FIFTH CONFERENCE ON NUCLEI AND MESOSCOPIC PHYSICS

6-10 March 2017 **Michigan State University**

On Bulk, Boundaries, and All That...

Characterization and Design of Topological Boundary Modes via Generalized Bloch's Theorem

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Broad context

<u>Challenge</u>: To fully characterize nature and implications of topological quantum matter.

- Fundamental significance across condensed-matter physics:
  - ➞ Can we find a *complete classification/unified theory* of TQM (beyond Landau paradigm)?
    - ✓ Topological quantum order – interacting systems, (holographic) symmetries?...
    - ✓ Interplay between bulk and boundary physics – surface states, band topology...
  - ➞ How to *experimentally, unambiguously* detect TQM, at equilibrium and beyond?
    - ✓ Spectroscopic and transport signatures...
    - ✓ Thermodynamic signatures?...

"for theoretical discoveries of topological phase transitions and topological phases of matter"

Broad context

Challenge: To fully characterize nature and implications of topological quantum matter.

- Conceptual and practical significance across quantum science:
  - ➞ Alternative, 'hardware-based' route to *fault-tolerant quantum computation*...

    Kitaev, Ann. Phys. **303**, 2 (2003).
    - ✔ What *exactly* is being 'topologically protected'?...
    - ✔ To what extent can topological protection function in *realistic* system-control settings?...

Topological quantum computation

Sankar Das Sarma, Michael Freedman, and Chetan Nayak

The search for a large-scale, error-free quantum computer is reaching an intellectual junction at which semiconductor physics, knot theory, string theory, anyons, and quantum Hall effects are all coming together to produce quantum immunity.

[Phys. Today (July 2006)]

PHYSICAL REVIEW X **6,** 031016 (2016)

Milestones Toward Majorana-Based Quantum Computing

David Aasen,[1] Michael Hell,[2,3] Ryan V. Mishmash,[1,4] Andrew Higginbotham,[5,3] Jeroen Danon,[3,6] Martin Leijnse,[2,3] Thomas S. Jespersen,[3] Joshua A. Folk,[3,7,8] Charles M. Marcus,[3] Karsten Flensberg,[3] and Jason Alicea[1,4]

Broad context

Challenge: To fully characterize nature and implications of topological quantum matter.

- Conceptual and practical significance across quantum science:
  - ➞ *Many-body quantum-control engineering:* Leverage control capabilities to design states of matter or phenomena not accessible otherwise [cf. Rudner's talk]...
    - ✔ Most general setting: *Open quantum-system dynamics*

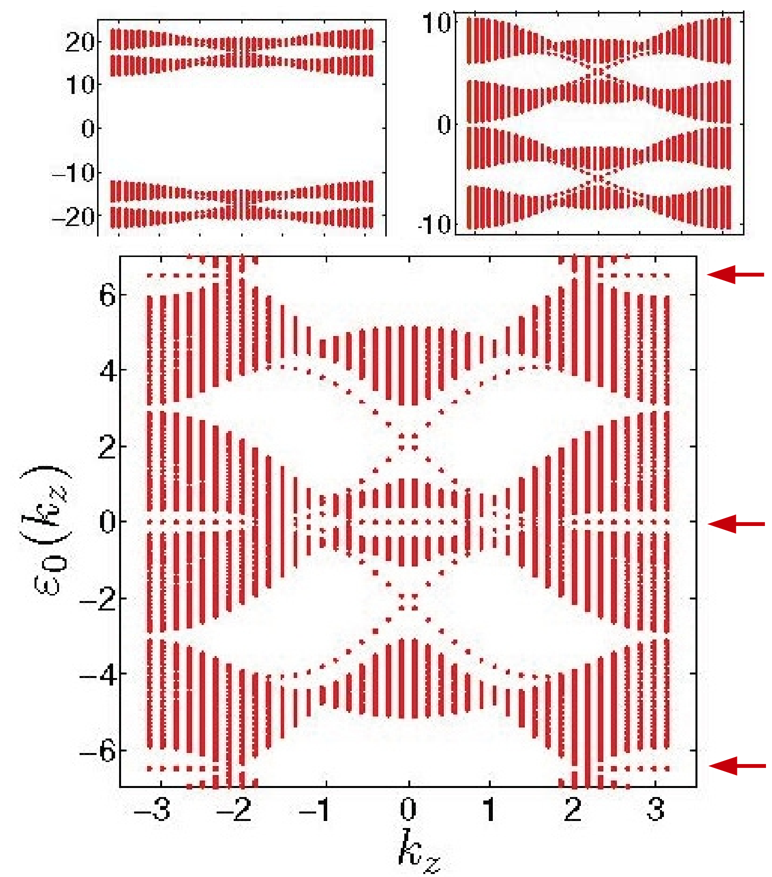
      LV & Lloyd, PRA(R) **65** (2001).

    - ✔ *Hamiltonian engineering* – e.g., Floquet driving

      A. Poudel, G. Ortiz & LV, *Dynamical generation of Floquet Majorana flat bands in s-wave superconductors*, EPL **110**, 17004 (2015).

    - ✔ *Dissipative engineering* – Lindblad or Kraus dynamics

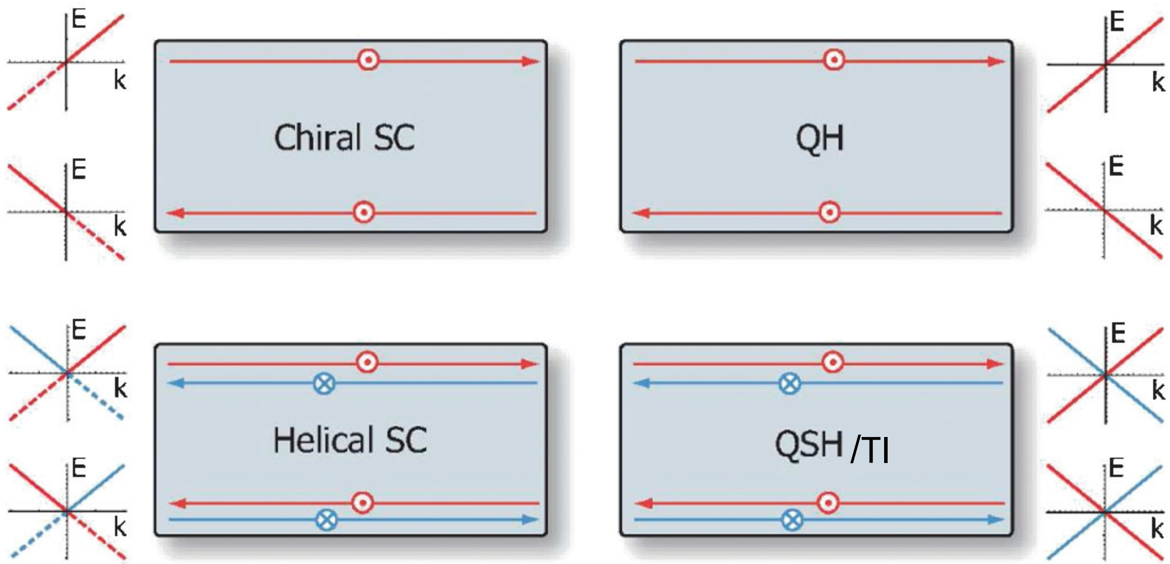
      P.D. Johnson, F. Ticozzi & LV, *General fixed points of quasi-local frustration-free quantum semigroups: From invariance to stabilization*, QIC **16**, 0657 (2016).



Focus: Topological fermionic matter

Topological insulators and superconductors are fully gapped [or nodal] phases of fermionic matter which support *'symmetry protected' mid-gap boundary modes.*

Qi & Zhang, Rev. Mod. Phys. **83** (2011); Chiu, Teo, Schnyder & Ryu, *ibid.* **88** (2016).



Broken TR invariance ⇒ *Chiral* boundary modes

TR invariance preserved ⇒ *Helical* boundary modes

➞ Phenomenological understanding based on *non-interacting* [mean-field] models

- ✓ TI ⇒ Odd number of pairs of helical edge modes/Dirac cone surface modes protected by TRI
- ✓ TSC ⇒ Odd number of [zero-energy] Bogoliubov-quasiparticle boundary modes obeying *Majorana statistics* $\gamma(\epsilon) = \gamma(-\epsilon)^\dagger \;\Rightarrow\; \gamma(0) = \gamma(0)^\dagger, \quad \{\gamma_\ell, \gamma_m\} = 2\delta_{\ell m}\mathbb{I}$

Bulk-boundary correspondence

<u>Key intuition</u>: Joining two topologically distinct *bulk* phases mandates the emergence of states localized near/on the *boundary* – in a way that is *robust* against 'local' perturbations...

- More formally, *bulk-boundary correspondence* (BBC) defines the relation between

Bulk topological invariants [e.g., Chern number]	⇔	*Number of boundary modes* [or *number of pairs* thereof], *mod 2*.

➞ Powerful principle, numerically validated in several cases, and rigorously established in a few special instances – 1D quantum walks, 2D TIs...

Fu & Kane, PRB **74**, (2006)...Kitagawa, QIP **11** (2012);
Graf & Porta, CMP **324** (2013), Cedzich et al, JPA **49** (2016)...

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- Still, no complete rigorous theory nor *general analytic, physical* insight available as yet...

→ *Genesis* of boundary modes: Exactly, how do they come about?...

→ *Robustness* of boundary modes: Exactly, what is the interplay between bulk/ boundary?...

✓ Response to boundary perturbations is key to topological robustness...

✓ Robustness against changes of BCs influences stationary bulk symmetries after a quench.

Isaev, Moon, Ortiz, PRB **84** (2011), Fagotti, J. Stat. Mech. (2016)...

→ Exactly, what does this all mean at the basic level of dynamical-system theory?...

Motivation & Outline

- Goal: Develop an analytic approach to the BBC, starting from the 'minimal setting' of clean, finite-range, non-interacting fermionic lattice systems at equilibrium.

  ➡ *Space-translation invariance broken only by boundary conditions.*

Outline: *A generalization of Bloch[-Floquet]'s theorem beyond torus topology...*

I. Exact solution of 1D free-fermion lattice models with boundaries – Characterization and design of topological boundary modes (⇒ 'power-law' modes), new indicators for BBC

- ✓ New perspectives on transfer-matrix approach...
- ✓ New role for *non-unitary representations of translation symmetry*...

A. Alase, E. Cobanera, G. Ortiz & LV, *Exact solution of quadratic fermionic Hamiltonians for arbitrary boundary conditions*, Phys. Rev. Lett. 117, 076804 (2016).

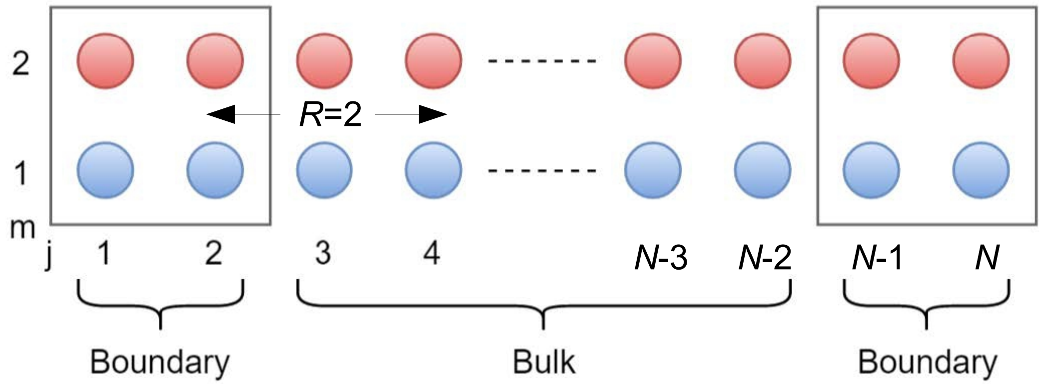
E. Cobanera, A. Alase, G. Ortiz & LV, *Exact solution of corner-modified banded block-Toeplitz eigensystems*, ArXiv:1612.05567, J. Phys. A: Math., in press (2017).

A. Alase, E. Cobanera, G. Ortiz & LV, *A generalization of Bloch's theorem for arbitrary boundary conditions: Theory*, Phys. Rev. B, in preparation (2017).

II. Further properties – D>1 extensions, topological band structure and mesoscopic applications

E. Cobanera, A. Alase, G. Ortiz & LV, *in progress...*

Tight-binding models with boundaries



$R \ll N$

PBC: $h_r = g_r$
OBC: $g_r \equiv 0$
...

- Case study: Finite-range disorder-free quadratic fermionic Hamiltonians on $D=1$ lattice ⇒ Diagonalizing single-particle (BdG) Hamiltonian suffices to diagonalize many-body problem

Hopping and pairing among fermions located r cells apart:
in the bulk
at the boundary

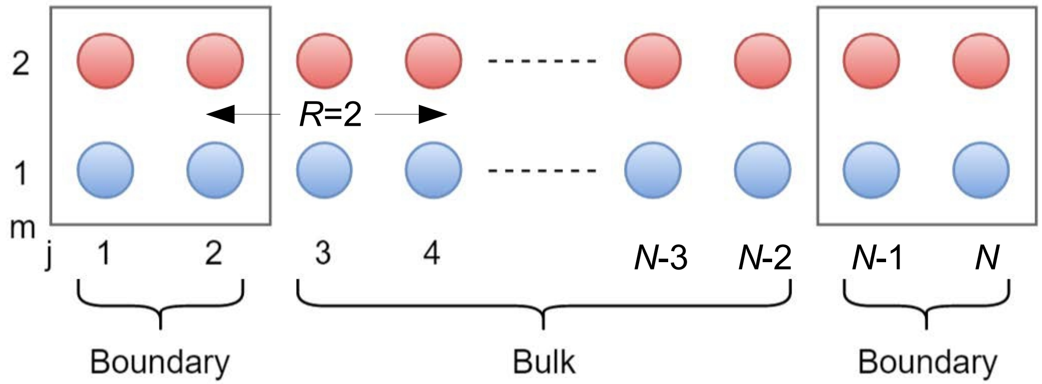
$$\widehat{H} = \frac{1}{2}\sum_{j=1}^{N} \psi_j^\dagger h_0 \psi_j + \frac{1}{2}\sum_{r=1}^{R}\left[\sum_{j=1}^{N-r}\left(\psi_j^\dagger h_r \psi_{j+r} + \text{h.c.}\right) + \sum_{b=N-R+1}^{N}\left(\psi_b^\dagger g_r \psi_{b+r-N} + \text{h.c.}\right)\right]$$

$$= \frac{1}{2}\Psi^\dagger H \Psi \equiv \frac{1}{2}\Psi^\dagger\left[H_N + W\right]\Psi$$

Nambu basis $\quad \Psi^\dagger \equiv [\psi_1^\dagger \ldots \psi_N^\dagger], \quad \psi_j^\dagger \equiv [c_{j,1}^\dagger c_{j,2}^\dagger \ldots c_{j,d}^\dagger c_{j,1} \ldots c_{j,d}]$

➞ This *non-conventional* ordering of the Nambu basis highlights the role of translation symmetry...

Tight-binding models with boundaries



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PBC: $h_r = g_r$
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...

- Single-particle Hamiltonian takes the structure of a banded, block quasi-Toeplitz matrix, with Toeplitz structure broken by boundaries ⇒ 'corner-modified' banded block-Toeplitz matrix:

$$H = H_N + W$$

$$H_N = \begin{bmatrix} h_0 & \dots & h_R & & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & & \ddots & \vdots \\ h_R^\dagger & & \ddots & & \ddots & & 0 \\ & \ddots & & & & & \\ & & & & & \ddots & \\ 0 & & \ddots & & \ddots & & h_R \\ \vdots & \ddots & & \ddots & & \ddots & \vdots \\ 0 & \cdots & 0 & & h_R^\dagger & \cdots & h_0 \end{bmatrix}, \quad W = \begin{bmatrix} w_{11}^{(l)} & \dots & w_{1R}^{(l)} & 0 & w_{11} & \dots & w_{1R} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{R1}^{(l)} & \dots & w_{RR}^{(l)} & \vdots & w_{R1} & \dots & w_{RR} \\ 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\ w_{11}^\dagger & \dots & w_{1R}^\dagger & \vdots & w_{11}^{(r)} & \dots & w_{1R}^{(r)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{R1}^\dagger & \dots & w_{RR}^\dagger & 0 & w_{R1}^{(r)} & \dots & w_{RR}^{(r)} \end{bmatrix}$$

Shift operators

- Strategy: Try to mimic the success story of Fourier transform by making it explicit that a translation-invariant Hamiltonian may *still* be constructed in a mathematically precise sense...

  ➞ Introduce *subsystem decomposition* on single-particle space:

$$\mathcal{H} \simeq \mathbb{C}^N \otimes \mathbb{C}^{2d} \equiv \text{span}\{|j\rangle|m\rangle \,|\, 1 \leq j \leq N;\, 1 \leq m \leq 2d\}$$

  ➞ Introduce *discrete translation operators* on the lattice degree of freedom:

$$T \equiv \sum_{j=1}^{N-1} |j\rangle\langle j+1|$$

Left-shift operator

$$V \equiv T + T^{\dagger N-1} = |N\rangle\langle 1| + \sum_{j=1}^{N-1} |j\rangle\langle j+1|$$

Cyclic- shift operator

  ➞ For *periodic BC* (torus topology): Single-particle Hamiltonian is invariant under *cyclic shifts*

  ⇒ Circulant block-Toeplitz matrix

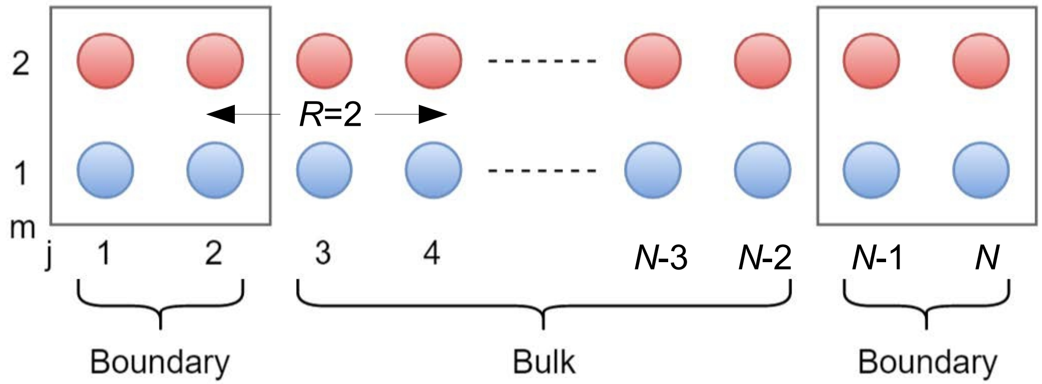
$$H = H_N + W = \mathbb{I}_N \otimes h_0 + \sum_{r=1}^{R} (V^r \otimes h_r + \text{h.c.}) \qquad (g_r = h_r)$$

  ✔ Diagonalization may be carried out via discrete Fourier transform to momentum basis.

  ✔ *Simultaneous* eigenstates of H, V are the familiar Bloch's states.

  ➞ For arbitrary boundary conditions, $H, T, T^\dagger$ no longer commute. However, we can *isolate the effect of translation-symmetry-breaking by 'projecting out' the boundary...*

Bulk-boundary separation



Bulk and boundary projectors:

$$P_B \equiv \sum_{j=R+1}^{N-R} |j\rangle\langle j| \otimes \mathbb{I}$$

$$P_\partial \equiv \mathbb{I} - P_B, \;\; P_B W = 0$$

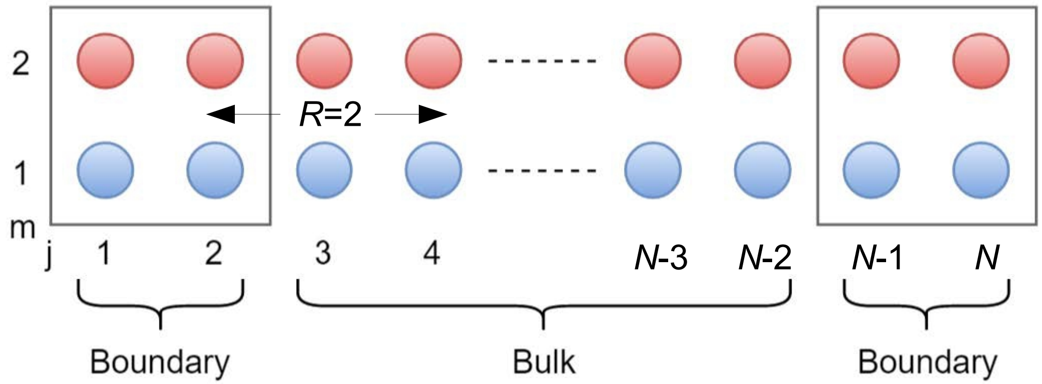
- Diagonalization problem for H may be *exactly* recast into the simultaneous solution of

$$\left\{ \begin{array}{r} P_B H_N|\epsilon\rangle = \epsilon\, P_B|\epsilon\rangle \\ (P_\partial H_N + W)|\epsilon\rangle = \epsilon\, P_\partial|\epsilon\rangle \end{array} \right.$$

BULK EQUATION

BOUNDARY EQUATION

Bulk-boundary separation



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$$\begin{cases} P_B H_N |\epsilon\rangle = \epsilon\, P_B |\epsilon\rangle & \text{BULK EQUATION} \\ (P_\partial H_N + W)|\epsilon\rangle = \epsilon\, P_\partial |\epsilon\rangle & \text{BOUNDARY EQUATION} \end{cases}$$

- Key advantage of this separation: We can naturally identify a *translation-invariant* Hamiltonian

$$H = H_N + W, \quad H_N = \sum_{r=0}^{R} (T^r \otimes h_r + T^{r\,\dagger} \otimes h_r^\dagger)$$

$$\mathbf{H} = \mathbb{I} \otimes h_0 + \sum_{r=1}^{R} [\mathbf{T}^r \otimes h_r + (\mathbf{T}^{-1})^r \otimes h_r^\dagger], \quad \mathbf{T} \equiv \sum_{j=-\infty}^{\infty} |j\rangle\langle j+1| \;, \quad [\mathbf{T}, \mathbf{H}] = 0$$

→ $\mathbf{H}$ is an infinite, *banded block-Laurent matrix*, whose eigensolutions *also* solve the bulk equation!

Solution approach: Bulk equation

- <u>Step 1</u>: Obtain *eigenvalue-dependent* Ansatz for the solutions to the bulk equation,

$$P_B H_N |\epsilon\rangle = \epsilon\, P_B |\epsilon\rangle \Leftrightarrow |\epsilon\rangle \in \mathrm{Ker}\, P_B (H_N - \epsilon)$$

➡ Key observation: For arbitrary ϵ, it is easy to compute and store a basis of the kernel of a corner-modified BBT matrix – *complexity is independent of system size,* N.

➡ For *generic* ('regular') ϵ and parameter values, *all* solutions arise as solutions of the infinite BBL system, which *<u>is</u>* translation-invariant: kernel determination entails solving a *polynomial equation of low degree:*

$$H_B(z) \equiv h_0 + \sum_{r=1}^{R} (z^r h_r + z^{-r} h_r^\dagger) \;\Rightarrow\; z^{2dR} \det(H(z) - \epsilon) \equiv c \prod_{\ell=0}^{n} (z - z_\ell)^{s_\ell} = 0$$

'Reduced bulk Hamiltonian' $H_B(z)$ is the analytical continuation of Bloch Hamiltonian off the BZ...

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✓ Generic case: $\mathcal{M}_N = \mathbf{P}_N \mathcal{M}_\infty, \quad \det h_R \neq 0$ $\Rightarrow$

Quasi-invariant solutions:
Generalized eigenvectors of $\mathbf{T}$
Extended spatial support

✓ Non-invertible case: *Additional* solutions may emerge because of projection from infinite-to-finite system,

$$\mathbf{H} \mapsto H_N,\ \det h_R = 0$$

$\Rightarrow$

Emergent solutions:
Finite spatial support
(Perfectly) boundary-localized

- Step 2: Impose BCs, by using Ansatz to select solutions that *also* solve the boundary equation,

$$(P_\partial H_N + W)|\epsilon\rangle = \epsilon\, P_\partial|\epsilon\rangle \Leftrightarrow P_\partial(H-\epsilon)|\epsilon\rangle = 0$$

➡ Parametrize a basis of solutions of the bulk equation in terms of $4dR$ amplitudes:

$$|\epsilon,\vec{\alpha}\rangle = \sum_{\ell=1}^{n}\sum_{s=1}^{s_\ell}\alpha_{\ell s}|\psi_{\ell s}\rangle + \sum_{s=1}^{s_0}\alpha_s^+|\psi_s^+\rangle + \sum_{s=1}^{s_0}\alpha_s^-|\psi_s^-\rangle, \quad \alpha_{\ell s},\alpha_s^+,\alpha_s^- \in \mathbb{C}$$

Translation-invariant

Emergent

$$\vec{\alpha} \equiv \left[\alpha_{11}\dots\alpha_{ns_n}\alpha_1^+\dots\alpha_{s_0}^+\alpha_1^-\dots\alpha_{s_0}^-\right]^T$$

Solution approach: Boundary equation

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$$\vec{\alpha} \equiv \left[\alpha_{11}\ldots\alpha_{ns_n}\alpha_1^+\ldots\alpha_{s_0}^+\alpha_1^-\ldots\alpha_{s_0}^-\right]^T$$

➞ Using the above Ansatz, recast the boundary equation as the kernel equation of a $4dR \times 4dR$ boundary matrix B,

$$H_\epsilon \equiv H_N + W - \epsilon\mathbb{I}$$

$$B = \begin{bmatrix} \langle 1|H_\epsilon|\psi_{11}\rangle & \cdots & \langle 1|H_\epsilon|\psi_{ns_n}\rangle & \langle 1|H_\epsilon|\psi_1^+\rangle & \cdots & \langle 1|H_\epsilon|\psi_{s_0}^-\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle R|H_\epsilon|\psi_{11}\rangle & \cdots & \langle R|H_\epsilon|\psi_{ns_n}\rangle & \langle R|H_\epsilon|\psi_1^+\rangle & \cdots & \langle R|H_\epsilon|\psi_{s_0}^-\rangle \\ \langle N-R+1|H_\epsilon|\psi_{11}\rangle & \cdots & \langle N-R+1|H_\epsilon|\psi_{ns_n}\rangle & \langle N-R+1|H_\epsilon|\psi_1^+\rangle & \cdots & \langle N-R+1|H_\epsilon|\psi_{s_0}^-\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle N|H_\epsilon|\psi_{11}\rangle & \cdots & \langle N|H_\epsilon|\psi_{ns_n}\rangle & \langle N|H_\epsilon|\psi_1^+\rangle & \cdots & \langle N|H_\epsilon|\psi_{s_0}^-\rangle \end{bmatrix}$$

➞ Bulk solution $|\epsilon,\vec{\alpha}\rangle$ is an eigenvector of H *if and only if* $\det B = 0$.

Generalized Bloch theorem

Theorem. Let $H = H_N + W$ denote the Hamiltonian of a clean, finite-range lattice system with boundary conditions described by W. If ϵ is a (regular) eigenvalue of H of degeneracy κ, then the associated energy eigenstates may be taken to be of the form $|\,\epsilon, \vec{\alpha}_k\,\rangle$, where the $\{\vec{\alpha}_k, k = 1,\ldots, \kappa\}$ are a basis of the kernel of the boundary matrix $B(\epsilon)$.

- For generic energy and parameter values, no emergent solution exists, and the *generalized, translation-invariant Bloch states* are generalized eigenvectors of the translation operator $\mathbf{T}$ [*invertible but not unitary* on the space of all lattice sequences...]

$$|\psi_{\ell s}\rangle = \sum_{v=1}^{s_\ell} |z_\ell, v\rangle |u_{\ell s v}\rangle, \quad \ell = 1, \ldots, n;\ s = 1, \ldots, s_\ell$$

$$|z_\ell, v\rangle = \begin{cases} \sum_{j=1}^N z_\ell^j\, |j\rangle,\ v = 1 \\ \sum_{j=1}^N [j(j-1)\ldots(j-v+2)] z_\ell^{j-v+1}\, |j\rangle,\ v \geq 2 \end{cases}$$

Exponential Bloch wave with complex momentum

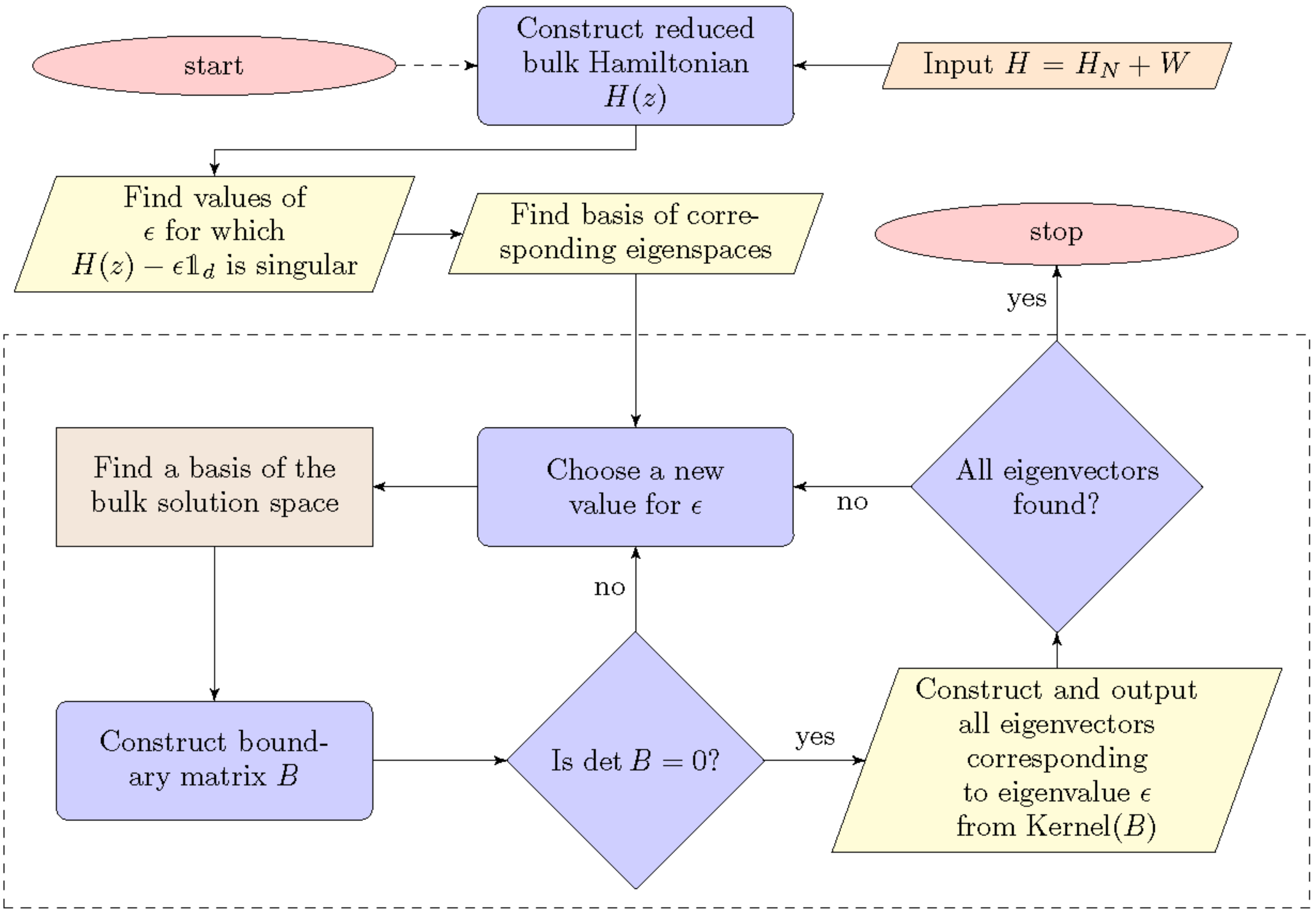
Power-law correction

➡ Power-law solutions exist, for fine-tuned parameter values, in *finite-range* lattice Hamiltonians.

➡ Standard Bloch's theorem is recovered for PBC: The only possible generalized Bloch's states that are eigenstates of cyclic shift V have the form

$$|\epsilon, \vec{\alpha}\rangle \equiv |\psi_{\ell 1}(\epsilon)\rangle = |z_\ell(\epsilon)\rangle\, |u_{\ell 11}(\epsilon)\rangle, \quad z_\ell \equiv e^{ik_\ell}$$

Diagonalization algorithm



- The algorithm may alternatively be cast in algebraic form, yielding *closed-form solution* for in the same spirit of Bethe Ansatz methods – albeit in terms of (only) *polynomial equations*.

Kitaev's Majorana chain revisited

- Paradigmatic tight-binding model of 1D [p-wave] topological superconductivity: under OBC,

$$\widehat{H} = -\sum_{j=1}^{N} \mu\, c_j^\dagger c_j - \sum_{j=1}^{N-1} \left(t\, c_j^\dagger c_{j+1} - \Delta\, c_j^\dagger c_{j+1}^\dagger + \text{h.c.} \right), \quad \mu, t, \Delta \in \mathbb{R}$$

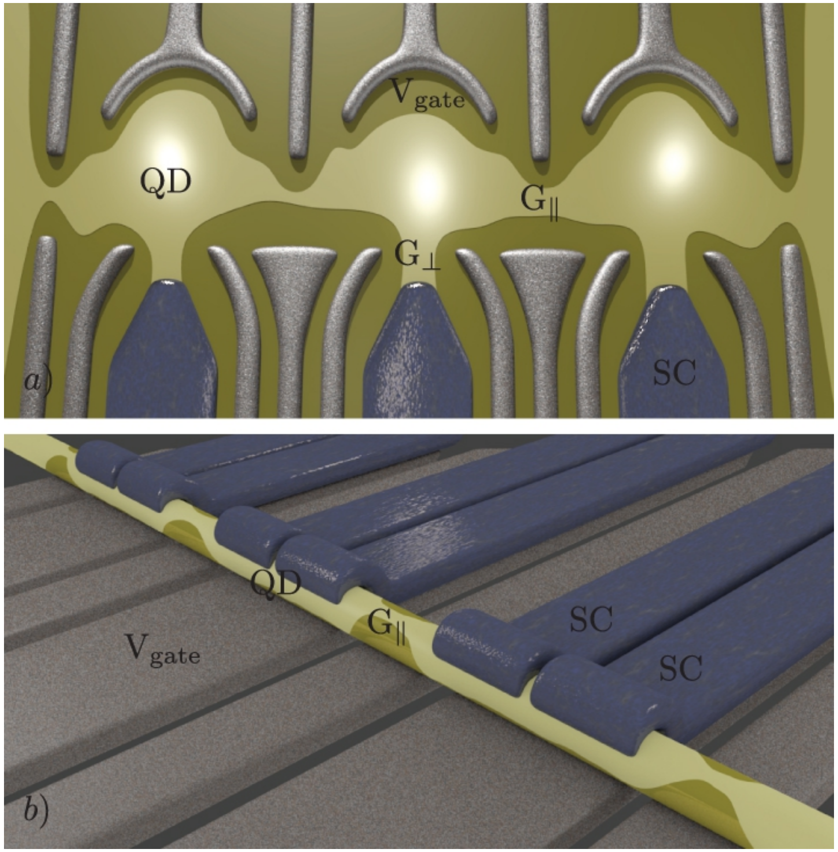
➞ Topologically non-trivial for $|\mu| < |2t|$, hosting *one zero-energy Majorana mode per edge*.

➞ Majoranas are known to be *perfectly localized* at the boundary at 'sweet spot', $\mu = 0,\ |t| = |\Delta|$.

Kitaev, Phys. Usp. **44** (2001).

➞ Experimental implementations with highly *tunable parameter values* are underway in chains of gate-tunable QDs, proximity-coupled to SCs.

Fulga et al, NJP **15** (2013).



➞ BdG Hamiltonian in terms of shift operators:

$$H_N = T \otimes h_1 + \mathbb{I} \otimes h_0 + T^\dagger \otimes h_1^\dagger,$$

$$h_0 = \begin{bmatrix} -\mu & 0 \\ 0 & \mu \end{bmatrix},\ h_1 = \begin{bmatrix} -t & \Delta \\ -\Delta & -t \end{bmatrix}$$

Kitaev's Majorana chain: New surprises

- Full range of possibilities predicted by the generalized Bloch theorem can be realized for different parameter regimes and energy values:

I. Non-invertible regime, $t = \Delta$:

$$\det h_R = \det h_1 = 0 \qquad (z + z^{-1})(2\mu t) + (\mu^2 + 4t^2 - \epsilon^2) = 0$$

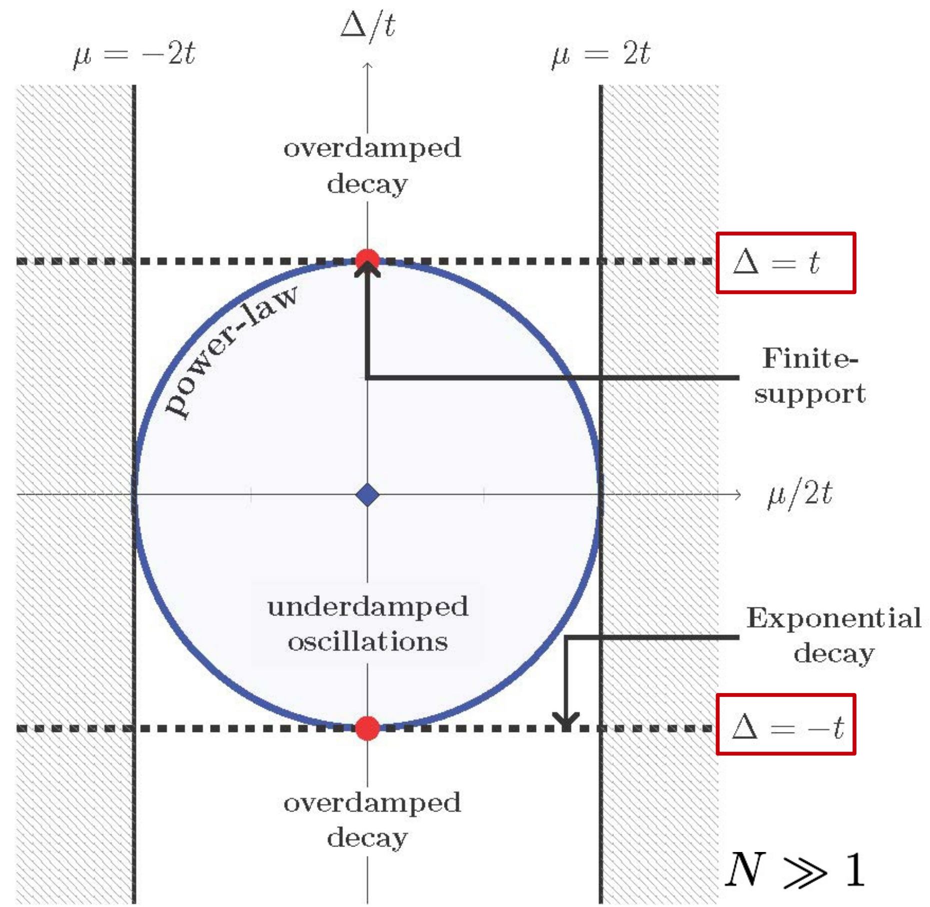
→ At sweet spots, $\mu = 0$, all solutions to the bulk equation have finite support:

✓ $2N$-2 perfectly bulk-localized solutions at $|\epsilon| = 2t$ $\Rightarrow$ *[bulk] flat bands.*

✓ 2 *perfectly boundary-localized solutions at* $\epsilon = 0$, irrespective of system size.

→ Away from sweet spots, doubly-degenerate roots can arise, and *power-law solutions* [with a linear pre-factor] belong to the physical spectrum for

$$\epsilon \in \{\pm(\mu \pm 2t)\} \qquad \frac{\mu}{2t} = -\frac{N}{N+1}$$



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II. Invertible regime, $t \neq \Delta$:

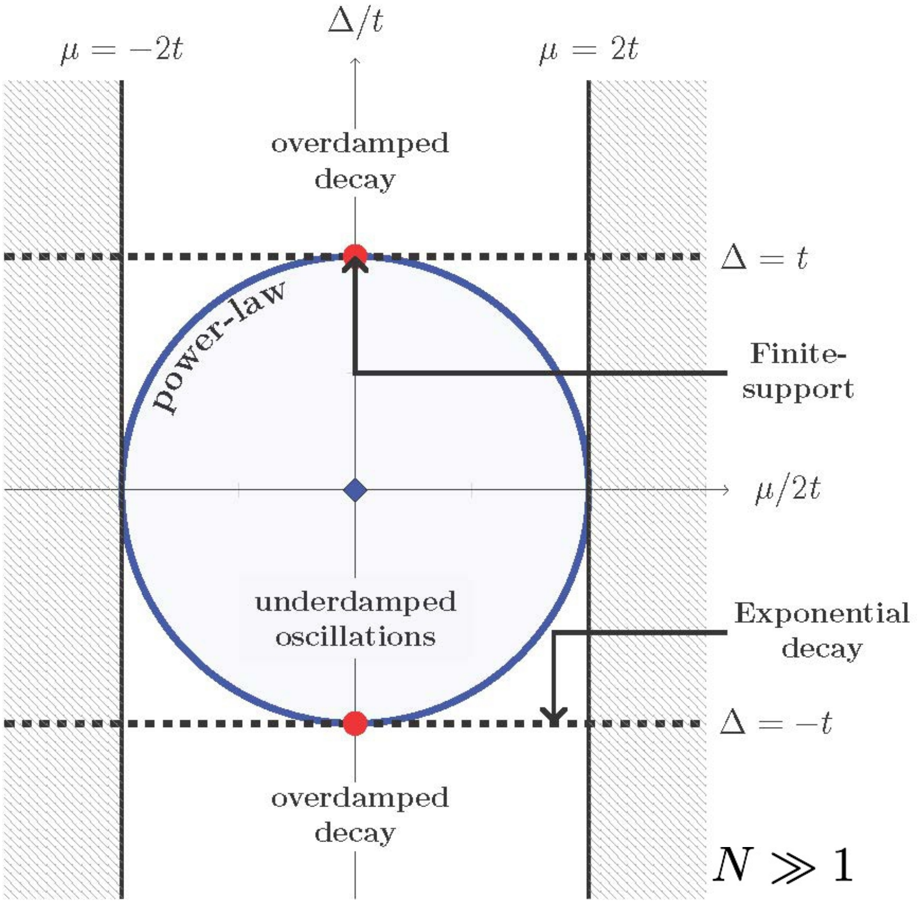
$$(z+z^{-1})^2(t^2-\Delta^2)+(z+z^{-1})(2\mu t)+(\mu^2+4\Delta^2-\epsilon^2)=0$$

➞ Exact solution explains observed presence/absence of *oscillatory behavior of Majorana wavefunctions*

$$\left(\frac{\mu}{2t}\right)^2+\left(\frac{\Delta}{t}\right)^2=1$$

Circle of oscillations

Hegde & Vishveshwara, PRB **94** (2016).



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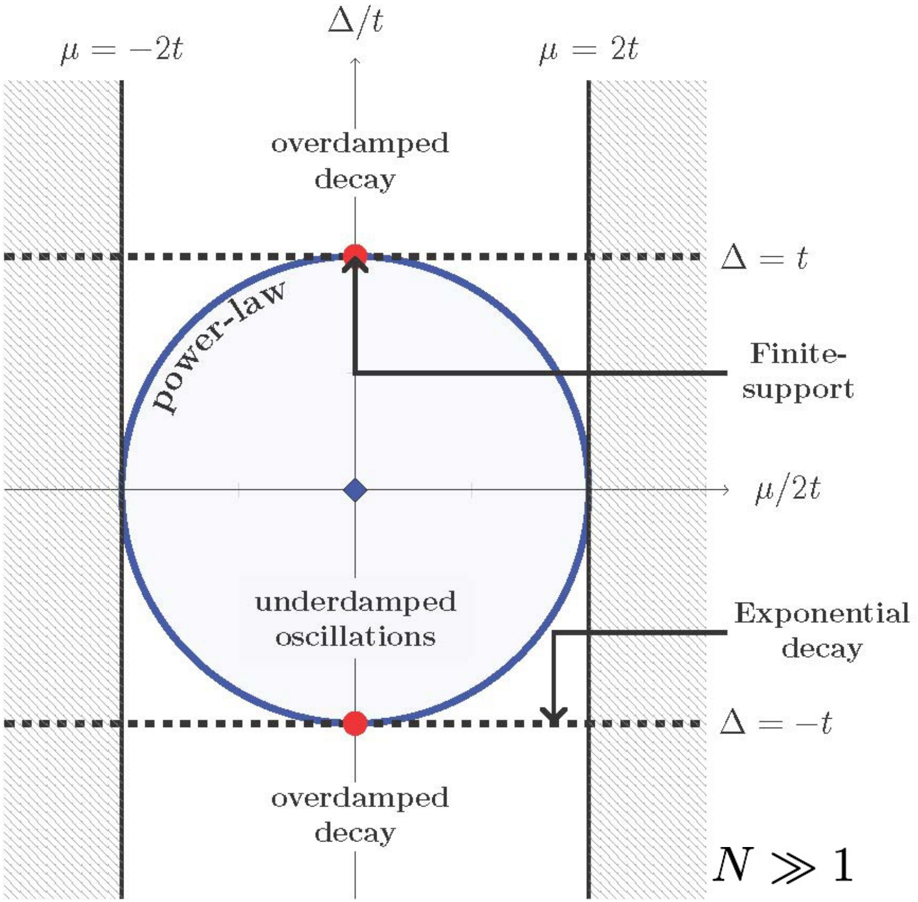
Circle of oscillations

Hegde & Vishveshwara, PRB **94** (2016).



➞ *On* the circle, power-law Majoranas are predicted:

$$|\epsilon=0\rangle \propto \sum_{j=1}^{\infty} j\,\zeta^{j-1}|j\rangle \begin{bmatrix}1\\1\end{bmatrix}$$

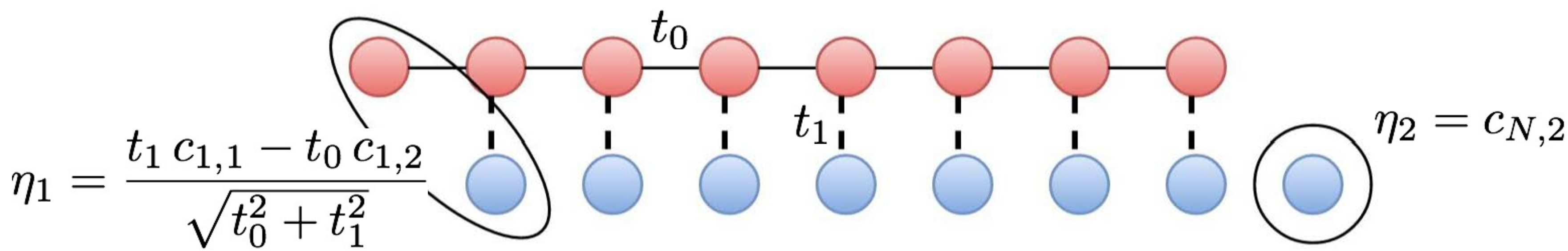
- ✓ Previously known *only* for long-range models.

Vodola *et al*, PRL **113** (2014).

- ✓ Power-law solutions are related to *generalized eigenvectors* of transfer matrix...

Engineering boundary modes: A topological comb

- The generalized Bloch theorem may be used to gain analytic insight and design 'exotic' topological boundary modes via parameter tuning…



- Case study: A fermionic ladder with intra- and inter-ladder NN hopping – also related to a tight-binding version of 1D Anderson lattice model for f electrons.

Tsutsui *et al*, PRL **76** (1996).

$$H_N = T \otimes h_1 + T^\dagger \otimes h_1^\dagger \qquad \Rightarrow \qquad H_N(|1\rangle|u^-\rangle) = 0 = H_N(|N\rangle|u^+\rangle)$$

$$h_1 = \begin{bmatrix} t_0 & 0 \\ t_1 & 0 \end{bmatrix}, \; |u^-\rangle \equiv \begin{bmatrix} t_1 \\ -t_0 \end{bmatrix}, \; |u^+\rangle \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

➞ A non-trivial *perfectly localized zero-energy mode exists, split over two boundary sites* – with weights controlled by the ratio t_0/t_1, independently of N.

➞ Full solution shows that the model is gapped, and *no flat bulk-localized band exists.*

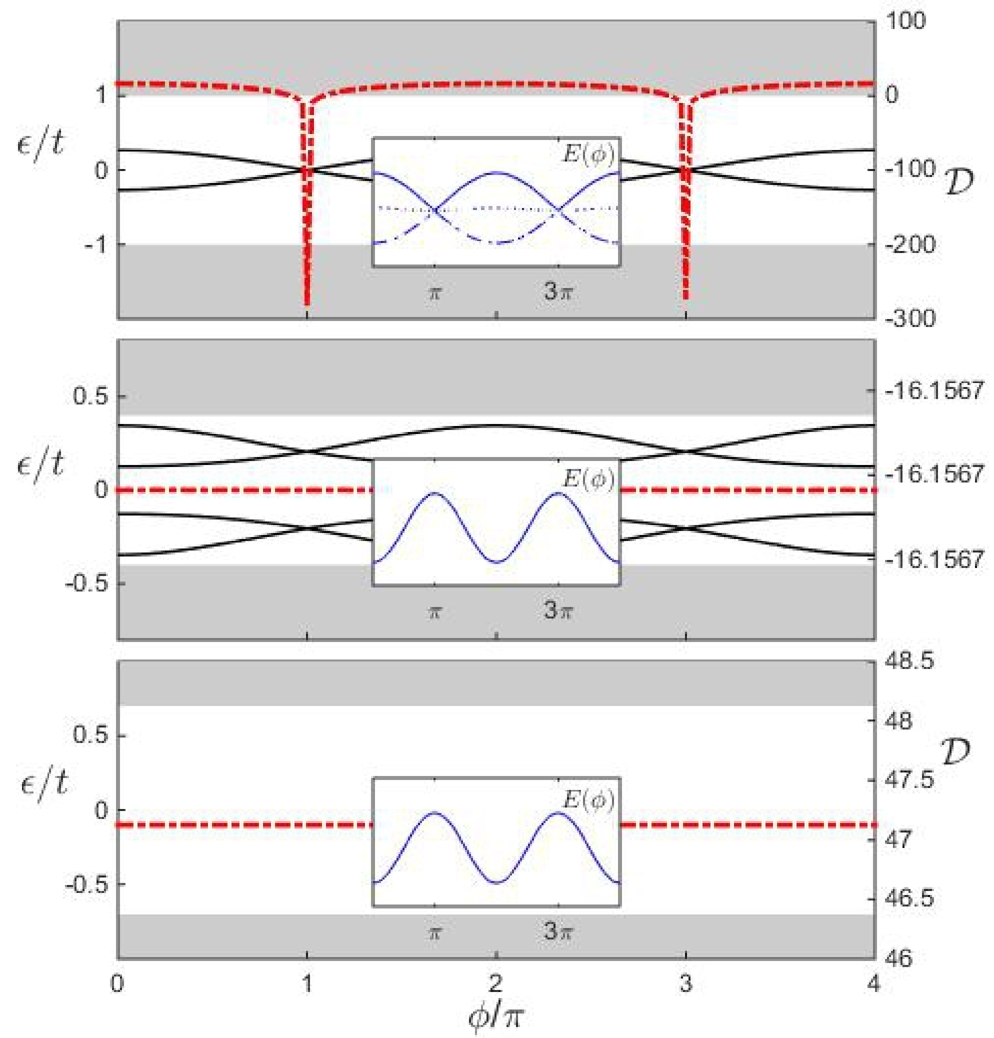
➞ Boundary mode is *robust*, despite the lack of a manifest protecting chiral symmetry…

A witness for bulk-boundary correspondence

- The boundary matrix may used to construct useful [computationally tractable] *indicators of bulk-boundary correspondence that include both bulk and boundary information:*

$$\mathcal{D} \equiv \log\{\det[B_\infty^\dagger(0)B_\infty(0)]\}$$

If either reduced bulk Hamiltonian or BCs are changed, a singularity develops *if and only if* the system hosts bound zero-energy modes.



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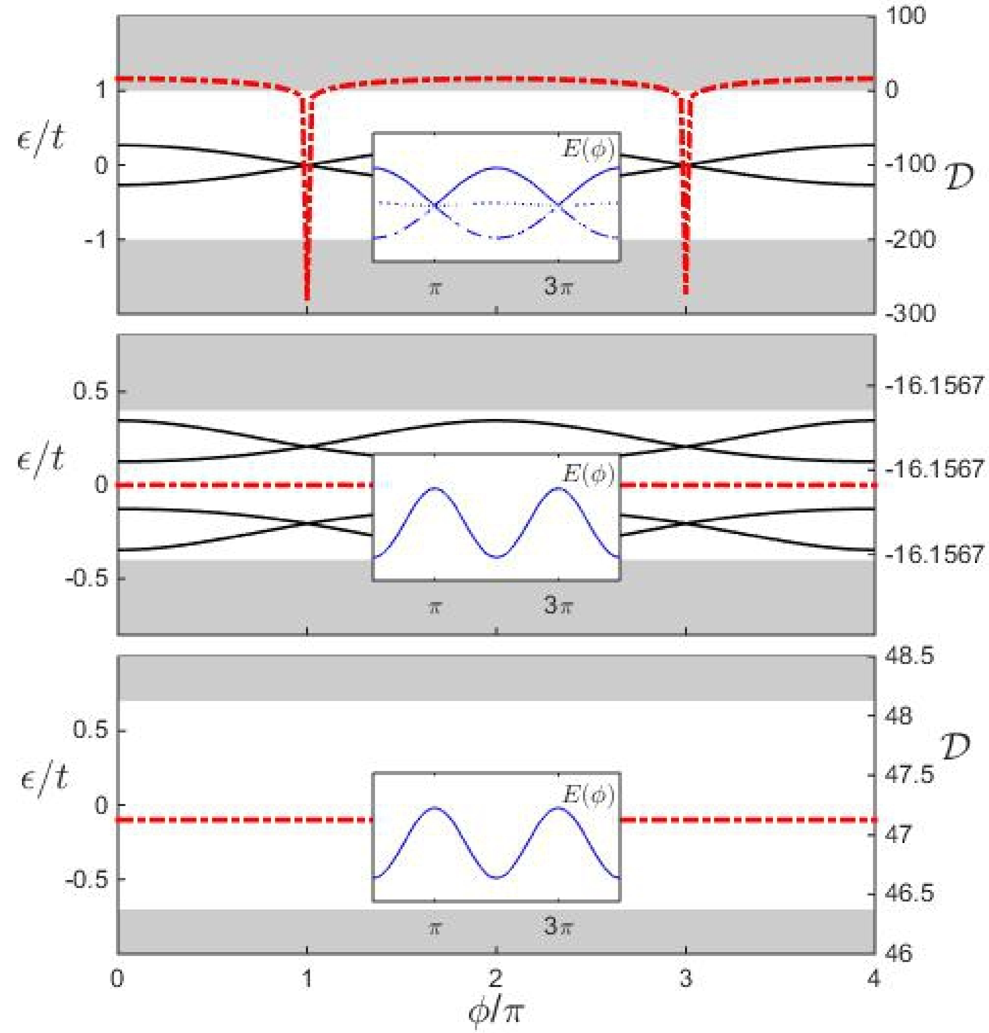
- Case study: Josephson response of s-wave, two-band topological SC wire.

Deng *et al*, PRL **108** (2012).

➡ Accounting for tunable flux requires non-trivial boundary interaction matrix $g_1 \equiv wh_1U(\phi)$.

➡ Fractional 4π-periodic Josephson effect occurs *only* if open chain hosts 1 Majorana pair/edge – explained by single-particle level crossings.

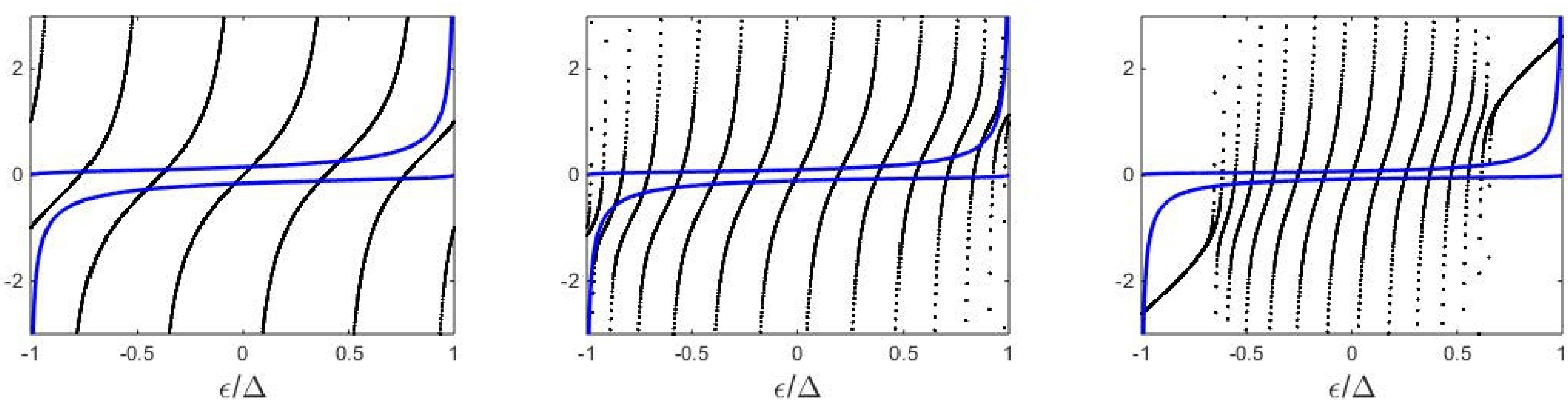
➡ Fractional 4π-periodic Josephson effect is *not* accompanied by fermionic parity switch!



Further implications...

- The approach may be extended to handle clean systems with a more complex structure:
  - ➞ Dimerized chain models (Aubrey-Harper and Peierls TIs, Creutz ladder...)
  - ➞ Systems with internal and/or multiple boundaries:
    - ✓ Impurity problems...
    - ✓ Bound states on tight-binding SN and SNS junctions...

- The approach may be extended to $D > 1$, as long as PBCs are imposed on $D-1$ directions:
  - ➞ Graphene and Weyl semi-metals, surface structure with arbitrary BCs
  - ➞ Surface band structure in topological superconductors:
    - ✓ Chiral, 2D $p + ip$ superconductors ...
    - ✓ 2D gapless s-wave superconductors and Majorana flat bands...



Summary & Outlook

- A natural generalization of Bloch's theorem is possible for 'almost translationally invariant' finite-range quadratic fermionic Hamiltonians – based on exact separation of eigenvalue problem into a translation-invariant bulk equation, and a boundary equation.

- The generalized Bloch theorem offers an analytic window into the bulk-boundary correspondence – including the origin of perfectly localized eigenstates and of both *exponential and power-law solutions* in short-range models, and the identification of *new bulk-boundary indicators* not solely based on bulk information.

- The generalized Bloch theorem provides new tools for understanding and engineering topological boundary modes, and an exact benchmark for more complex physical scenarios.

Summary & Outlook

- A natural generalization of Bloch's theorem is possible for 'almost translationally invariant' finite-range quadratic fermionic Hamiltonians – based on exact separation of eigenvalue problem into a translation-invariant bulk equation, and a boundary equation.

- The generalized Bloch theorem offers an analytic window into the bulk-boundary correspondence – including (1) the origin of perfectly localized eigenstates and of both *exponential and power-law solutions* in short-range models; (2) the identification of *new bulk-boundary indicators* not solely based on bulk information.

- The generalized Bloch theorem provides new tools for understanding and engineering topological boundary modes, and an exact benchmark for more complex physical scenarios.

- Plenty of directions call for further investigation...
  - ➡ Generalized Bloch theorem for *driven Floquet systems with boundaries*...
  - ➡ Generalized Bloch theorem for *mildly broken time-translation symmetry*?...
  - ➡ Relationship between bulk-boundary separation and *entanglement spectrum*...
  - ➡ Generalized Bloch theorem for *quadratic systems of bosons*...
  - ➡ Approach is <u>*not*</u> restricted to Hamiltonian operators ⇒ Diagonalization of quadratic *non-Hermitian Hamiltonians* or *Lindblad dynamics with boundaries*...

  ⋮

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...Thanks for your attention!