac{\Delta \sigma_\gamma}{\sigma_\gamma} = \frac{\sigma_\gamma \ \Delta s \ \frac{d(d\gamma/ds)}{d\gamma}}{\sigma_\gamma} = \Delta s \ \frac{d(d\gamma/ds)}{d\gamma}$$

and

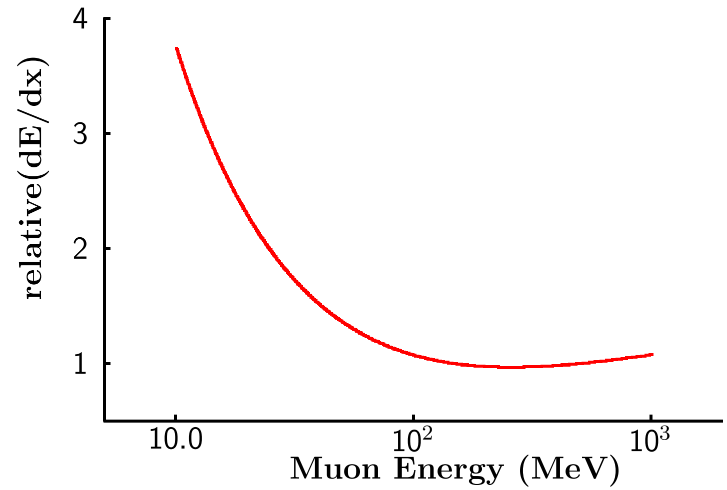
$$\frac{\Delta p}{p} = \frac{\Delta\gamma}{\beta_v^2\gamma} = \frac{\Delta s}{\beta_v^2\gamma}\left(\frac{d\gamma}{ds}\right)$$

So from the definition of the partition function J_z:

$$J_z = \frac{\frac{\Delta\epsilon_z}{\epsilon_z}}{\frac{\Delta p}{p}} = \frac{\left(\Delta s\ \frac{d(d\gamma/ds)}{d\gamma}\right)}{\frac{\Delta s}{\beta_v^2\gamma}\left(\frac{d\gamma}{ds}\right)} = \frac{\left(\beta_v^2\ \frac{d(d\gamma/ds)}{d\gamma/\gamma}\right)}{\left(\frac{d\gamma}{ds}\right)} \tag{32}$$

A typical energy loss, relative to its minimum, as a function of energy, is shown at right (this example is for Lithium).

Note rapid rise for $E < 100$ MeV



$$\text{Approximately} \qquad \frac{d\gamma}{ds} = B\,\frac{1}{\beta_v^2}\left(\frac{1}{2}\,ln(A\ \beta_v^4\gamma^4\ -\ \beta_v^2\right) \tag{33}$$

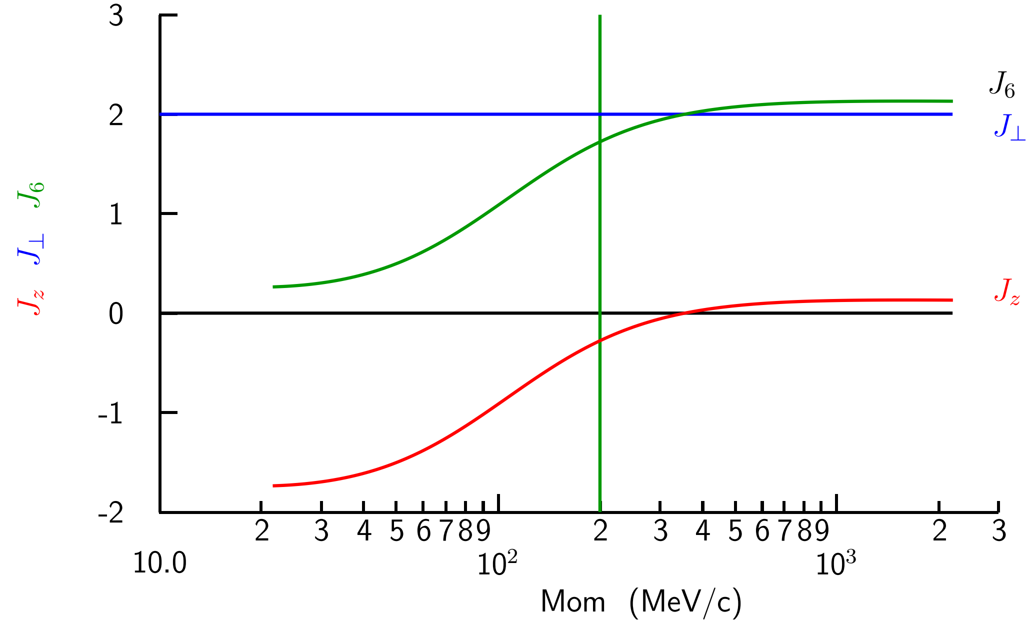
where

$$A = \frac{(2m_ec^2/e)^2}{I^2} \qquad B \approx \frac{0.0307}{(m_\mu c^2/e)}\,\frac{Z}{A} \tag{34}$$

where Z and A are for the nucleus of the material, and I is the ionization potential for that material.

Differentiating the above and substituting this into equation 32, we get J_z and $J_6 = J_z + 2$ vs. momentum.

It is seen that J_z is strongly negative at low energies (longitudinal heating), and is only barely positive at momenta above 300 MeV/c. To avoid rapid longitudinal heating we often cool at a moderate momentum around 200 MeV/c, where $J_z \approx 0$, and $J_6 \approx 2$.



However 6D cooling remains finite even at very low momenta

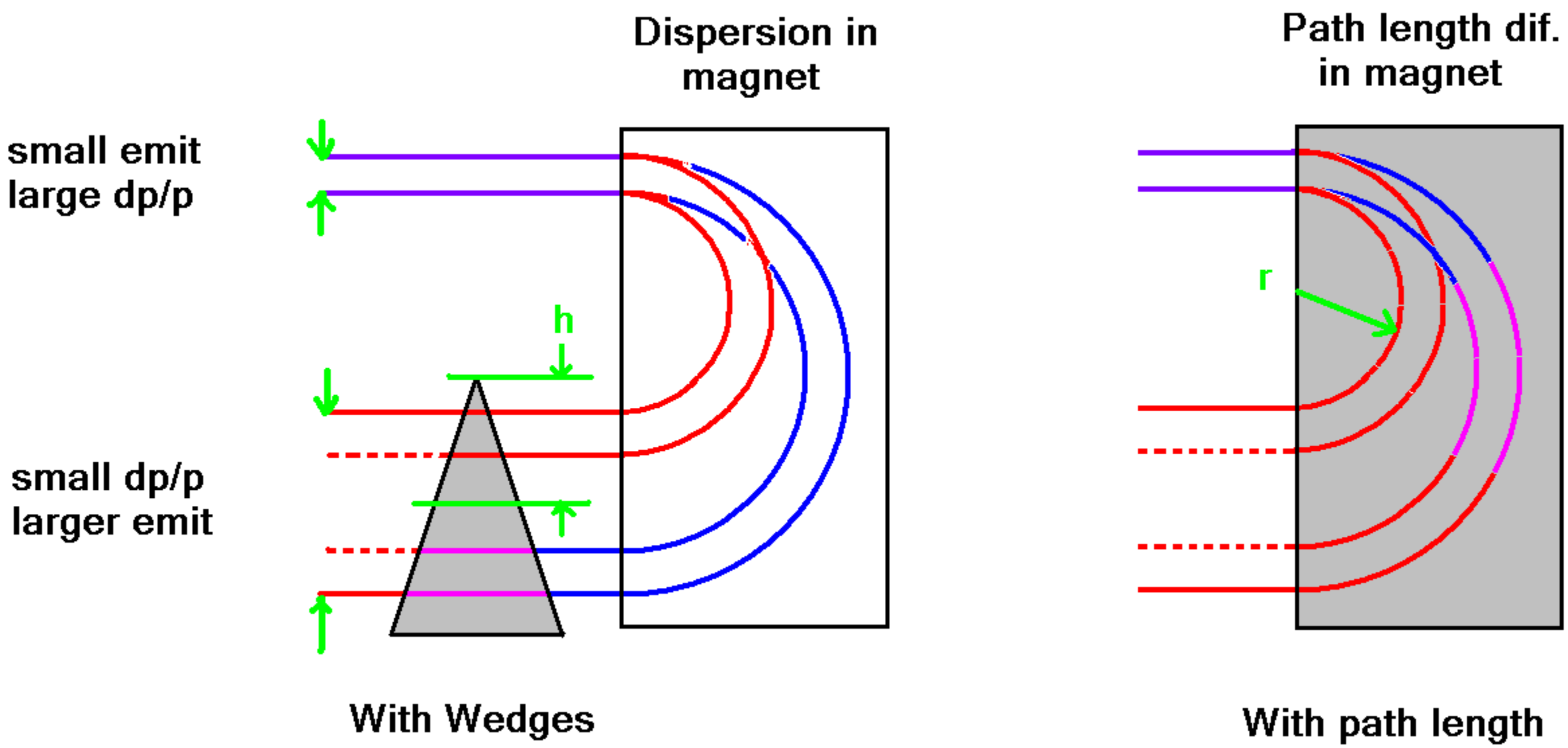
In order to cool ϵ_z we need a method to exchange cooling between the transverse and longitudinal directions. This can be done in synchrotron cooling if focusing and bending is combined, but in this case, and in general, one can show that such mixing can only increase one J at the expense of the others: J_6 is conserved.

$$\Delta J_x + \Delta J_x + \Delta J_x = 0 \tag{35}$$

and for typical operating momenta (p $\approx$ 200 MeV/c:

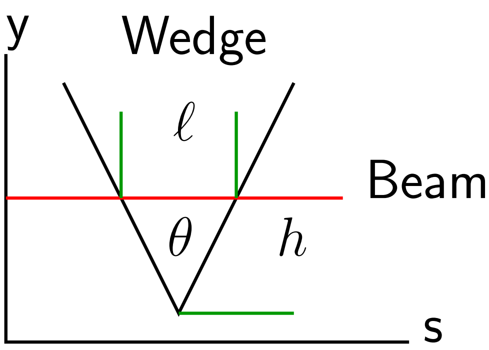
$$J_x = J_z = 1.0 \qquad J_z \approx -0.3 \qquad J_6 = J_x + J_y + J_z \approx 1.7 \tag{36}$$

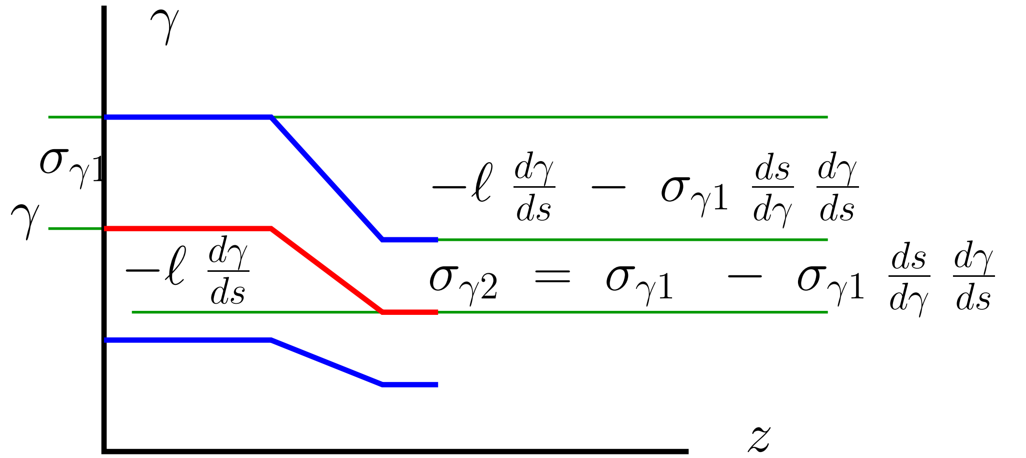
2.3 Emittance Exchange



- dp/p reduced — But σ_y increased
- Long Emittance reduced — Trans Emittance Increased
- "Emittance Exchange"

2.4 Longitudinal cooling with wedges and Dispersion





i.e.

$$\Delta\sigma_\gamma = \sigma_\gamma\ \frac{ds}{d\gamma}\ \frac{d\gamma}{ds}$$

$$\Delta\sigma_\gamma = \sigma_\gamma\ \frac{\ell}{h}\ \frac{dy}{d\gamma}\ \frac{d\gamma}{ds}$$

For a wedge with center thickness ℓ and height from center h ($=\frac{\ell}{2\tan(\theta/2)}$), in dispersion D

$$D = \frac{dy}{dp/p} \quad \text{so with eq. 2} \quad D = \beta_v^2\, \frac{dy}{d\gamma/\gamma} \qquad \frac{dy}{d\gamma} = \frac{D}{\gamma\beta_v^2}$$

$$\frac{\Delta\epsilon_z}{\epsilon_z} = \frac{\Delta\sigma_\gamma}{\sigma_\gamma} = \frac{\sigma_\gamma\ \frac{ds}{d\gamma}\left(\frac{d\gamma}{ds}\right)}{\sigma_\gamma} = \frac{ds}{d\gamma}\left(\frac{d\gamma}{ds}\right) = \left(\frac{\ell}{h}\right)\frac{D}{\beta_v^2\ \gamma}\ \left(\frac{d\gamma}{ds}\right)$$

and

$$\frac{\Delta p}{p} = \frac{\Delta\gamma}{\beta_v^2\gamma} = \frac{\ell}{\beta_v^2\gamma}\ \left(\frac{d\gamma}{ds}\right)$$

$$\Delta J_z(\text{wedge}) \ = \frac{\frac{\Delta\epsilon_z}{\epsilon_z}}{\frac{\Delta p}{p}} = \frac{\left(\frac{\ell}{h}\right)\frac{D}{\beta_v^2\ \gamma}\ \left(\frac{d\gamma}{ds}\right)}{\frac{\ell}{\beta_v^2\gamma}\ \left(\frac{d\gamma}{ds}\right)} = \frac{D}{h} \qquad (\text{for simple bend \& gas } \Delta J_z(\text{wedge}) = 1) \tag{37}$$

$$J_z \ = \ J_z(\text{no wedge}) \ + \ \Delta J_z(\text{wedge}) \tag{38}$$

But from eq.35, for any finite $J_z(\text{wedge})$, J_x or J_y will change in the opposite direction.

2.5 Longitudinal Heating Terms

Since (eq. 31) $\epsilon_z \ = \ c\ \sigma_\gamma\ \sigma_t$, and t and thus σ_t is conserved in an interaction

$$\frac{\Delta\epsilon_z}{\epsilon_z} = \frac{\Delta\sigma_\gamma}{\sigma_\gamma}$$

Straggling, from Perkins text book ***, converted to MKS:

$$\Delta(\sigma_\gamma) \ = \ \frac{\Delta\sigma_\gamma^2}{2\sigma_\gamma} \approx \frac{1}{2\sigma_\gamma}\, 0.06\, \frac{Z}{A} \left(\frac{m_e}{m_\mu}\right)^2 \gamma^2 \left(1-\frac{\beta_v^2}{2}\right) \rho\, \Delta s$$

$$\text{From eq. 2}: \quad \Delta \mathrm{E} \ = \ \mathrm{E}\beta_{\mathrm{v}}^2 \frac{\Delta \mathrm{p}}{\mathrm{p}}, \text{so}: \qquad \Delta s \ = \ \frac{\Delta E}{dE/ds} \ = \ \frac{1}{dE/ds} E\ \beta_v^2\ \frac{\Delta p}{p}$$

so

$$\frac{\Delta\epsilon_z}{\epsilon_z} = \frac{0.06}{2\sigma_\gamma^2}\ \frac{Z}{A} \left(\frac{m_e}{m_\mu}\right)^2 \gamma^2 \left(1-\frac{\beta_v^2}{2}\right) \rho\ \frac{\beta_v^2\ E}{dE/ds}\ \frac{\Delta p}{p}$$

This can be compared with the cooling term

$$\frac{\Delta\epsilon_z}{\epsilon_z} = -\ J_z\ \frac{dp}{p}$$

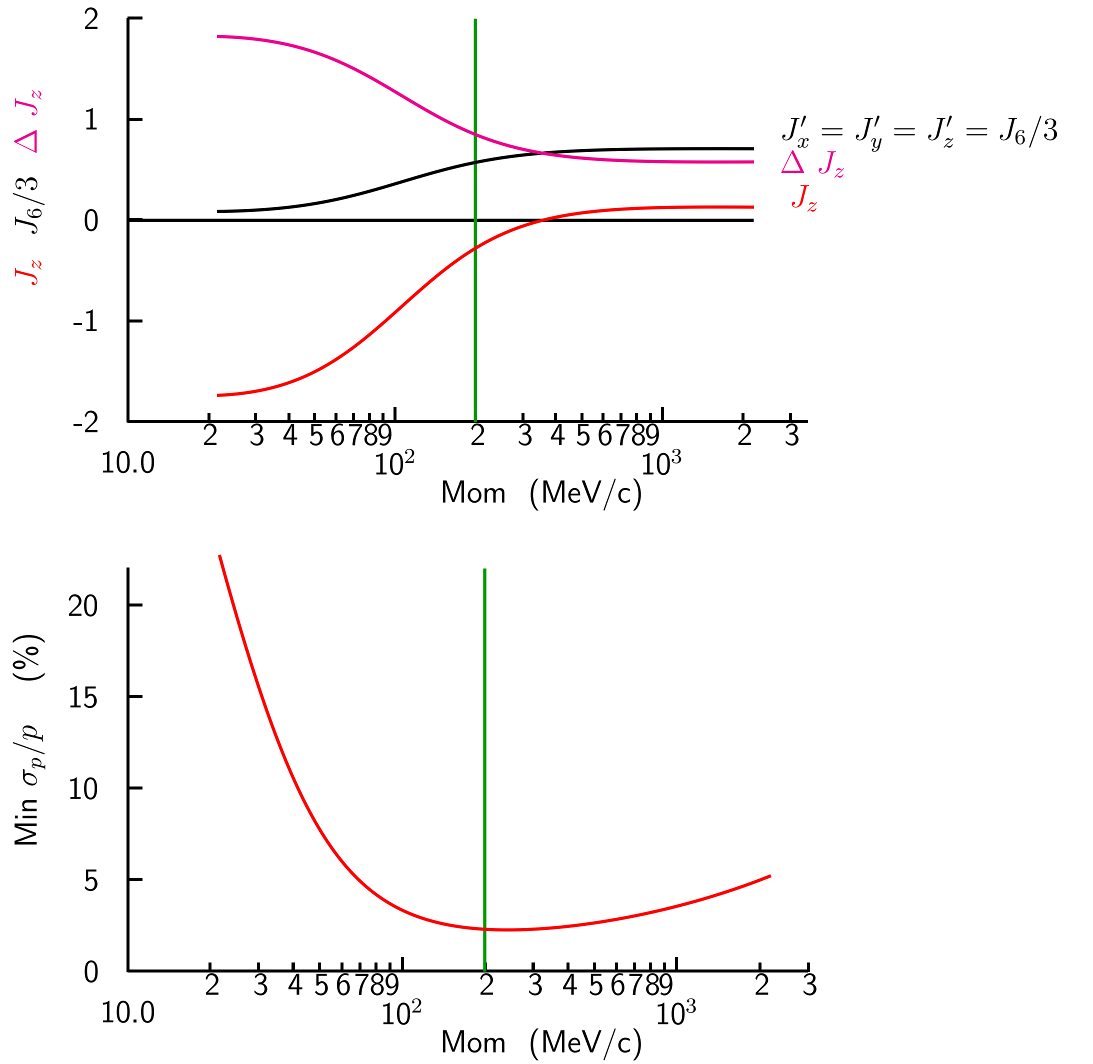
giving an equilibrium:

$$\frac{\sigma_p}{p} = \left(\left(\frac{m_e}{m_\mu}\right) \sqrt{\frac{0.06\ Z\ \rho}{2\ A\ (d\gamma/ds)}}\right) \sqrt{\frac{\gamma}{\beta_v^2} \left(1-\frac{\beta_v^2}{2}\right) \quad \frac{1}{J_z}} \tag{39}$$

For Hydrogen, the value of the first parenthesis is ≈1.36 %.

.

Without coupling, J_z is small or negative, and the equilibrium does not exist. But with wedges to generate $deltaJ_z$s to give equal partition functions (see plot), then, since J_6 remains positive at all momenta, $J_z = J_6/3$ will also remain positive, and there will be equilibrium longitudinal emittances plotted here.



It is seen to favor cooling at around 200 MeV/c, but has a broad minimum.

2.5.1 rf and bunch length

Above we have determined the equilibrium σ_p/p, but to obtain the Longitudinal emittance $\epsilon_z = \beta_v \gamma \sigma p/p \sigma_z$ we need σ_z.

If the rf acceleration is relatively uniform along the lattice, then the synchrotron wavelength[1] :

$$\lambda_s \;=\; \sqrt{\frac{2\pi\beta_v^2\lambda_{rf}\gamma \;\; m_\mu)}{-\mathcal{E}_{\rm rf}\eta\; \cos(\phi)}} \tag{40}$$

where $\mathcal{E}_{\rm rf}$ is the rf accelerating field, ϕ is the rf phase, ($\phi = 0$ has zero acceleration) and η is the frequency slip factor

$$\eta \;=\; -\frac{\frac{dv_z}{v_z}}{\frac{dp}{p}} \;=\; \alpha \;-\; \frac{1}{\gamma^2} \tag{41}$$

For a linear lattice the momentum compaction $\alpha = 0$ and

$$\lambda_s \;=\; \sqrt{\frac{2\pi\beta_v^2\lambda_{rf}^3\gamma \;\; m_\mu)}{\mathcal{E}_{\rm rf}\; \cos(\phi)}}$$

The bunch length, given the relative momentum spread $dp/p = \delta$, is given by[2]:

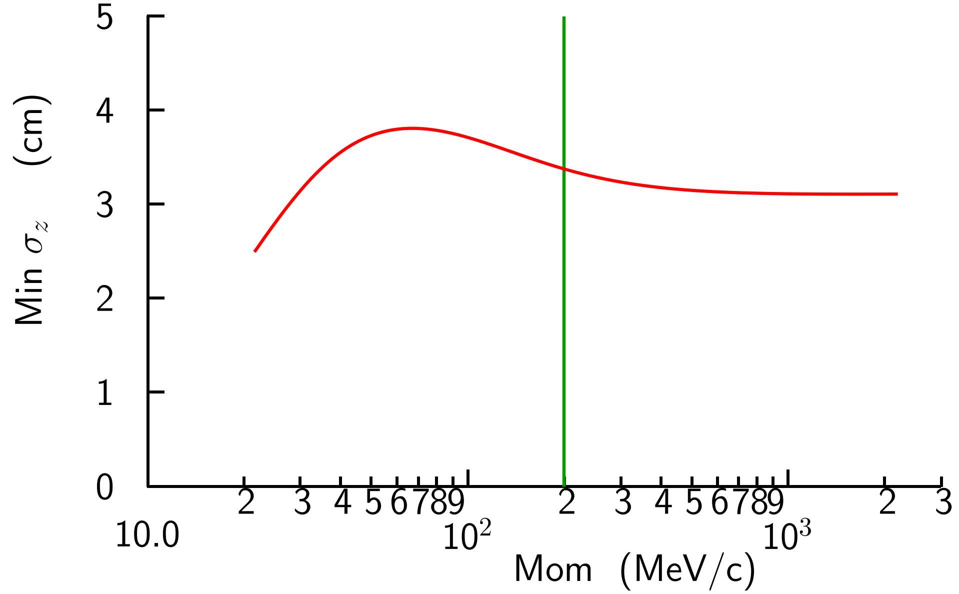
$$\sigma_z \;=\; \delta\;\beta_v\frac{-\eta\;\lambda_s}{2\pi} \;=\; \delta\;\beta_v^2\sqrt{\frac{-\eta\lambda_{rf}\;m_\mu\gamma}{2\pi\;\mathcal{E}\;\;\cos(\phi)}} \tag{42}$$

[1] e.g. S.Y. Lee "Accelerator Physics", eq 3.27
[2] e.g. S.Y. Lee "Accelerator Physics", eq 3.55

which for a linear lattice gives

$$\sigma_z = \delta \, \beta_v^2 \sqrt{\frac{\lambda_{rf} \, m_\mu}{2\pi \, \mathcal{E} \, \cos(\phi) \, \gamma}} \qquad (43)$$

This is only weakly dependent on the energy.

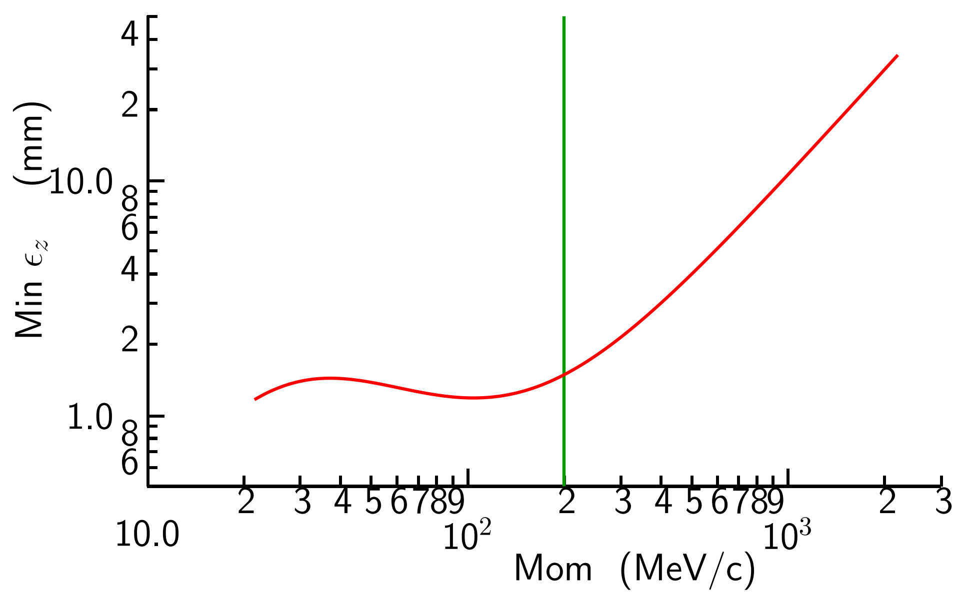


The longitudinal emittance

$$\epsilon_z \;=\; \beta_v\gamma\,\frac{\sigma_p}{p}\,\sigma_z$$

is seen to be rather flat below 200 MeV/c, but rises linearly with momentum above, thus favoring a momentum around 200 MeV/c

The equilibrium longitudinal emittance could be reduced if $(-\eta)$ could be decreased, e.g. by having a suitable positive momentum compaction α. In principle $(-\eta)$, and thus σ_z and ϵ_z(min) could be reduced to arbitrarily small values as the lattice approaches its "transition energy". A simple bend, or a helical lattice, would do this, but a linear wiggler would have a negative α and make things worse



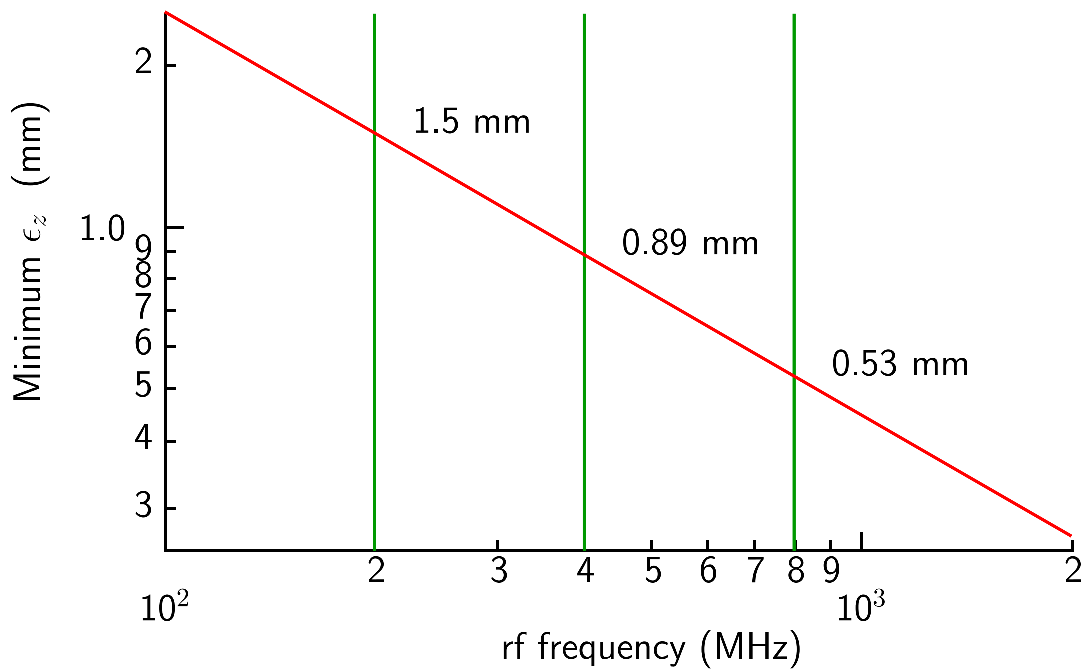
We see that the equilibrium longitudinal emittance

$$\epsilon_z \;\propto\; \sigma_z \;\propto\; \sqrt{\frac{\lambda_{\rm rf}}{\mathcal{E}}}$$

In general the maximum accelerating gradient limiting $\mathcal{E}$ is $\propto\ 1/\sqrt{\lambda_{\rm rf}}$ so

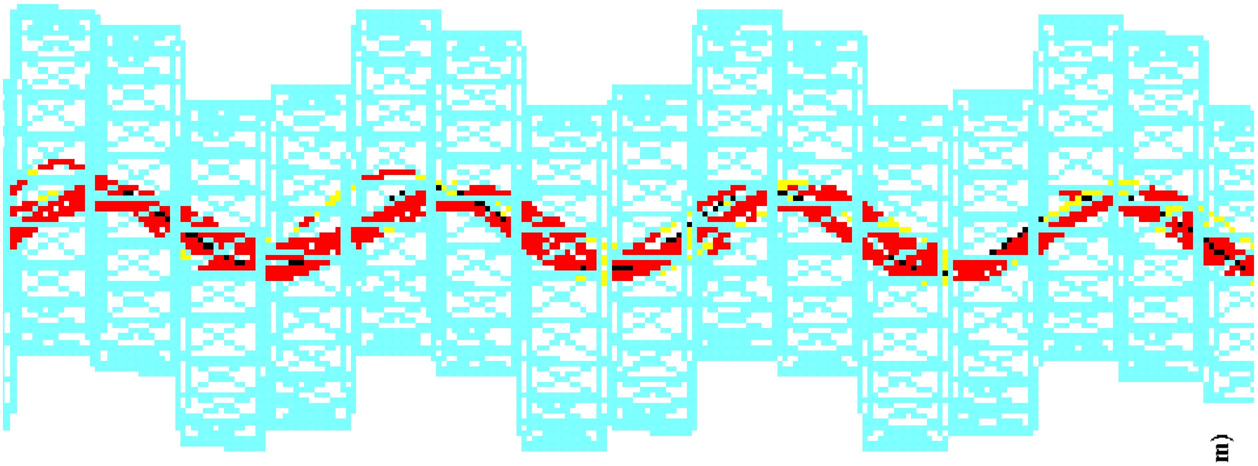
$$\epsilon_z \;\propto\; \gamma^{3/4}$$

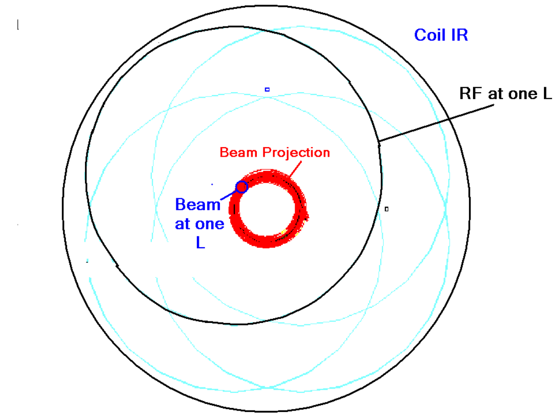
We have plotted this minimum for 200 MeV/c

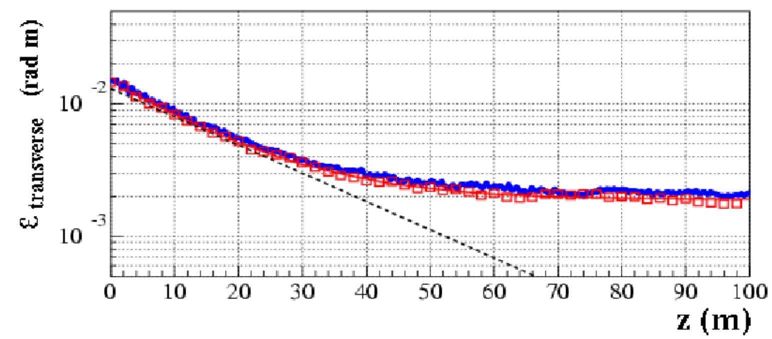


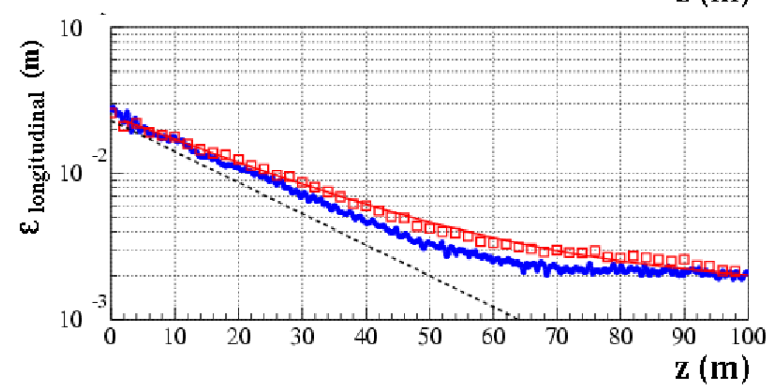
2.6 Example 1) In Gas with Helical Field

(Derbenev, Rol Johnson, Muons Inc.) [3]







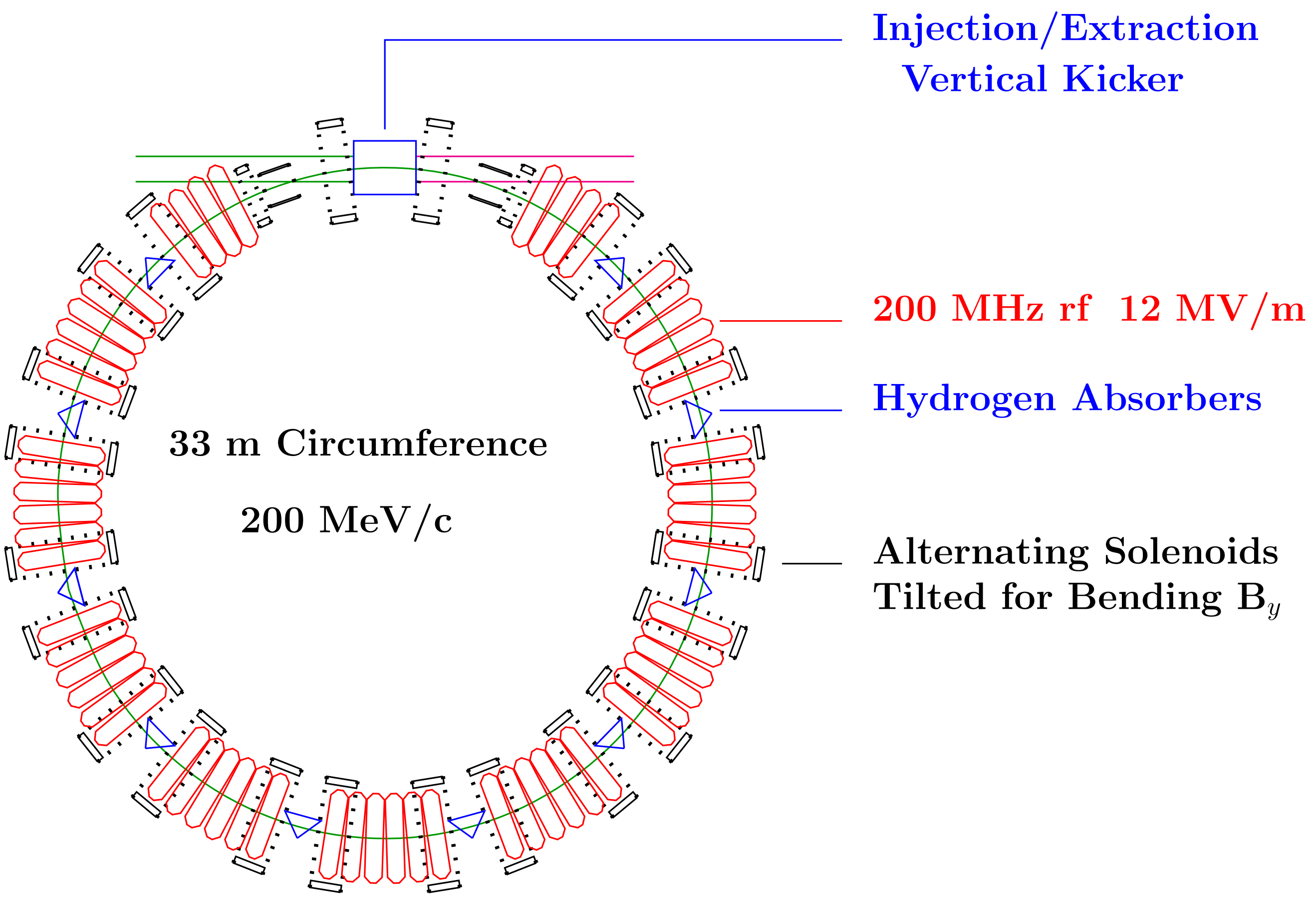


- Hydrogen gas filled
- Higher momenta have longer paths gives cooling in 6 dimensions
- Moderate fields at beam
  Bz=3.5 T. Br=.5 T
- But very high fields if outside rf
- Problem of integrating rf not yet solved

[3] MUC 185 and 284

2.7 Example 2) RFOFO Ring

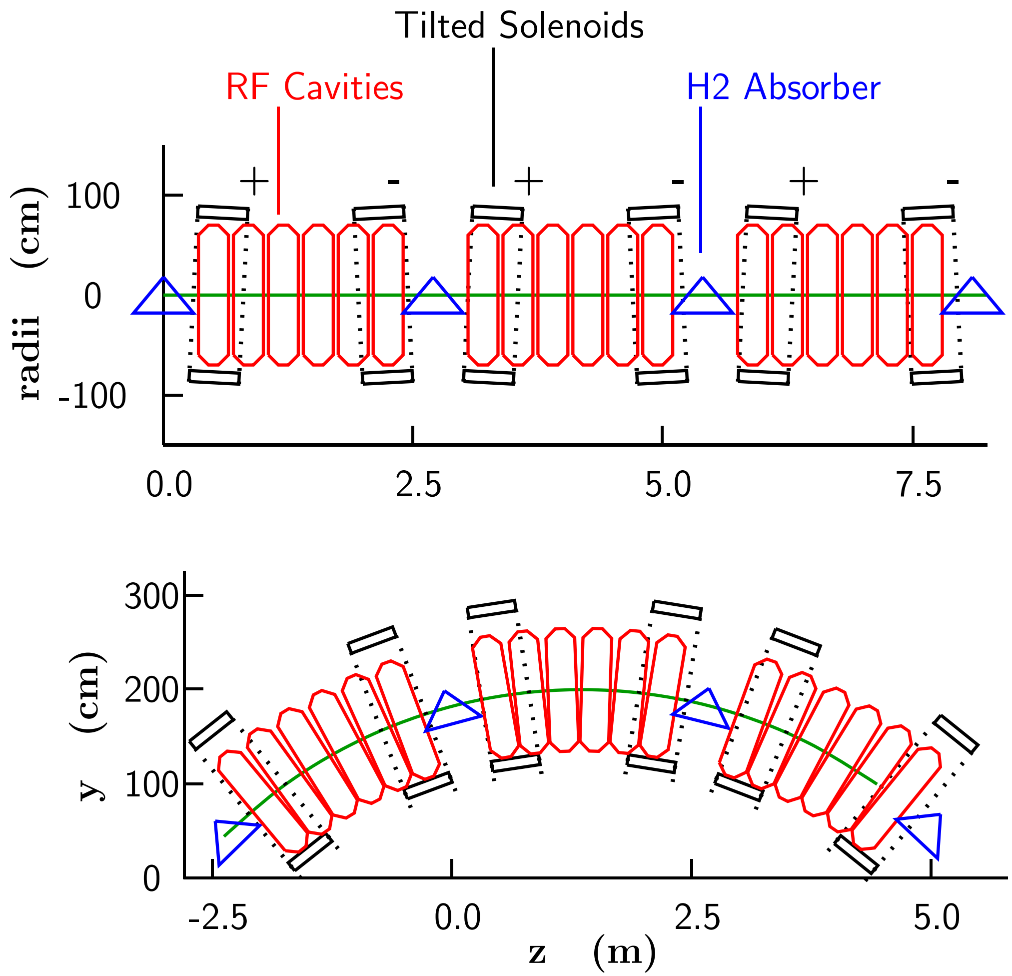
R.B. Palmer R. Fernow J. Gallardo[4], and Balbekov[5]



[4] Fernow and others: MUC-232, 265, 268, & 273
[5] V.Balbekov "Simulation of RFOFO Ring Cooler with Tilted Solenoids" MUC-CONF–0264

(

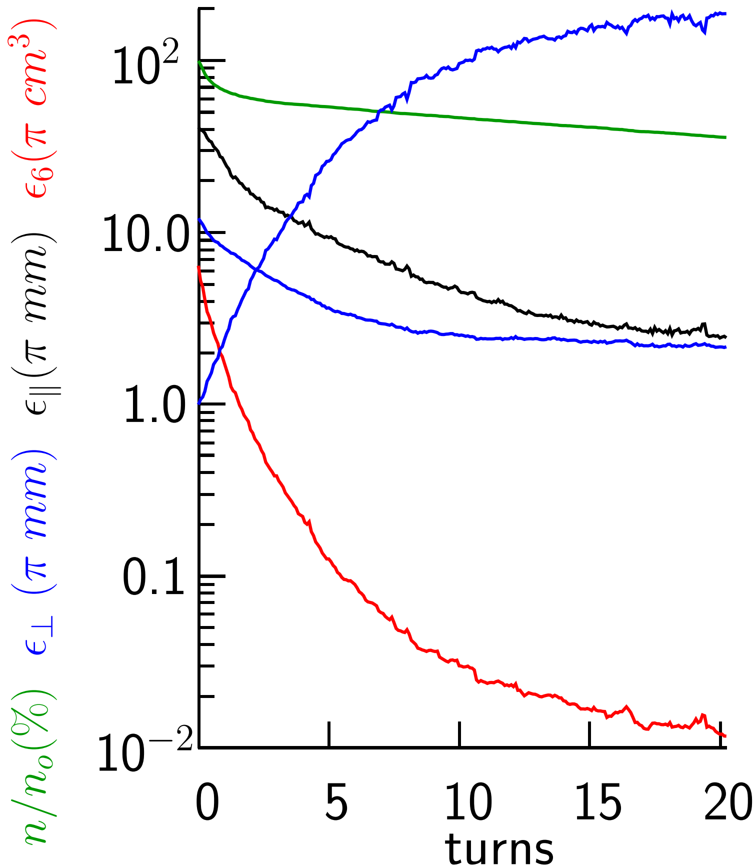
- Coils tilted to generate 0.125 T average bending
- SFOFO would have new resonances because periodicity 5.5 m vs. 2.75 m
- RFOFO cells all the same so new resonances avoided



Performance

Using Real Fields, but no windows or injection insertion

$$\text{Merit} = \frac{n}{n_o}\,\frac{\epsilon_{6,o}}{\epsilon_6} = \frac{\text{Initial phase density}}{\text{final phase density}}$$



final rms $dp/p \approx 3.5$ %

5) Space Charge Effects

From S Y Lee (p109),for a uniform charge density, where $\epsilon_\perp$ is the normalized transverse emittance:

$$\frac{\Delta\nu_{\rm flat}}{L} = \left(\frac{N_\mu}{\sqrt{2\pi}\ \sigma_z}\right)\ \frac{r_\mu}{2\pi\ \epsilon_\perp\ \beta_v\gamma^2}$$

For a Gaussian distribution:

$$\frac{\Delta\nu_{\rm Gausian}}{L} = \left(\frac{N_\mu}{2\sqrt{2\pi}\ \sigma_z}\right)\ \frac{r_\mu}{2\pi\ \epsilon_\perp\ \beta_v\gamma^2}$$

This is true INDEPENDENT of $\beta_\perp$

For convenience I define

$$\beta_{\perp\ \rm ave} = \left(\frac{L_{\rm cell}}{2\pi\ \nu_{\rm cell}}\right)$$

Then:

$$\frac{\Delta\nu_{\rm Gaussian}}{\nu_{\rm cell}} = \left(\frac{N_\mu}{\epsilon_\perp}\right)\ \frac{\beta_{\perp\ \rm ave}\ r_\mu}{2\sqrt{2\pi}\sigma_z\ \ \beta_v\gamma^2}$$

where $r_\mu = 1.35\ 10^{-17}$ (mm),

Examples

$R_\mu = 1.35\ 10^{-17}$ (m)

Note that N_μ is larger at earlier cooling stages to allow for losses

case	N_μ 10^{12}	$< \beta_\perp >$ m	σ_z m	$\epsilon_\perp$ mm mrad	p MeV/c	$\Delta\nu/\nu$
Last 50 T cooling	2.8	0.3	4	25	50	0.05
Last RFOFO Guggenheim	4	0.19	0.025	400	200	0.11
First RFOFO Guggenheim after merge	6	0.6	0.02	2000	200	0.12

- Negligible problem in the 50 T solenoids
  They operate in the first pass band & can tolerate large $\Delta\nu/\nu$

- Finite effect in Guggenheim RFOFO lattices
  The accepted $\Delta\nu/\nu$ between the resonaces at $\nu = .5$ and $\nu = 1.0$

$$\frac{\Delta\nu(\text{accepted})}{\nu} \approx \frac{0.5}{0.75} \approx 0.67$$

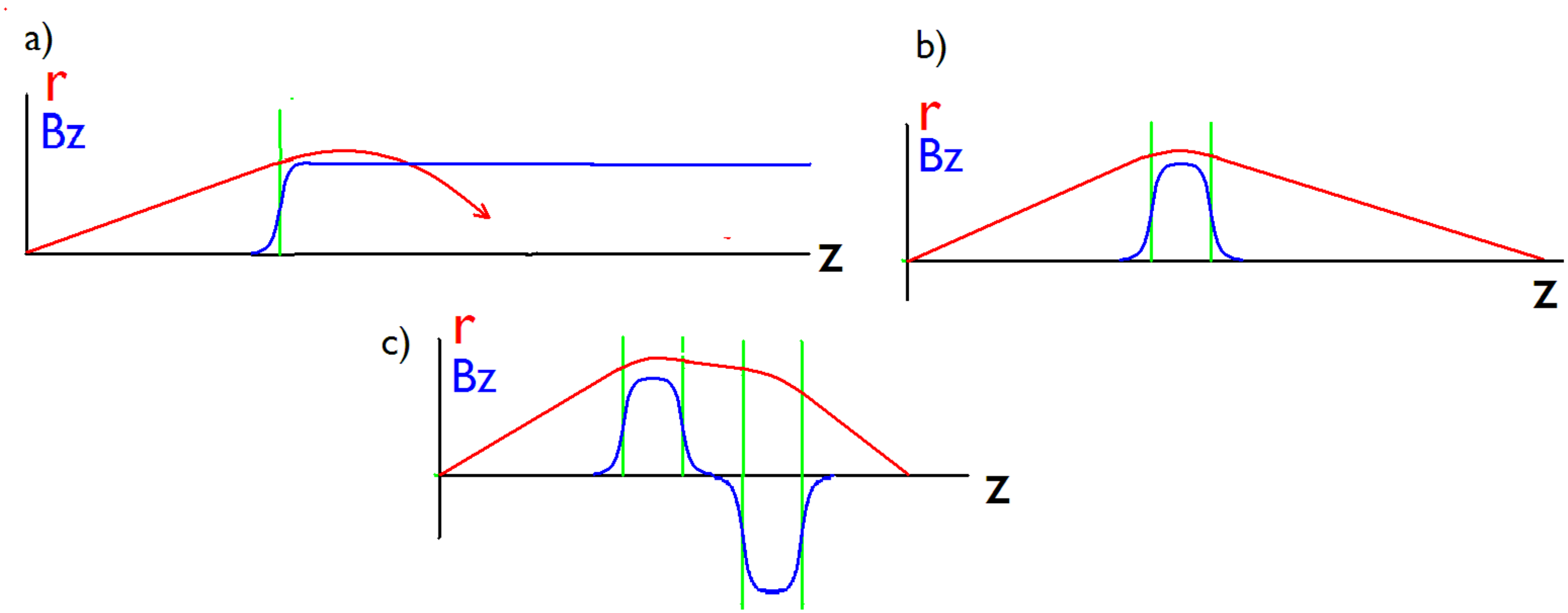
  so tune spreads of 0.11 & 0.12 will somewhat reduce momentum acceptance
  This needs to be included in simulations

3 TUTORIALS

1. Depth of Focus

If a depth of focus is defined as that distance from a focus where the spot size has increased by $\sqrt{2}$ then what is this depth in terms of $\beta_\perp$ and/or $\epsilon_\perp$?

2. Solenoid focus



Assuming the axis of symmetry is z

(a) Sketch the xy view of a track starting on axis and then entering a long solenoid

(b) Sketch the xy view of a track starting on axis, then passing through a thinnish solenoid

(c) Sketch the xy view of a track starting on axis then passing through two thinnish solenoids with opposite fields, with the second having a somewhat lower field than the first

(d) If in a long solenoid a paraxial particle makes two helix turns in a length $\lambda_o \quad (= 4\pi\, p/Bc)$, what is λ_θ for $\theta = 45^o$, and approximately for $\theta = 0.3$?

(e) What is the fractional momentum changes that would generate the same changes in focusing ?

3. Transverse cooling in a continuous solenoid

(a) What is the $\beta_\perp$ for $B_z = 10\ T$ and $p\ =\ 200\ MeV/c$?

(b) What is the approximate equilibrium emittance ?

(c) And for $B = 50\ T$ and muons of kinetic energy $E = 10\ MeV$

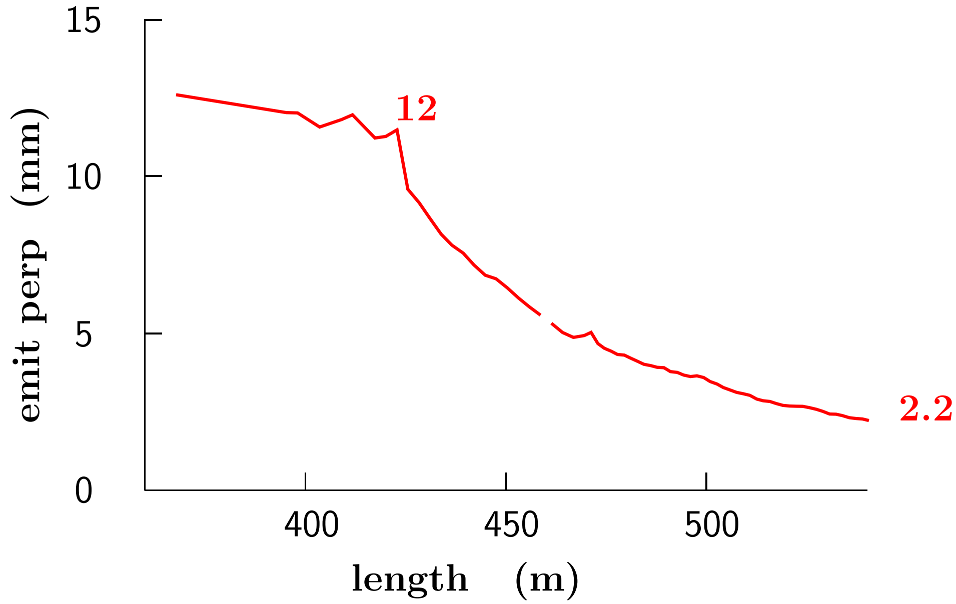
4. Transverse cooling in a Lithium Lens

(a) What is the $\beta_\perp$ in a current carrying rod B_{max}=10 T, p=176 MeV/c and radius=2.35 cm?

(b) What is the current

(c) What is the equilibrium emittance and rms beam size ?

5. Study 2 performance



mom=200 MeV/c

final emittance = 2.2 π mm radians

final $\beta_\perp$=18 cm

With RF and Hydrogen Windows, C$_o$ $\approx$ $45\ 10^{-4}$

(a) What is equilibrium emittance

(b) What is final rate of cooling

$$rate = \frac{d\epsilon_\perp/\epsilon_\perp}{dp/p}$$

6. Choice of cooling energy

Without scattering, the rate of cooling strongly favors low energy to minimize the amount of rf needed to re accelerate (eq. 18:

$$\frac{d\epsilon}{\epsilon} = \frac{dp}{p} = \frac{dE}{E}\,\frac{1}{\beta_v^2}$$

But if total acceleration were not important, e.g. if the cooling is done in a ring, then there is another criterion: The cooling per fractional loss of particles by decay:

(a) Calculate the rate of cooling vs particle loss:

$$Q = \frac{d\epsilon/\epsilon}{dn/n}$$

(b) How does it depend on the cooling momentum

(c) Would the criterion change if we were interested in rate of 6 dimensional cooling

7. RFOFO performance

Assuming

$D = 7$ cm, $\ell = 28.6$ cm,

Wedge opening angle 110 degrees

Momentum 200 MeV/c where $J_z \approx -0.3$, and for hydrogen $C_o = 38\ 10^{-4}$

$\beta_\perp =$ 40 cm

(a) what is ΔJ_z

(b) what are $J_x = J_y$ and J_z

(c) what is the calculated equilibrium transverse emittance

(d) what is the calculated rms dp/p

(e) compare with simulated performance

8. Space charge tune shift

(a) express the space charge tune shift $\Delta\nu/\nu$ as a function of $(N_\nu/\epsilon_\perp)$, $\epsilon_\parallel$ and dp/p

(b) Assuming a beam beam tune shift limited collider ring, is the space charge tune shift problem during cooling reduced if, we use more bunches with smaller numbers of muons per bunch (N_ν) ?

(c) Are there other reasons one prefers fewer muons per bunch ?

(d) Again assuming a beam beam tune shift limited collider ring, is the space charge tune shift problem during cooling reduced if we cool more bunches with smaller numbers of muons per bunch (N_ν) and recombine them after cooling, before injecting them into the collider ring ?

(e) What can one do to minimize the space charge tune shift problem while attempting to reach the smallest possible $\epsilon_6 = \epsilon_x\epsilon_y\epsilon_z$?