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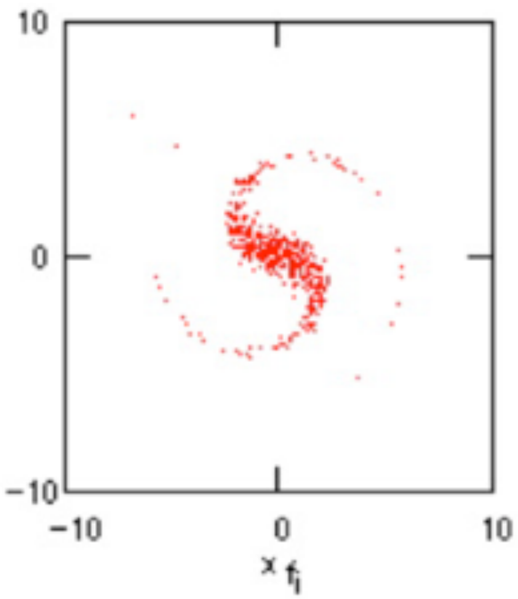
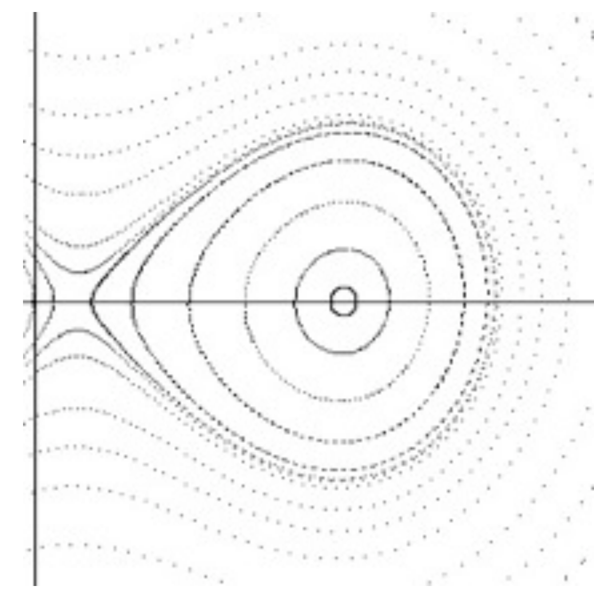
An Introduction to Hadron Colliders

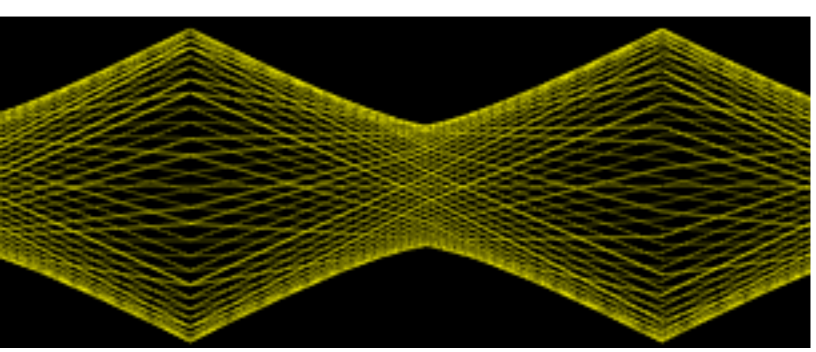
Lecture 2

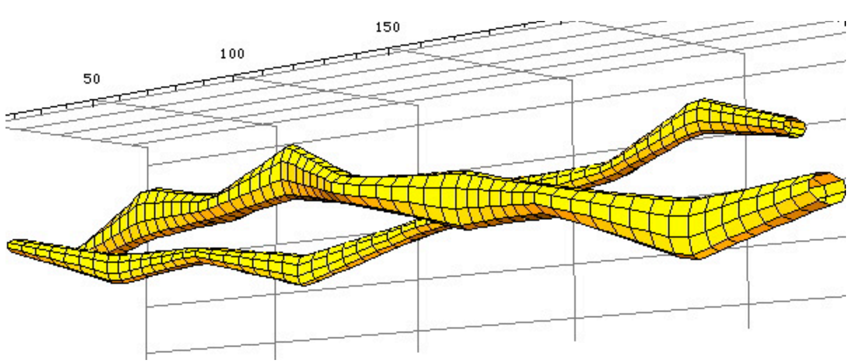




Mike Syphers

Michigan State University





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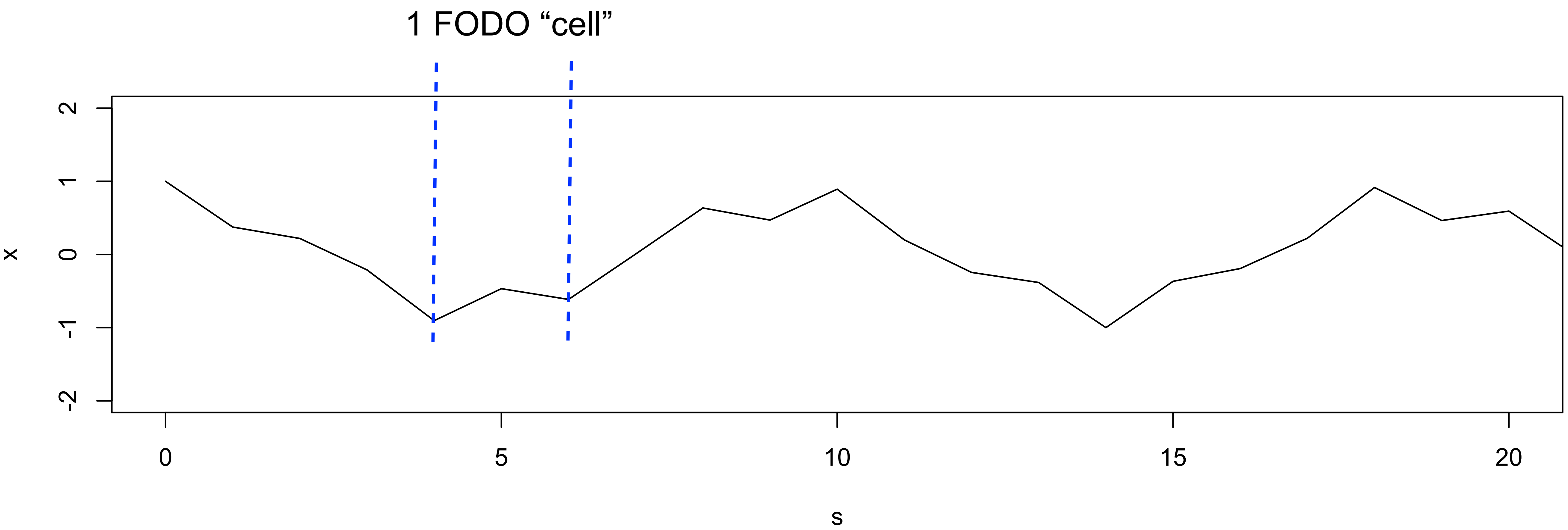
Outline

- Day One:
  - luminosity
  - a little history -- the modern synchrotron
  - magnets and cavities
  - longitudinal dynamics
  - transverse dynamics
- Day Two:
  - Courant-Snyder variables (the 'beta' function)
  - transverse emittance
  - momentum dispersion and chromaticity
  - linear errors and adjustments
- Day Three:
  - beam-beam interactions
  - hour glass and crossing angles
  - diffusion and emittance growth
  - luminosity optimization
  - future directions

Particle Trajectories



- Let's develop an analytical description:

$$\frac{dx'}{ds} = \frac{d^2x}{ds^2} = -\frac{eB'(s)}{p}x$$

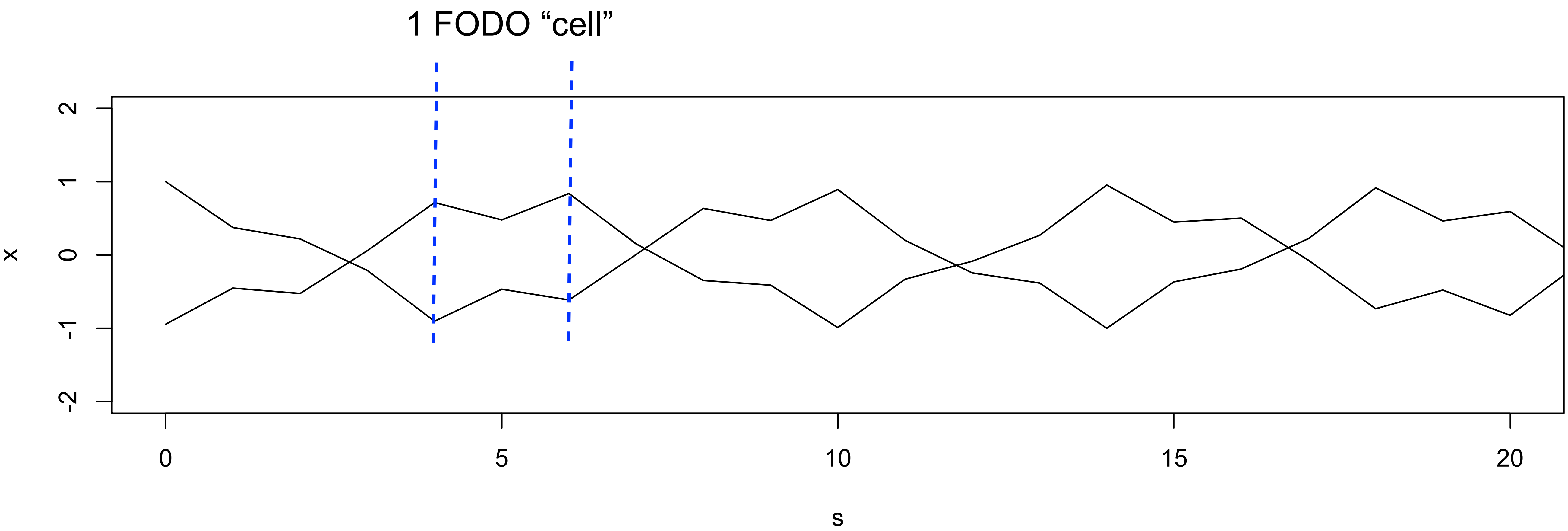
(Hill's Equation)

$$\boxed{x'' + K(s)x = 0}$$

$$\left[K(s) = \frac{e}{p}\frac{\partial B_y}{\partial x}(s)\right]$$

- Look for oscillatory solution with modified amplitude ...

Particle Trajectories



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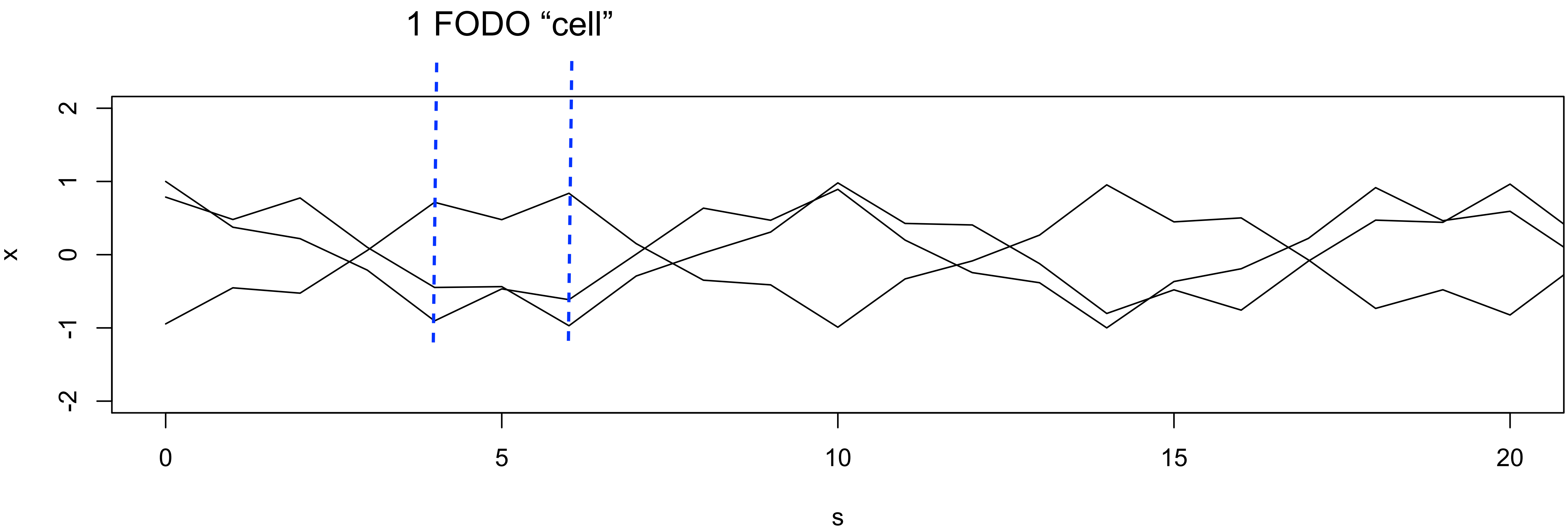

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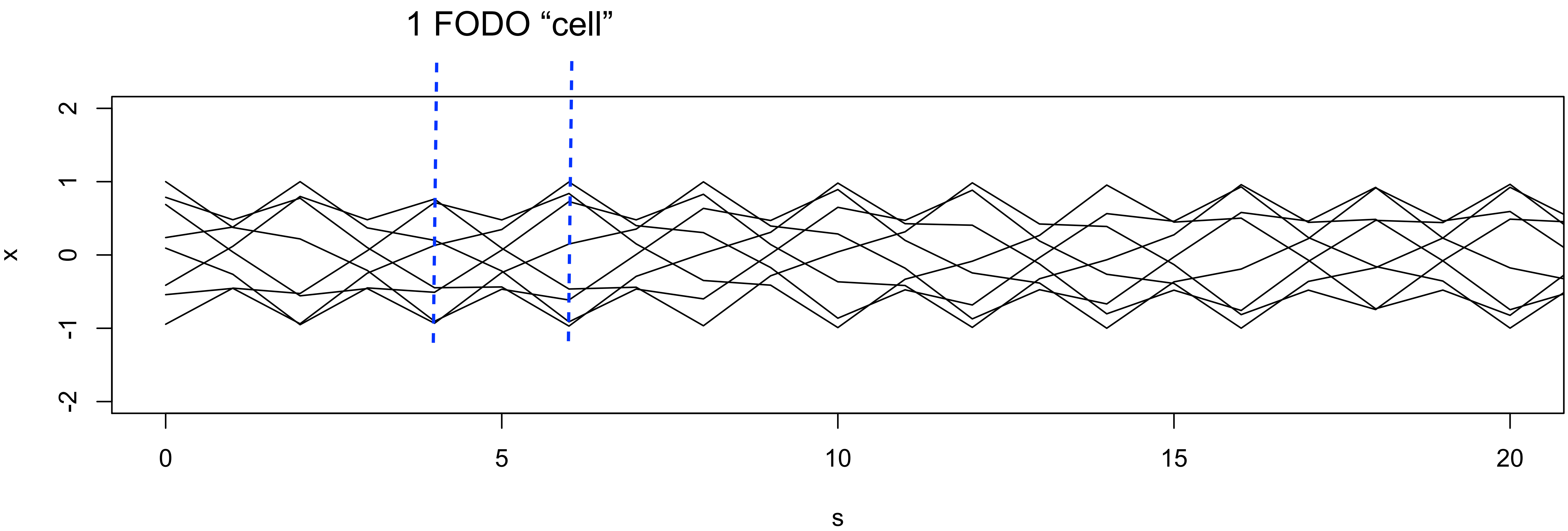

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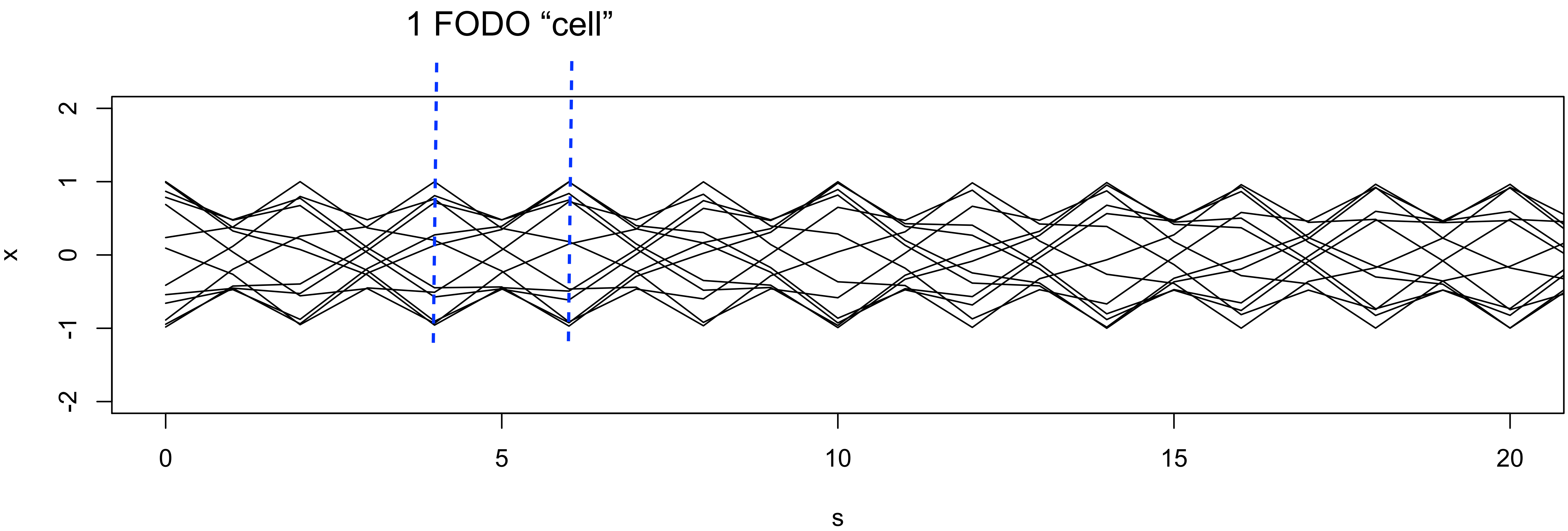

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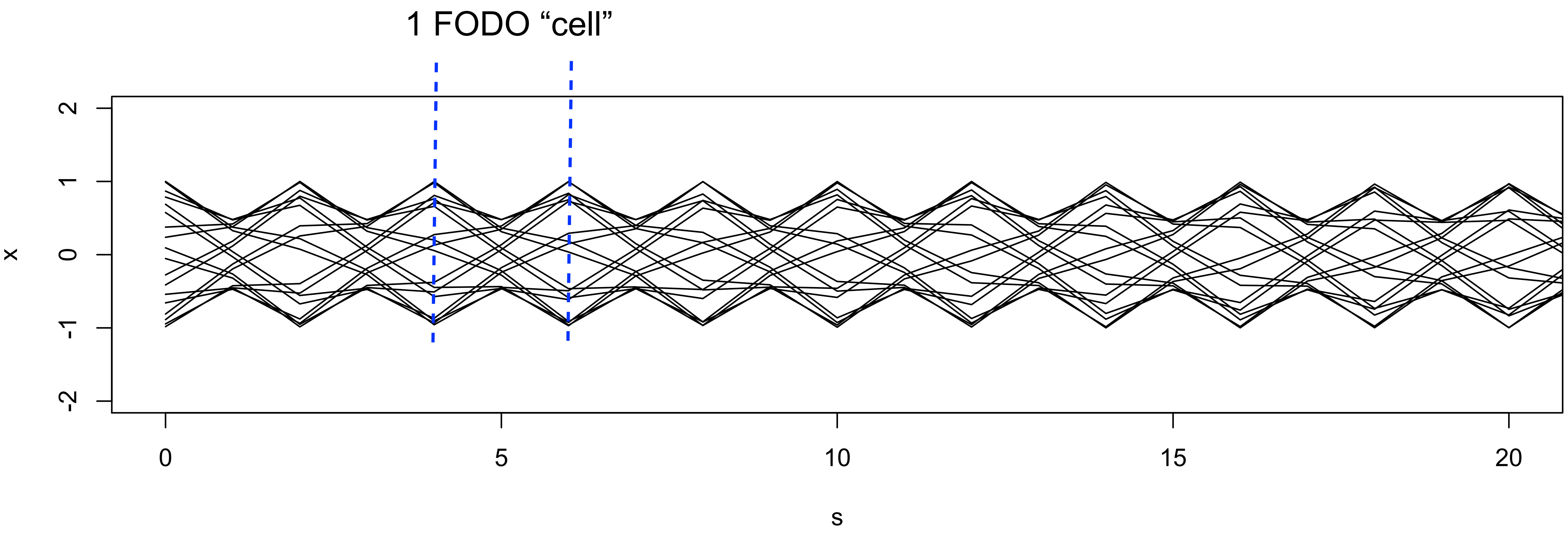

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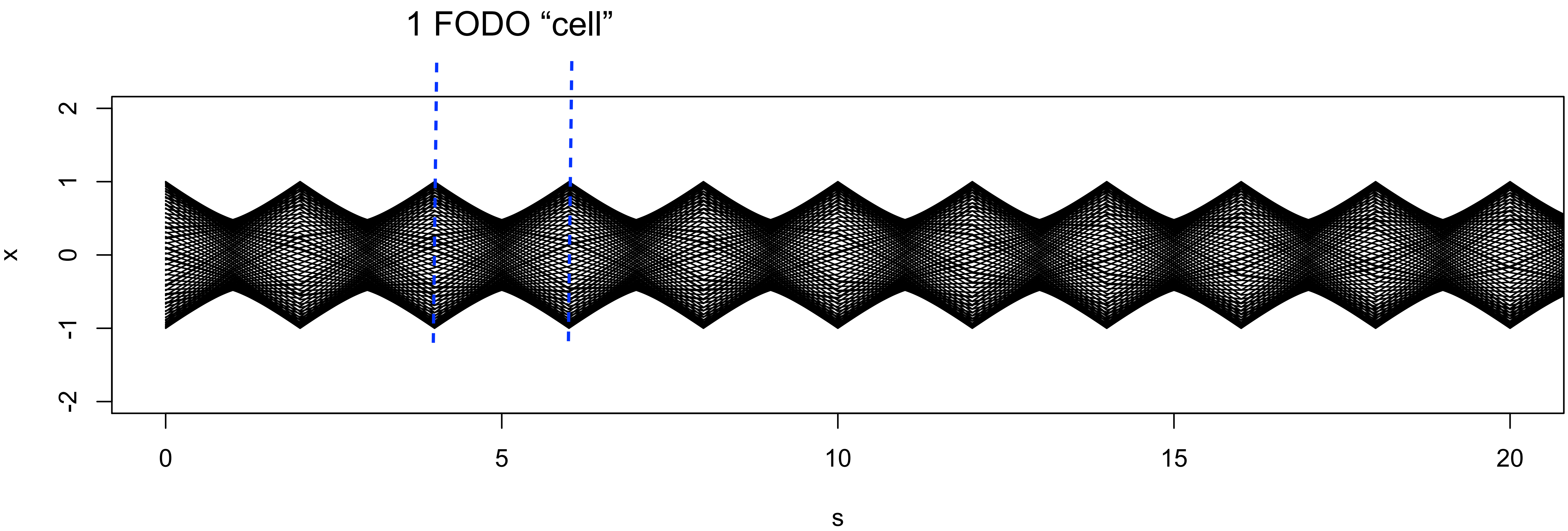

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$$\left[K(s) = \frac{e}{p}\frac{\partial B_y}{\partial x}(s)\right]$$

- Look for oscillatory solution with modified amplitude ...

Analytical Solution

- our assumption:

$$
\begin{aligned}
x(s) &= A\sqrt{\beta(s)}\sin[\psi(s)+\delta] \\
x' &= \frac{1}{2}A\beta^{-\frac{1}{2}}\beta'\sin[\psi(s)+\delta] + A\sqrt{\beta}\cos[\psi(s)+\delta]\psi' \\
x'' &= \ldots
\end{aligned}
$$

- take 1st, 2nd derivatives..

Plug into Hill's Equation, and collect terms...

$$
\begin{aligned}
x'' + K(s)x &= A\sqrt{\beta}\left[\psi'' + \frac{\beta'}{\beta}\psi'\right]\cos[\psi(s)+\delta] \\
&\quad + A\sqrt{\beta}\left[-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K\right]\sin[\psi(s)+\delta] = 0
\end{aligned}
$$

A and δ are constants of integration, defined by the initial conditions (x_0, x_0') of the particle. For arbitrary A, δ, must have contents of each [ ] = 0 simultaneously.

Analytical Solution (cont'd)

- Thus, we must have ...

$$\psi'' + \frac{\beta'}{\beta}\psi' = 0$$

$$\beta\psi'' + \beta'\psi' = 0$$

$$(\beta\psi')' = 0$$

$$\beta\psi' = const$$

$$\psi' = 1/\beta$$

and

$$-\frac{1}{4}\frac{(\beta')^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta} - (\psi')^2 + K = 0$$

$$2\beta\beta'' - (\beta')^2 - 4\beta^2(\psi')^2 + 4K\beta^2 = 0$$

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

The function $\beta(s)$ is the local wavelength $(\lambda/2\pi)$ of the oscillatory motion.

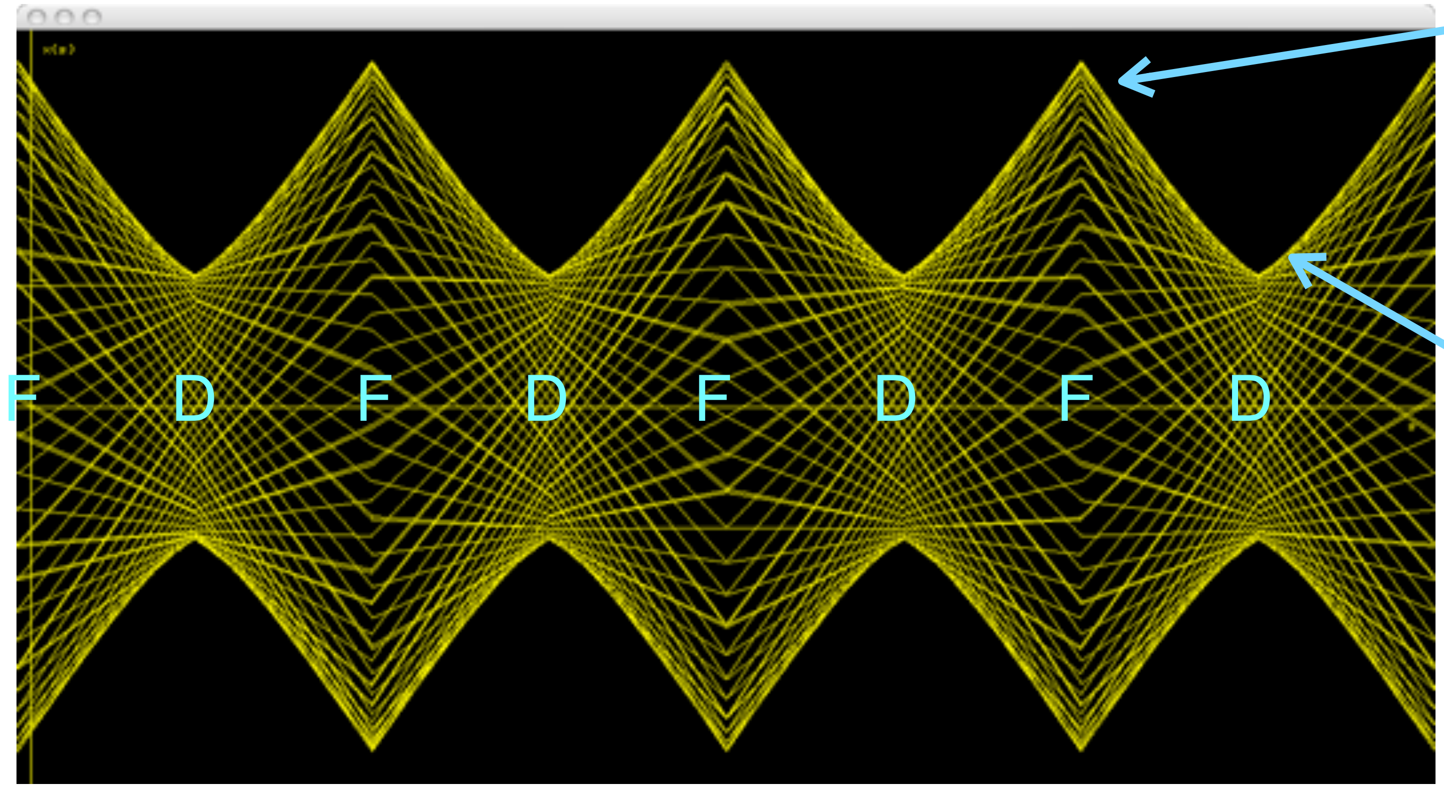
Differential equation that the amplitude function must obey

Note: the phase advance is an observable quantity. So, while could choose different value of *const*, then would just scale β accordingly; thus, valid to choose *const* = 1.

Some Comments

- We chose the amplitude function to be a positive definite function in its definition, since we want to describe real solutions.
- The square root of the amplitude function determines the shape of the envelope of a particle's motion. But the amplitude function also is a local wavelength of the motion.
- This seems strange at first, but ...
  - Imagine a particle oscillating within our focusing lens system; if the lenses are suddenly spaced further apart, the particle's motion will grow larger between lenses, and additionally it will travel further before a complete oscillation takes place. If the lenses are spaced closer together, the oscillation will not be allowed to grow as large, and more oscillations will occur per unit distance travelled.
  - Thus, the spacing and/or strengths (i.e., $K(s)$) determine both the rate of change of the oscillation phase as well as the maximum oscillation amplitude. These attributes must be tied together.

The Amplitude Function, β



Higher β--

smaller phase advance rate
larger beam size

Lower β--

greater phase advance rate
smaller beam size

- Since the amplitude function is a wavelength, it will have numerical values of many meters, say. However, typical particle transverse motion is on the scale of mm. So, this means that the constant *A* must have units of $\mathrm{m}^{1/2}$, and it must be numerically small. More on this subject coming up...

Equation of Motion of Amplitude Function

From

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4$$

we get

$$2\beta'\beta'' + 2\beta\beta''' - 2\beta'\beta'' + 4K'\beta^2 + 8K\beta\beta' = 0$$

$$\beta''' + 4K\beta' + 2K'\beta = 0.$$

Typically, $K'(s) = 0$, and so

$$(\beta'' + 4K\beta)' = 0$$

or

$$\boxed{\beta'' + 4K\beta = const.}$$

is the general equation of motion for the amplitude function, β.

(in regions where K is either zero or constant)

Piecewise Solutions

- *K* = 0:

$$\beta'' = const \longrightarrow \beta(s) = \beta_0 + \beta_0' s + \frac{1}{2}\beta_0'' s^2$$

Parabola!

- since $\beta > 0$, then from original diff. eq.: $2\beta\beta'' - (\beta')^2 = 4$
- the parabola is always concave up $\beta'' > 0$

- *K* > 0, *K* < 0:

$$\beta(s) \sim \sin / \cos \quad \text{or} \quad \sinh / \cosh \ + \ const$$

Courant-Snyder Parameters, & Connection to Matrix Approach

- Suppose, for the moment, that we know the value of the amplitude function and its slope at two points along our particle transport system.
- Have seen how to write the motion of a single particle in one degree of freedom between two points in terms of a matrix. We can now recast the elements of this matrix in terms of the local values of the amplitude function.
- Define two new variables,

$$\boxed{\alpha \equiv -\frac{1}{2}\beta', \quad \gamma \equiv \frac{1+\alpha^2}{\beta}}$$

- Collectively, β, α, γ are called the Courant-Snyder Parameters (sometimes called "Twiss parameters" or "lattice parameters")

$$2\beta\beta'' - (\beta')^2 + 4K\beta^2 = 4 \qquad == \qquad K\beta = \gamma + \alpha'$$

The Transport Matrix

- We can write: $$x(s) = a\sqrt{\beta}\sin\Delta\psi + b\sqrt{\beta}\cos\Delta\psi$$

- Solve for *a* and *b* in terms of initial conditions and write in matrix form

  - we get:

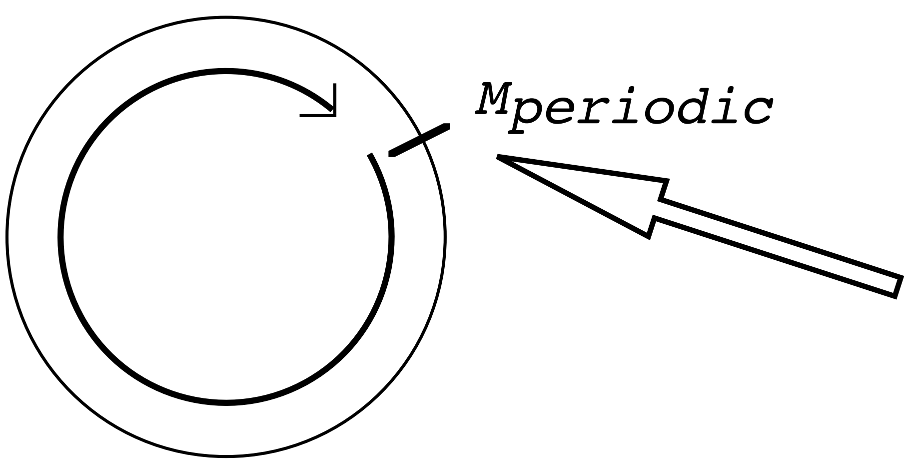
$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \left(\frac{\beta}{\beta_0}\right)^{1/2}(\cos\Delta\psi + \alpha_0\sin\Delta\psi) & \sqrt{\beta_0\beta}\sin\Delta\psi \\ -\frac{1+\alpha_0\alpha}{\sqrt{\beta_0\beta}}\sin\Delta\psi - \frac{\alpha-\alpha_0}{\sqrt{\beta_0\beta}}\cos\Delta\psi & \left(\frac{\beta_0}{\beta}\right)^{1/2}(\cos\Delta\psi - \alpha\sin\Delta\psi) \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

$\Delta\psi$ is the phase advance from point s_0 to point s in the beam line

Periodic Solutions

- Within a system made up of periodic sections it is natural to want the beam envelope to have the same periodicity.
- Taking the previous matrix to be that of a periodic section, and demanding the C-S parameters be periodic yields...

$$M_{periodic} = \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix}$$



Natural choice in a circular accelerator, when values of β, α above correspond to one particular point in the ring

Propagation of Courant-Snyder Parameters

- We can write the matrix of a *periodic* section as:

$$
\begin{aligned}
M_0 &= \begin{pmatrix} \cos\Delta\psi + \alpha\sin\Delta\psi & \beta\sin\Delta\psi \\ -\gamma\sin\Delta\psi & \cos\Delta\psi - \alpha\sin\Delta\psi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cos\Delta\psi + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}\sin\Delta\psi \\
&= I\cos\Delta\psi + J\sin\Delta\psi \qquad = e^{J\Delta\psi}
\end{aligned}
$$

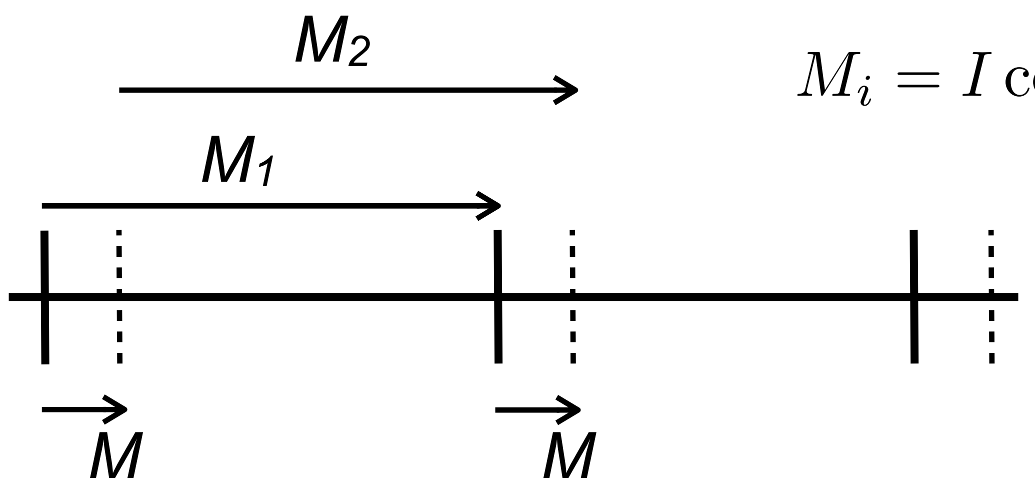
- where

$$
J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \qquad det J = 1, \quad trace(J) = 0; \quad J^2 = -I
$$

α, β are values at the beginning/end of the periodic section described by matrix M

Tracking β, α, γ ...

- Let M_1 and M_2 be the “periodic” matrices as calculated at two points, and M propagates the motion between them. Then,



$$M_i = I \cos \Delta\psi + J_i \sin \Delta\psi$$

$$M_2 = M \; M_1 \; M^{-1}$$

- Or, equivalently,
  - if know C-S parameters (i.e., J) at one point, can find them at another point if given the matrix for motion in between:

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

$$J_2 = M \; J_1 \; M^{-1}$$

- Doesn’t have to be part of a periodic section; valid between any two points of an arbitrary arrangement of elements

Evolution of the Phase Advance

- Again, if know parameters at one point, and the matrix from there to another point, then

$$M_{1\to 2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Longrightarrow \frac{b}{a\beta_1 - b\alpha_1} = \tan \Delta\psi_{1\to 2}$$

- So, from knowledge of matrices, can "transport" phase *and* the Courant-Snyder parameters along a beam line from one point to another

Simple Examples

- Propagation through a Drift:

$$M = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = \tan^{-1}\left(\frac{L}{\beta_1 - L\alpha_1}\right)$$

$$\beta = \beta_0 - 2\alpha_0 L + \gamma_0 L^2$$

$$\alpha = \alpha_0 - \gamma_0 L$$

$$\gamma = \gamma_0$$

- Propagation through a Thin Lens:

$$M = \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix}$$

$$\Longrightarrow \quad \Delta\psi = 0$$

$$\beta = \beta_0$$

$$\alpha = \alpha_0 + \beta_0/F$$

$$\gamma = \gamma_0 + 2\alpha_0/F + \beta_0/F^2$$

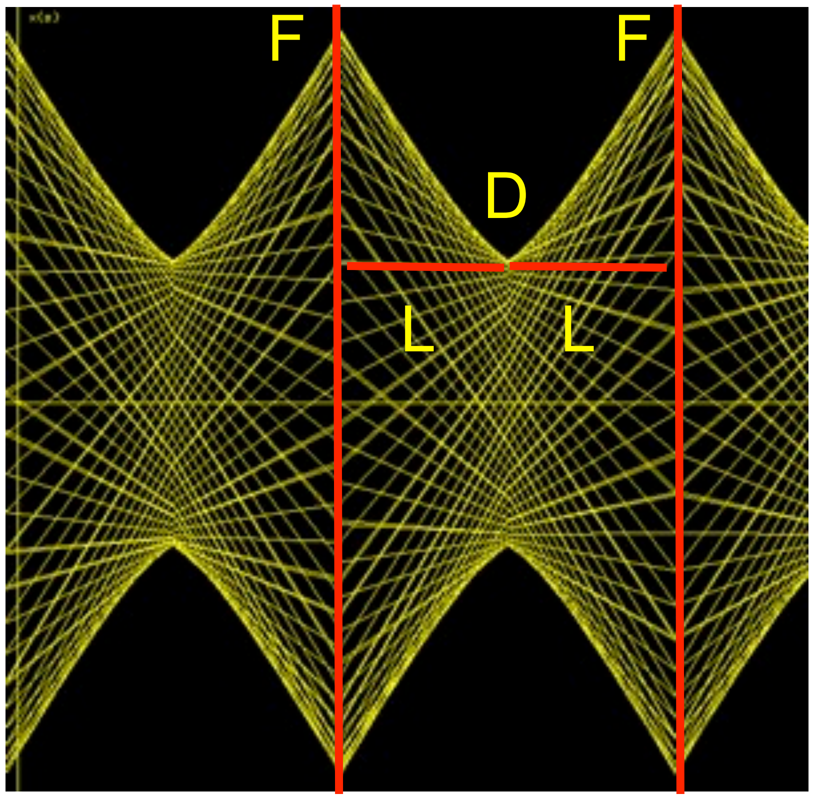
- Given *α, β* at one point, can calculate *α, β* at all downstream points

Choice of Initial Conditions

- Have seen how β can be propagated from one point to another. Still, have the choice of initial conditions...
- If periodic system, like a "ring," then natural to choose the periodic solution for β, α
- If beam line connects one ring to another ring, or a ring to a target, then we take the periodic solution of the upstream ring as the initial condition for the beam line
- In a system like a linac, wish to "match" to desired initial conditions at the input to the system (somewhere after the source, say) using an arrangement of focusing elements

Computation of Courant-Snyder Parameters

- As an example, consider again the FODO system



$$
\begin{aligned}
M &= \begin{pmatrix} 1 & 0 \\ -1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/F & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & L \\ -1/F & 1 - L/F \end{pmatrix} \begin{pmatrix} 1 & L \\ 1/F & 1 + L/F \end{pmatrix} \\
&= \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix}
\end{aligned}
$$

- Thus, use above matrix of the periodic section to compute functions at the *exit* of the F quad..

FODO Cell

- From the matrix:

$$M = \begin{pmatrix} 1 + L/F & 2L + L^2/F \\ -L/F^2 & 1 - L/F - L^2/F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

call $\mu = \Delta\psi$

Here, μ is phase advance through one periodic cell

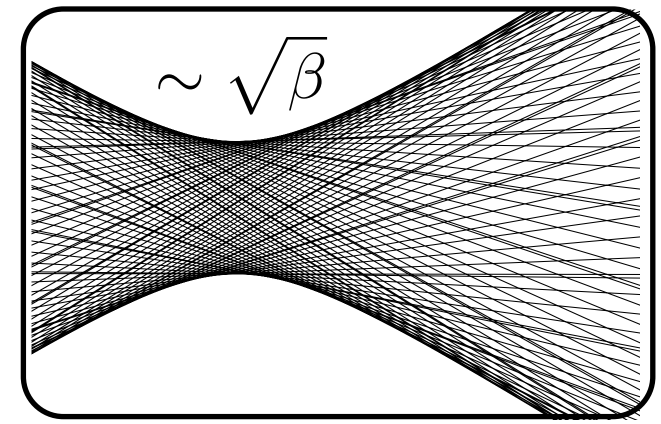
$$trace M = a + d = 2 - L^2/F^2 = 2\cos\mu \quad \Rightarrow \quad \sin\frac{\mu}{2} = \frac{L}{2F}$$

$$\beta = \frac{b}{\sin\mu} = 2F\sqrt{\frac{1 + L/2F}{1 - L/2F}} \qquad \alpha = \frac{a - d}{2\sin\mu} = \sqrt{\frac{1 + L/2F}{1 - L/2F}}$$

- If go from D quad to D quad, simply replace *F* --> *-F* in matrix *M*
  - at exit:

$$\beta = 2F\sqrt{\frac{1 - L/2F}{1 + L/2F}} \qquad \alpha = -\sqrt{\frac{1 - L/2F}{1 + L/2F}}$$

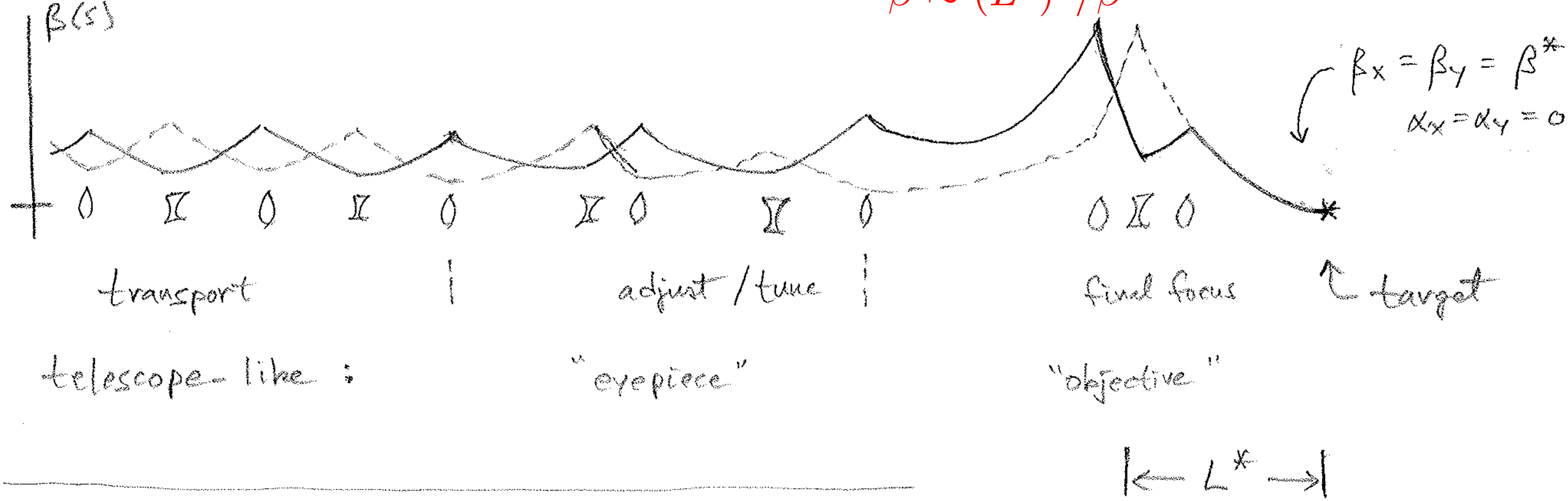
- In drift, amplitude function is a parabola:



$$\beta(s) = \beta^* \left[1 + (s/\beta^*)^2\right]$$

- Very small beam at IP requires very large beam in the final focus triplet:

$$\hat{\beta} \approx (L^*)^2/\beta^*$$



Betatron Tune

- Since $x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$ and $\psi' = 1/\beta$

  then the total phase advance around the circumference is given by

  $$\psi_{tot} \equiv 2\pi\nu = \oint \frac{ds}{\beta}$$

  The tune, ν, is the number of transverse "*betatron oscillations*" per revolution. The phase advance through one FODO cell is given by

  $$\psi_{cell} = 2\sin^{-1}\left(\frac{L}{2F}\right)$$

  Example: For the Tevatron, *L*/2*F* = 0.6, and since there are about 100 cells, the total tune is about 100 x (2 x 0.6)/2π ~ 20

- Note: since betatron tune ~ 20, and synchrotron tune ~ 0.002, it *is* (relatively) safe to consider these effects independently
- "circular" accels --> resonance conditions; choose *tunes* carefully!

FODO Cells (arcs)

max, min values of β:

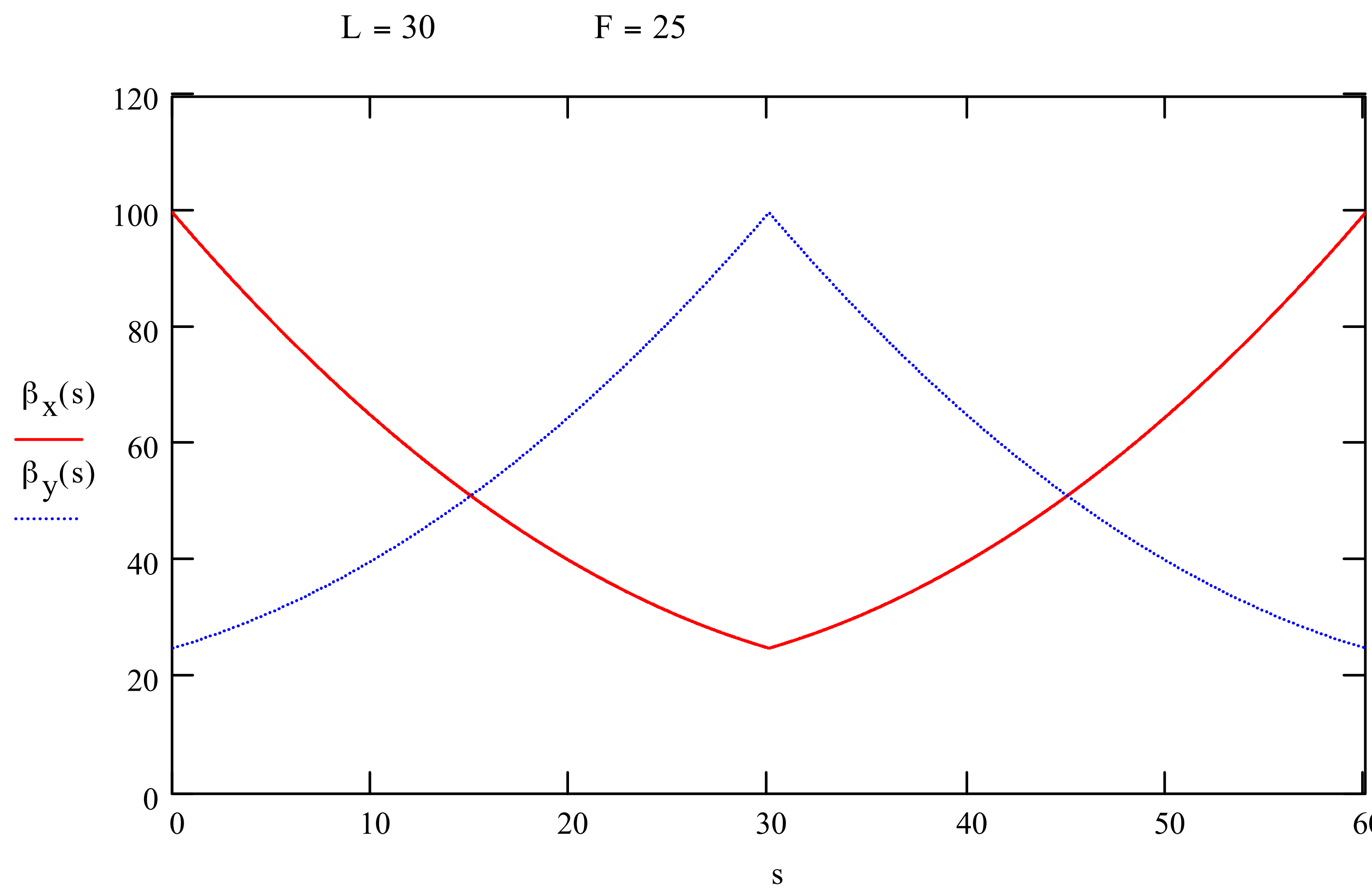
$$\beta_{max,min} = 2F\sqrt{\frac{1 \pm L/2F}{1 \mp L/2F}}$$

entering, exiting a thin lens quad:

$$\Delta\beta' = \mp 2\beta/F$$

between the quadrupoles:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

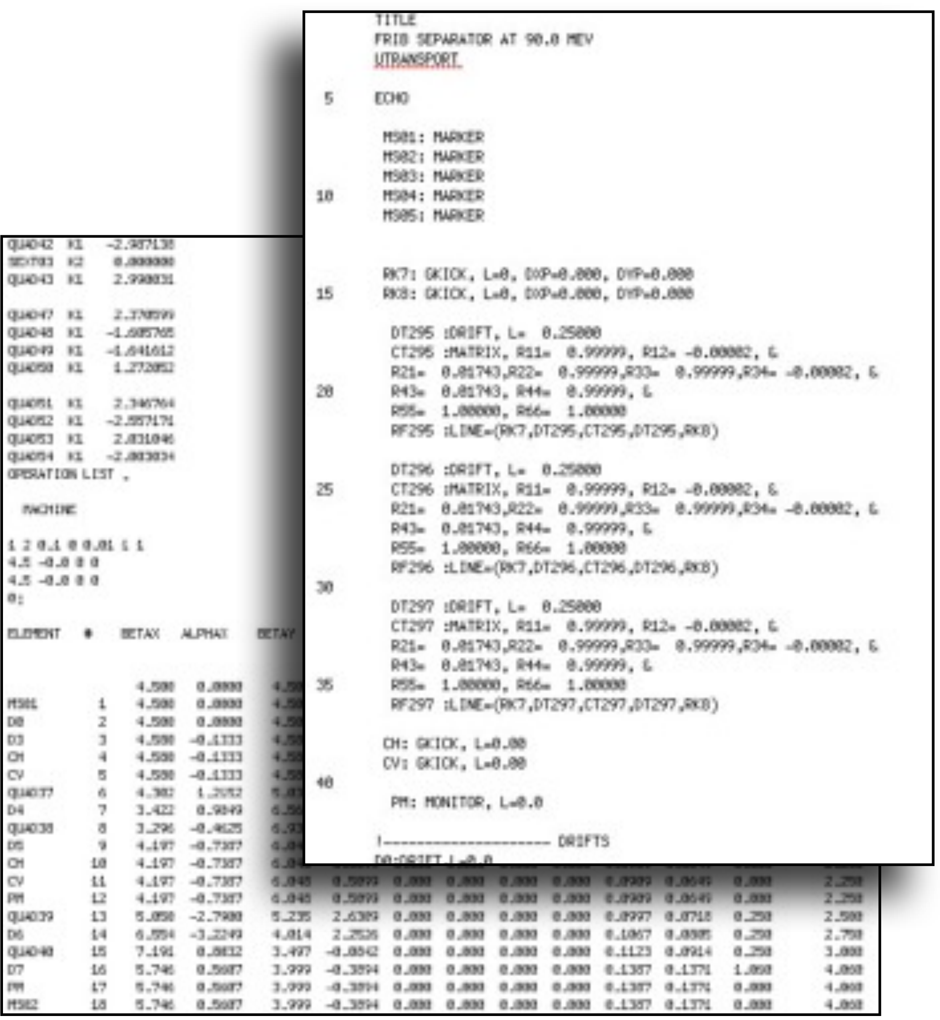


Ex: Tevatron Cell

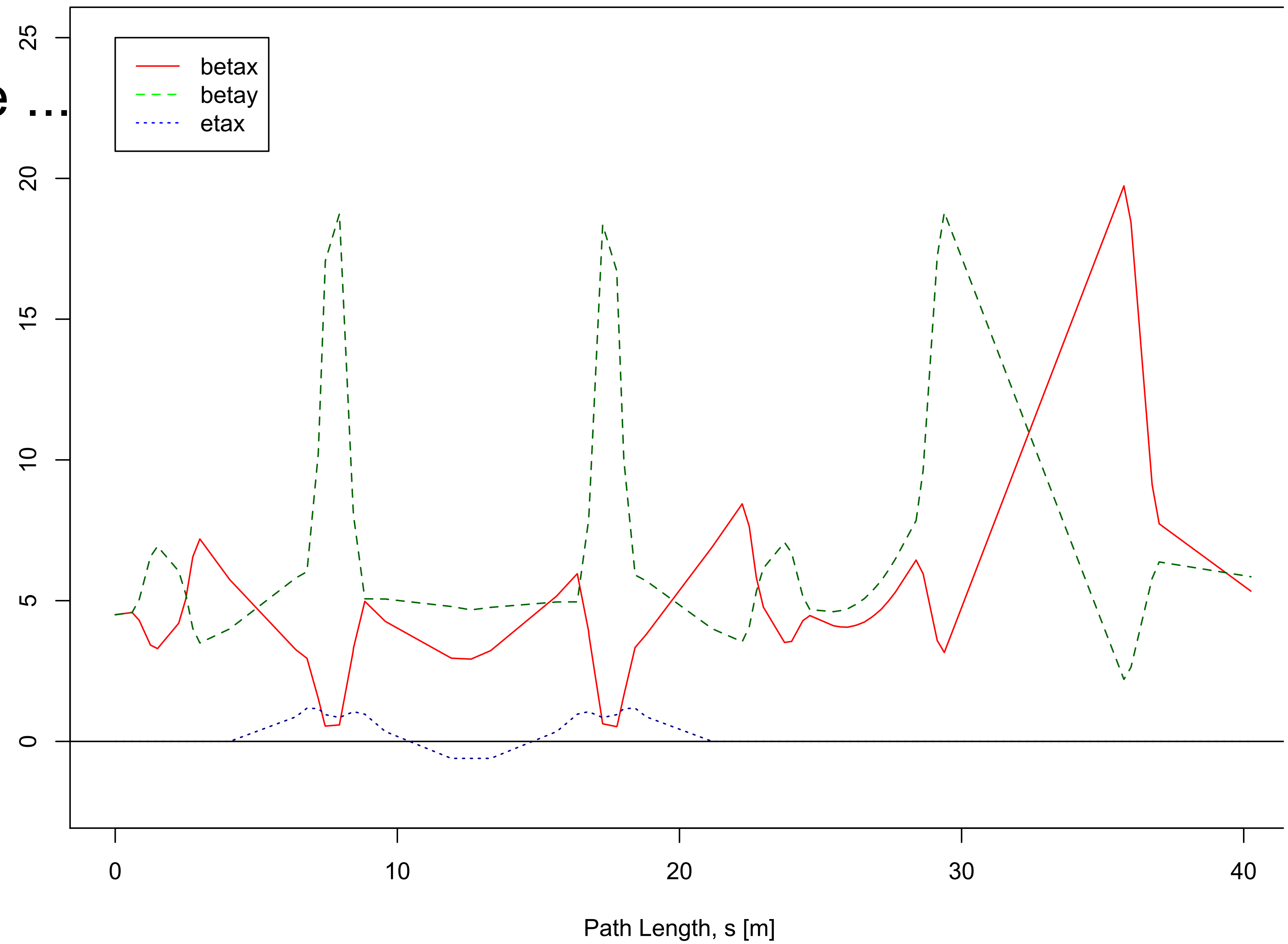
$$\begin{aligned}
&\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ) \\
&\beta_{max} = 2(25\text{ m})\sqrt{1.6/0.4} = 100\text{ m} \\
&\beta_{min} = 2(25\text{ m})\sqrt{0.4/1.6} = 25\text{ m} \\
&\nu \approx 100 \times 1.2/2\pi \sim 20
\end{aligned}$$

Computer Codes

- Complicated arrangements can be fed into now-standard computer codes for analysis
  - TRANSPORT, MAD, DIMAD
  - TRACE, TRACE3D, COSY
  - SYNCH, CHEF, many more …



```
TITLE
FRIB SEPARATOR AT 90.0 MEV
UTRANSPORT

ECHO

 MS01: MARKER
 MS02: MARKER
 MS03: MARKER
 MS04: MARKER
 MS05: MARKER


 RK7: GKICK, L=0, DXP=0.000, DYP=0.000
 RK8: GKICK, L=0, DXP=0.000, DYP=0.000

  DT295 :DRIFT, L=  0.25000
  CT295 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF295 :LINE=(RK7,DT295,CT295,DT295,RK8)

  DT296 :DRIFT, L=  0.25000
  CT296 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF296 :LINE=(RK7,DT296,CT296,DT296,RK8)

  DT297 :DRIFT, L=  0.25000
  CT297 :MATRIX, R11=  0.99999, R12= -0.00002, &
  R21=  0.01743,R22=  0.99999,R33=  0.99999,R34= -0.00002, &
  R43=  0.01743, R44=  0.99999, &
  R55=  1.00000, R66=  1.00000
  RF297 :LINE=(RK7,DT297,CT297,DT297,RK8)

 CH: GKICK, L=0.00
 CV: GKICK, L=0.00

  PM: MONITOR, L=0.0

!--------------------- DRIFTS
D0:DRIFT,L=0.0
```



Review

- Found analytical solution to Hill's Equation:

$$x(s) = A\sqrt{\beta(s)}\sin[\psi(s) + \delta]$$

- So far, discussed amplitude function, β

- What about A?
  - Given $\beta(s)$, how big is the beam at a particular location? mm? cm? m?
  - If perturb the beam's trajectory, how much will it move downstream?

- Want to go from discussing single particle behavior to discussing a "beam" of particles

Betatron Oscillation Amplitude

- Transverse oscillations in a synchrotron (or beam line) are called Betatron Oscillations (first observed/analyzed in a betatron)
- Write *x*,*x'* in terms of initial conditions x_0, x'_0 :

$$x(s) = a\sqrt{\beta}\cos\Delta\psi + b\sqrt{\beta}\sin\Delta\psi$$
$$x' = \frac{1}{\sqrt{\beta}}([b - a\alpha]\cos\Delta\psi - [a + b\alpha]\sin\Delta\psi)$$
$$\downarrow$$
$$a = \frac{x_0}{\sqrt{\beta_0}}, \qquad b = \frac{\alpha_0 x_0 + \beta_0 x'_0}{\sqrt{\beta_0}}$$

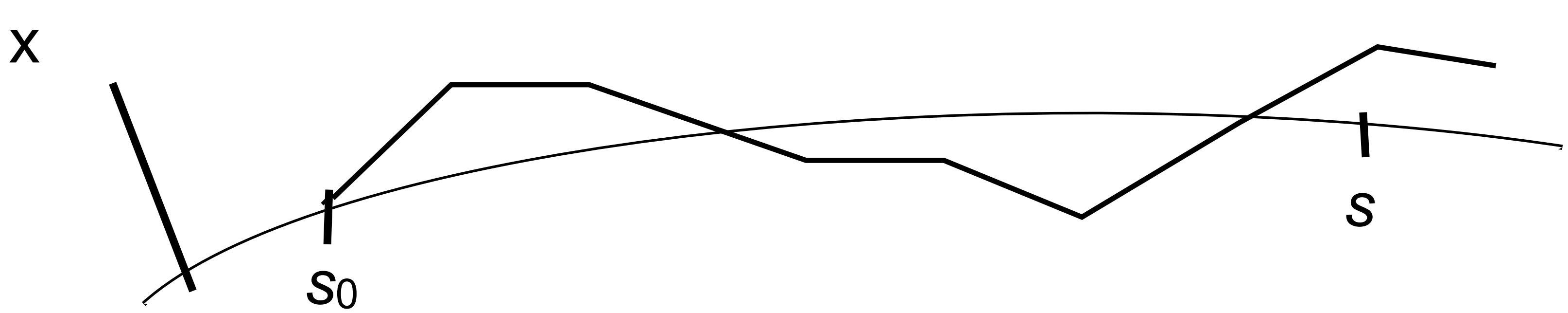
$$\boxed{\Longrightarrow x(s) = \sqrt{\frac{\beta(s)}{\beta_0}}\,[x_0\cos\Delta\psi + (\alpha_0 x_0 + \beta_0 x'_0)\sin\Delta\psi]}$$

$$\text{amplitude: } A = \sqrt{\frac{x_0^2 + (\alpha_0 x_0 + \beta_0 x'_0)^2}{\beta_0}}$$

Free Betatron Oscillation

- Suppose a particle traveling along the design path is given a sudden (impulse) deflection through angle $\Delta x' = x'_0 = \Delta\theta$
- Then, downstream, we have

$$x(s) = \Delta\theta\sqrt{\beta_0\beta(s)}\sin[\psi(s) - \psi_0]$$



Example:
Suppose $\Delta\theta = 0.4$ mrad, $\beta_0 = 4.0$ m, $\beta(s) = 6.4$ m, and $\Delta\psi = n \times 2\pi + 30°$. Then $x(s) = 1$ mm.

Courant-Snyder Invariant

- In general,

$$\begin{aligned} x &= A\sqrt{\beta}\sin\psi \\ x' &= \frac{A}{\sqrt{\beta}}[\cos\psi - \alpha\sin\psi] \\ \beta x' &= A\sqrt{\beta}[\cos\psi - \alpha\sin\psi] \\ &= A\sqrt{\beta}\cos\psi - \alpha x \\ \boxed{\beta x' + \alpha x} &\boxed{= A\sqrt{\beta}\cos\psi} \end{aligned}$$

$$\begin{aligned} x^2 + (\beta x' + \alpha x)^2 &= A^2\beta \\ A^2 &= \frac{x^2 + (\beta x' + \alpha x)^2}{\beta} \\ &= \frac{x^2 + \alpha^2 x^2 + 2\alpha\beta x x' + \beta^2 x'^2}{\beta} \end{aligned}$$

$$\boxed{A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2}$$

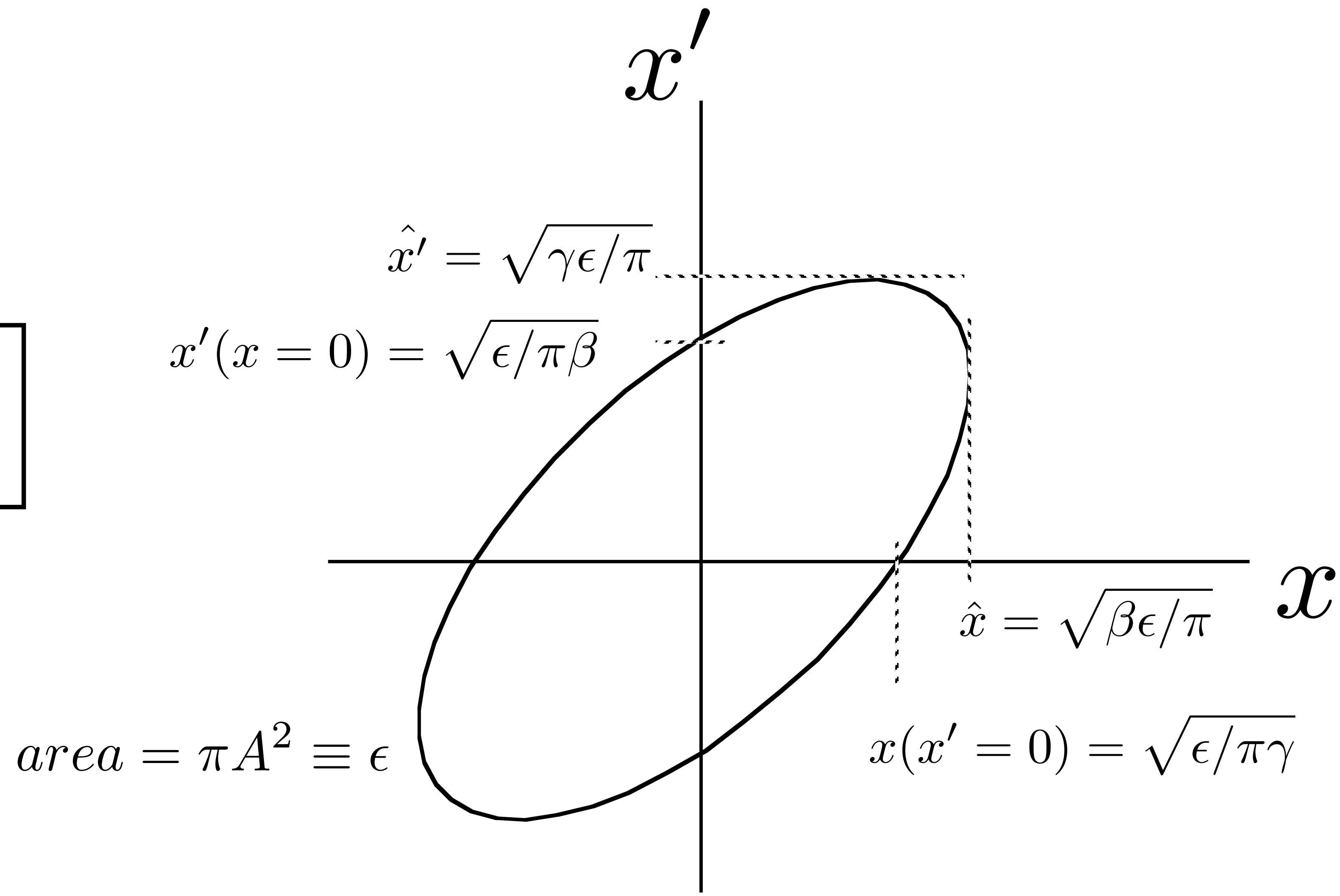
While C-S parameters evolve along the beam line, the combination above remains constant.

Properties of the Phase Space Ellipse

- The eqn. for the C-S invariant is that of an ellipse.
- If compute the area of the ellipse, find that

$$\boxed{area = \pi A^2}$$

i.e., while the ellipse changes shape along the beam line, its area remains constant



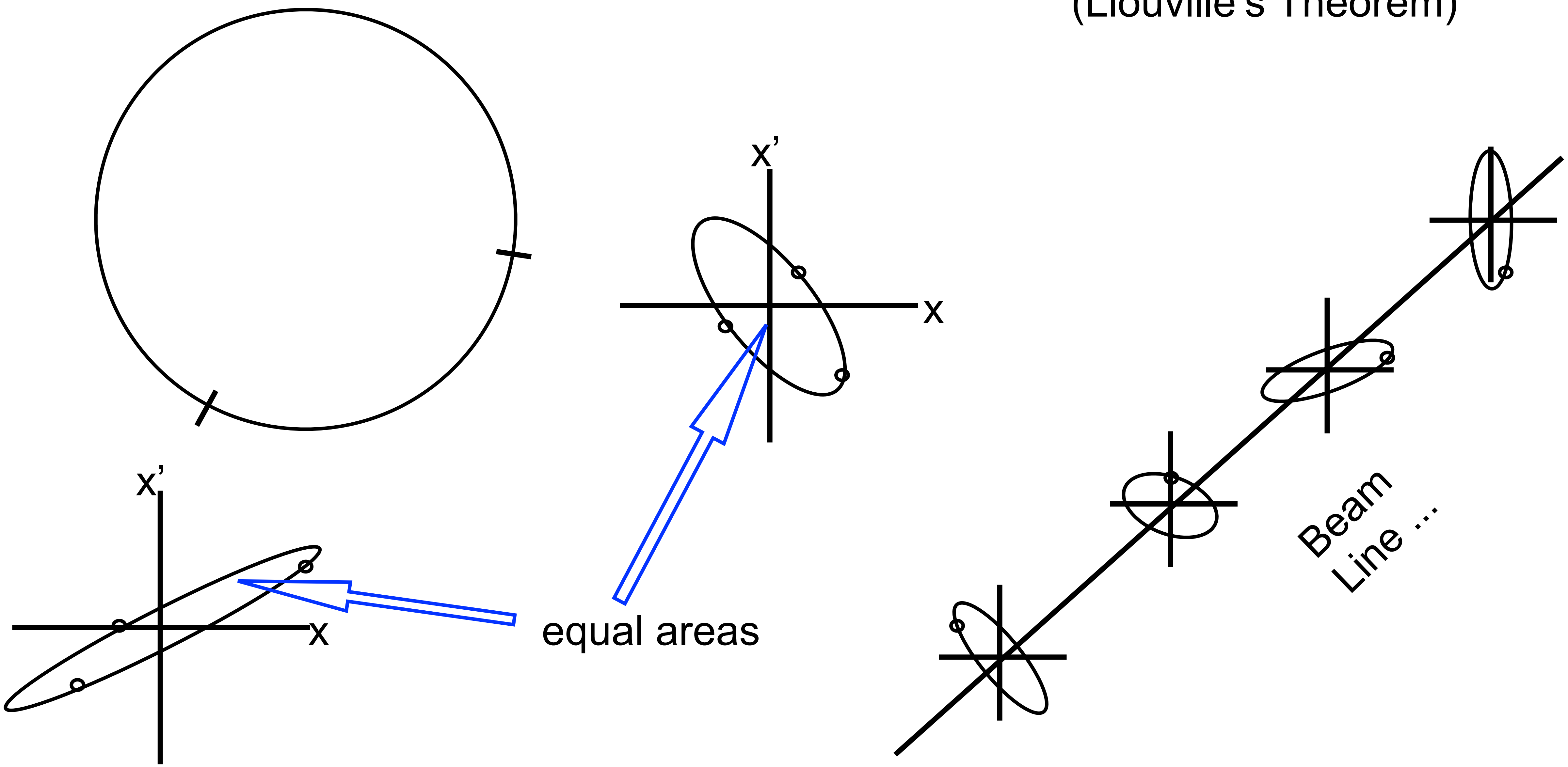
Emittance = area within a phase space trajectory

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = A^2$$

Motion in Phase Space

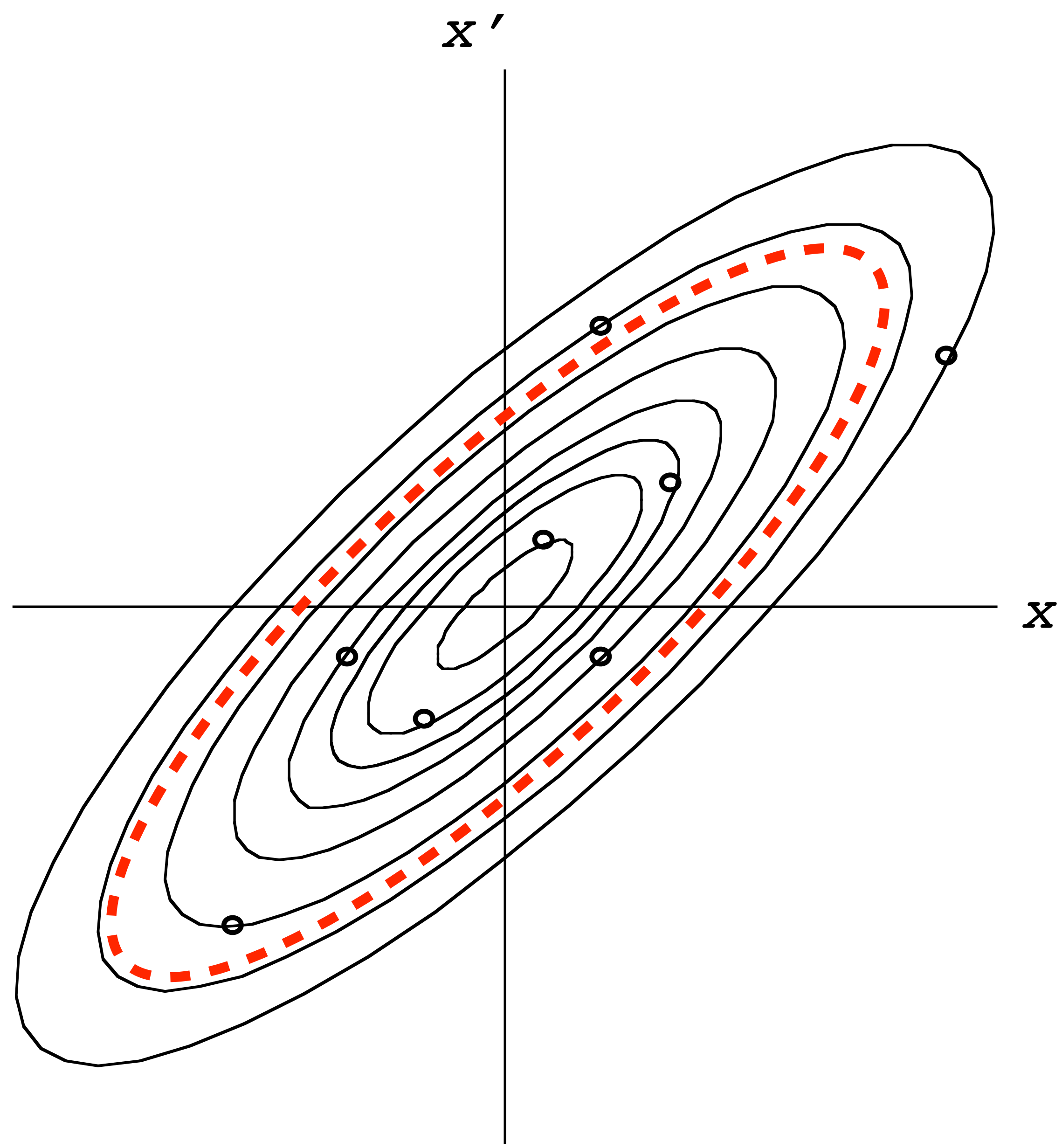
- Follow phase space trajectory...

Phase Space area is preserved
(Liouville's Theorem)



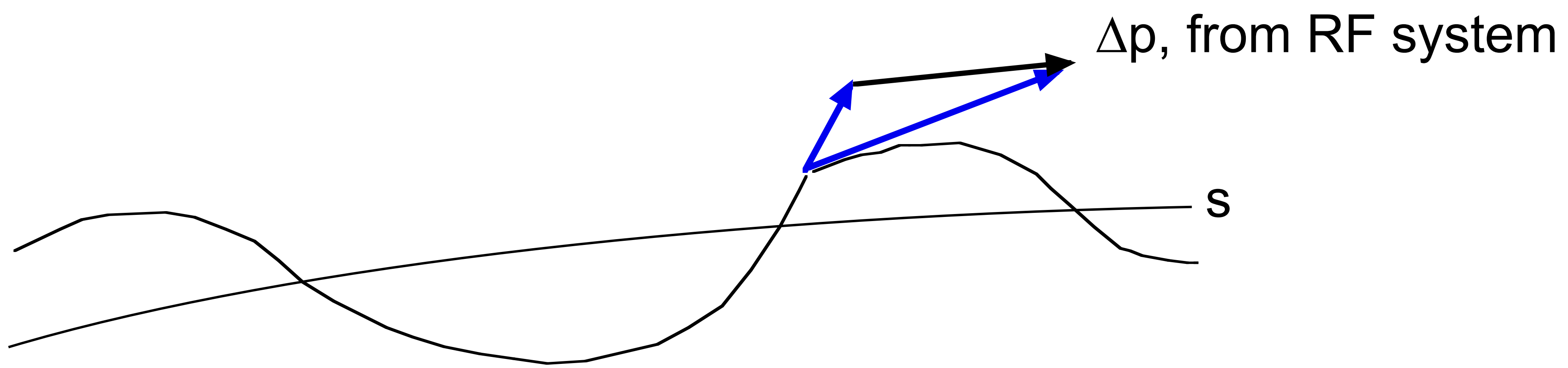
Beam Emittance

- Phase space area which contains a certain fraction of the beam particles
- Popular Choices:
  - 95%
  - 39%
  - 15%



Adiabatic Damping from Acceleration

- Transverse oscillations imply transverse momentum. As accelerate, momentum is "delivered" in the longitudinal direction (along the *s*-direction). Thus, on average, the angular divergence of a particle will decrease, as will its oscillation amplitude, during acceleration.



- The coordinates *x-x'* are not canonical conjugates, but x-p_x are; thus, the area of a trajectory in x-p_x phase space is invariant for adiabatic changes to the system.

Normalized Beam Emittance

- Hence, as particles are accelerated, the area in *x-x'* phase space is not preserved, while area in x-p_x is preserved. Thus, we define a "normalized" beam emittance, as

$$\epsilon_N \equiv \epsilon \cdot (\beta\gamma)$$

- In principle, the normalized beam emittance should be preserved during acceleration, and hence along the chain of accelerators from source to target. Thus it is a measure of beam quality, and its preservation a measure of accelerator performance.
- In practice, it is not preserved -- non-adiabatic acceleration, especially at the low energy regime; non-linear field perturbations; residual gas scattering; charge stripping; field errors and setting errors; *etc.* -- all contribute at some level to increase the beam emittance. Best attempts are made to keep this as small as possible.

Gaussian Emittance

- So, normalized emittance that contains a fraction *f* of a Gaussian beam is:

$$\epsilon_N = -2\pi \ln(1-f)\ \frac{\sigma_x^2(s)}{\beta(s)}\ (\beta\gamma)$$

Lorentz!

- Common values of *f* :

f	$\epsilon_N/(\beta\gamma)$
95%	$6\pi\sigma^2/\beta$
86.5%	$4\pi\sigma^2/\beta$
39%	$\pi\sigma^2/\beta$
15%	σ^2/β

Perhaps most commonly used, sometimes called the "rms" emittance; but, always ask if not clear in context!

Emittance in Terms of Moments

- For each particle, $x = A\sqrt{\beta}\sin\psi$ $\qquad x' = \frac{A}{\sqrt{\beta}}(\cos\psi - \alpha\sin\psi)$
- Average over the distribution...

$$x^2 = A^2\beta\sin^2\psi \qquad x'^2 = \frac{A^2}{\beta}(\cos^2\psi + \alpha^2\sin^2\psi - \alpha\sin 2\psi)$$

$$\langle x^2\rangle = \frac{1}{2}\langle A^2\rangle\beta \qquad \langle x'^2\rangle = \frac{\langle A^2\rangle}{2\beta}(1+\alpha^2) = \frac{1}{2}\langle A^2\rangle\gamma$$

$$\beta\gamma - \alpha^2 = 1$$

and ... $$xx' = A^2(\frac{1}{2}\sin 2\psi - \alpha\sin^2\psi)$$

$$\langle xx'\rangle = -\frac{1}{2}\langle A^2\rangle\alpha$$

From which the average of all particle emittances will be $\pi\langle A^2\rangle = 2\pi\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}$

and the "normalized rms emittance" can be defined as:

$$\boxed{\epsilon_N = \pi(\beta\gamma)\sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}}$$

TRANSPORT of Beam Moments

- For simplicity, define $\tilde{\epsilon} \equiv \frac{1}{2}\langle A^2 \rangle$; then,
- Note that:

$$\tilde{\epsilon} J = \begin{pmatrix} \tilde{\epsilon}\alpha & \tilde{\epsilon}\beta \\ -\tilde{\epsilon}\gamma & -\tilde{\epsilon}\alpha \end{pmatrix} = \begin{pmatrix} -\langle x x' \rangle & \langle x^2 \rangle \\ -\langle x'^2 \rangle & \langle x x' \rangle \end{pmatrix}$$

- Correlation Matrix:

$$\Sigma \equiv \begin{pmatrix} \langle x^2 \rangle & \langle x x' \rangle \\ \langle x x' \rangle & \langle x'^2 \rangle \end{pmatrix}$$

$$\boxed{\Sigma_2 = M \ \Sigma_1 \ M^T}$$

Here, *M* is from point 1 to point 2 along the beam line (same *M* as previously)

Summary

- So, can look at propagation of amplitude function through beam line given matrices of individual elements. Beam size at a particular location determined by

$$x_{rms}(s) = \sqrt{\beta(s)\epsilon_N/\pi(\beta\gamma)}$$

- Or, given an initial particle distribution, can look at propagation of second moments (of position, angle) given the same element matrices, and hence the propagation of the beam size, $\sqrt{\langle x^2\rangle(s)}$.

- Either way, can separate out the inherent properties of the beam distribution from the optical properties of the hardware arrangement

Effects due to Momentum Distribution

- Beam will have a distribution in momentum space
- Trajectories of individual particles will spread out when pass through magnetic fields

  - B is constant; thus $\Delta\theta/\theta \sim -\Delta p/p$
  - path is also altered by the gradient fields...
  -

- These trajectories are described by the Dispersion Function:

$$D(s) \equiv \Delta x_{c.o.}(s)/(\Delta p/p)$$

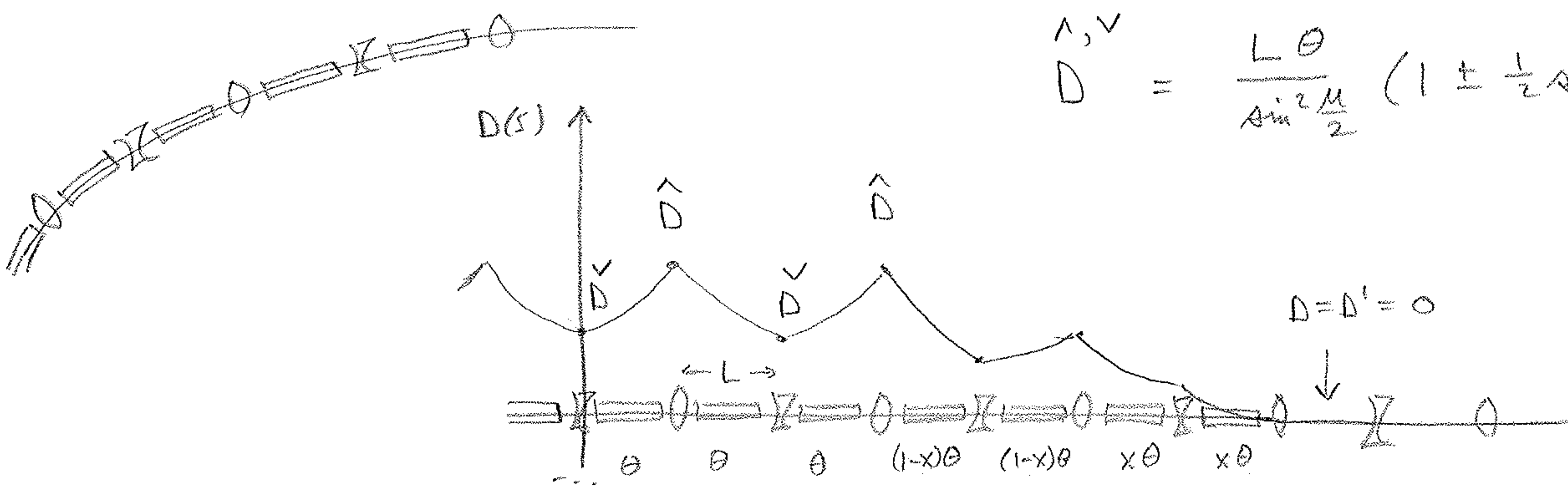
- Consequently, affects beam size:

$$\langle x^2 \rangle = \epsilon_N \beta(s)/(\pi \gamma v/c) + D(s)^2 \langle (\Delta p/p)^2 \rangle$$

-

Dispersion Suppressor

$$\hat{D},\check{D} = \frac{L\theta}{\sin^2\frac{\mu}{2}}\left(1 \pm \frac{1}{2}\sin\frac{\mu}{2}\right)$$



So, dispersion of a periodic FODO lattice w/ bending can be cancelled when $x = \frac{1}{2(1-\cos\mu)}$ [where μ = cell (F-to-F) phase advance]

common ex's:

μ	x	$(1-x)$
60°	1	0
90°	$\frac{1}{2}$	$\frac{1}{2}$

Chromaticity

- Focusing effects from the magnets will also depend upon momentum:

$$x'' + K(s,p)x = 0 \qquad K = e(\partial B_y(s)/\partial x)/p$$

- To give all particles the similar optics, regardless of momentum, need a "gradient" which depends upon momentum. Orbits spread out horizontally (or vertically) due to dispersion, can use a sextupole field:

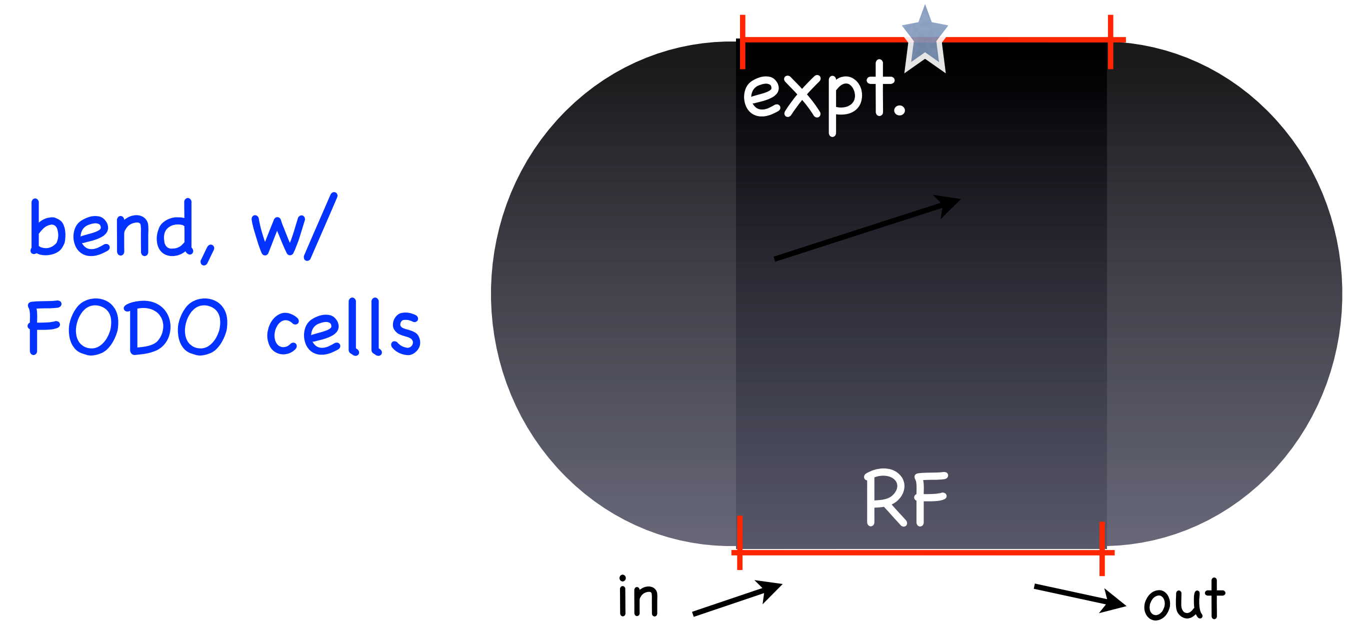
$$\vec{B} = \frac{1}{2}B''[2xy\;\hat{x} + (x^2 - y^2)\;\hat{y}]$$

- which gives $\partial B_y/\partial x = B''x = B''D(\Delta p/p)$
  -
  - i.e., a field gradient which depends upon momentum

- *Chromatic aberrations* are the variation of optics with momentum; *chromaticity* is the variation of tune with momentum. We use sextupole magnets to control/adjust; but, now introduces nonlinear fields ...
- can create a transverse *dynamic aperture*!

Collider Accelerator Lattice

- can build up out of modules



- check for overall stability -- *x*/*y*
- meets all requirements of the program
  - Energy --> circumference, fields, etc.
  - spot size at interaction point: β minimized, D=0
  - etc...

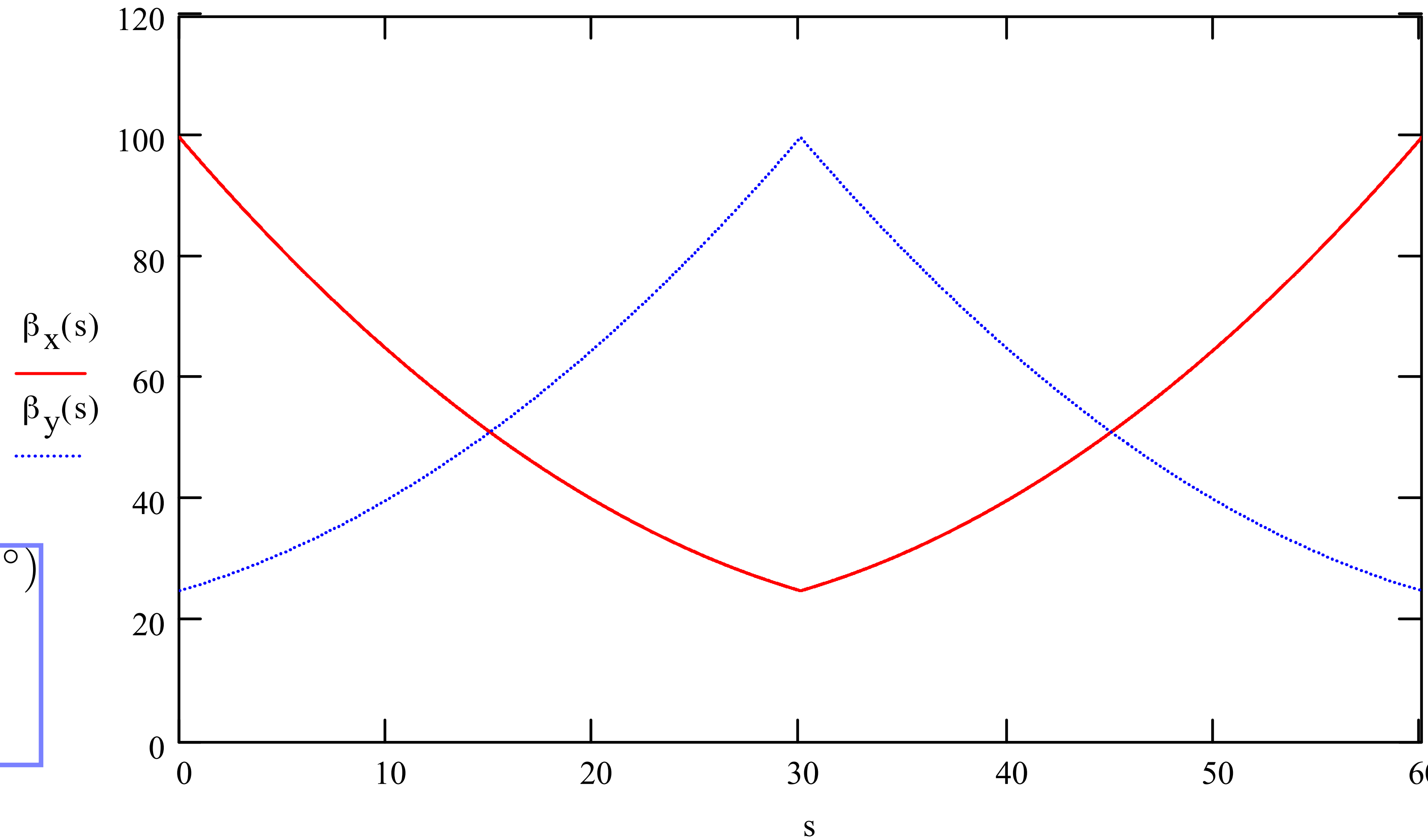
FODO Cells (arcs)

$$\Delta\beta' = \mp 2\beta/F$$ through a thin quad

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$ between quadrupoles

$$\beta_{max,min} = 2F\sqrt{\frac{1 \pm L/2F}{1 \mp L/2F}}$$

L = 30 F = 25

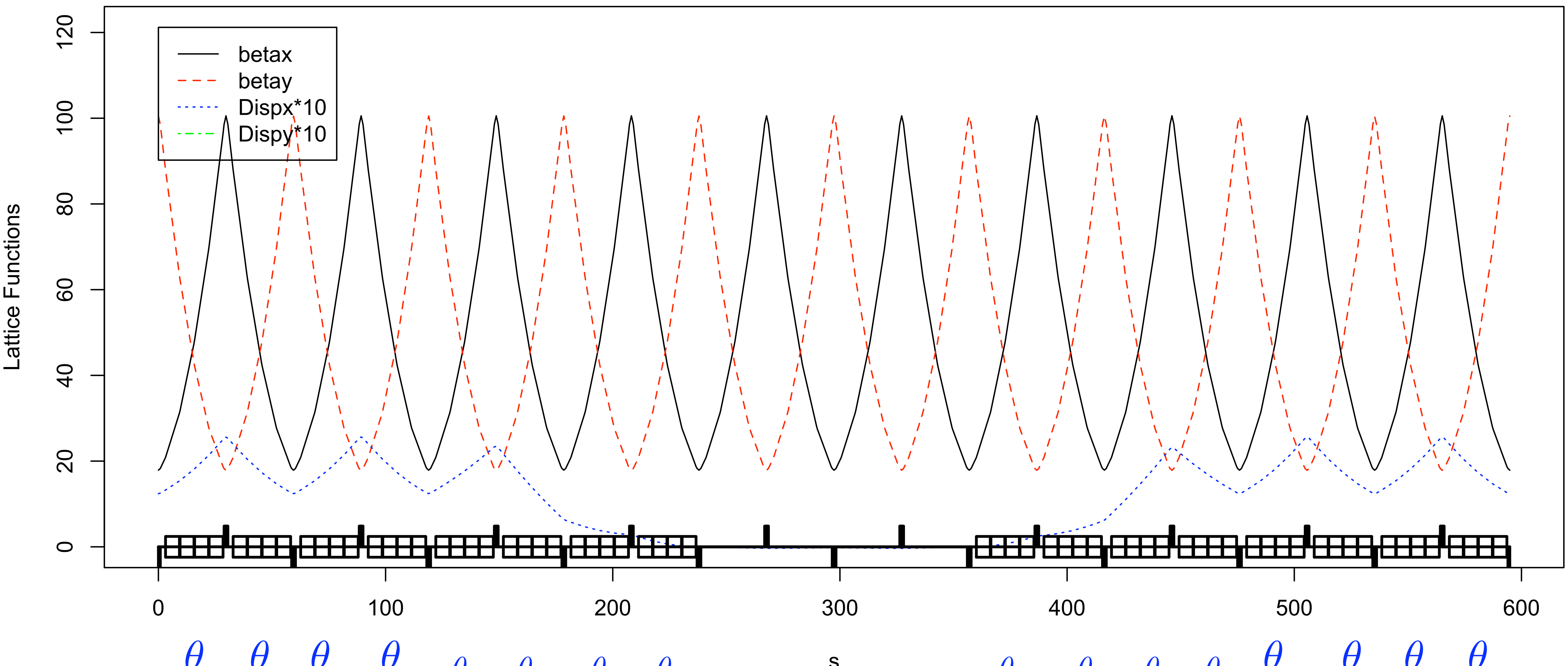


Ex: Tevatron Cell

$$\boxed{\begin{aligned}&\sin(\mu/2) = L/2F = 0.6 \longrightarrow \mu \approx 1.2(69^\circ)\\&\beta_{max} = 2(25\ \mathrm{m})\sqrt{1.6/0.4} = 100\ \mathrm{m}\\&\beta_{min} = 2(25\ \mathrm{m})\sqrt{0.4/1.6} = 25\ \mathrm{m}\\&\nu \approx 100 \times 1.2/2\pi \sim 20\end{aligned}}$$

Dispersion Suppression

phase advance $= 90°$ per cell



$\theta \quad \theta \quad \theta \quad \theta \quad \frac{\theta}{2} \quad \frac{\theta}{2} \quad \frac{\theta}{2} \quad \frac{\theta}{2} \qquad \frac{\theta}{2} \quad \frac{\theta}{2} \quad \frac{\theta}{2} \quad \frac{\theta}{2} \quad \theta \quad \theta \quad \theta \quad \theta$

Dispersion Suppression

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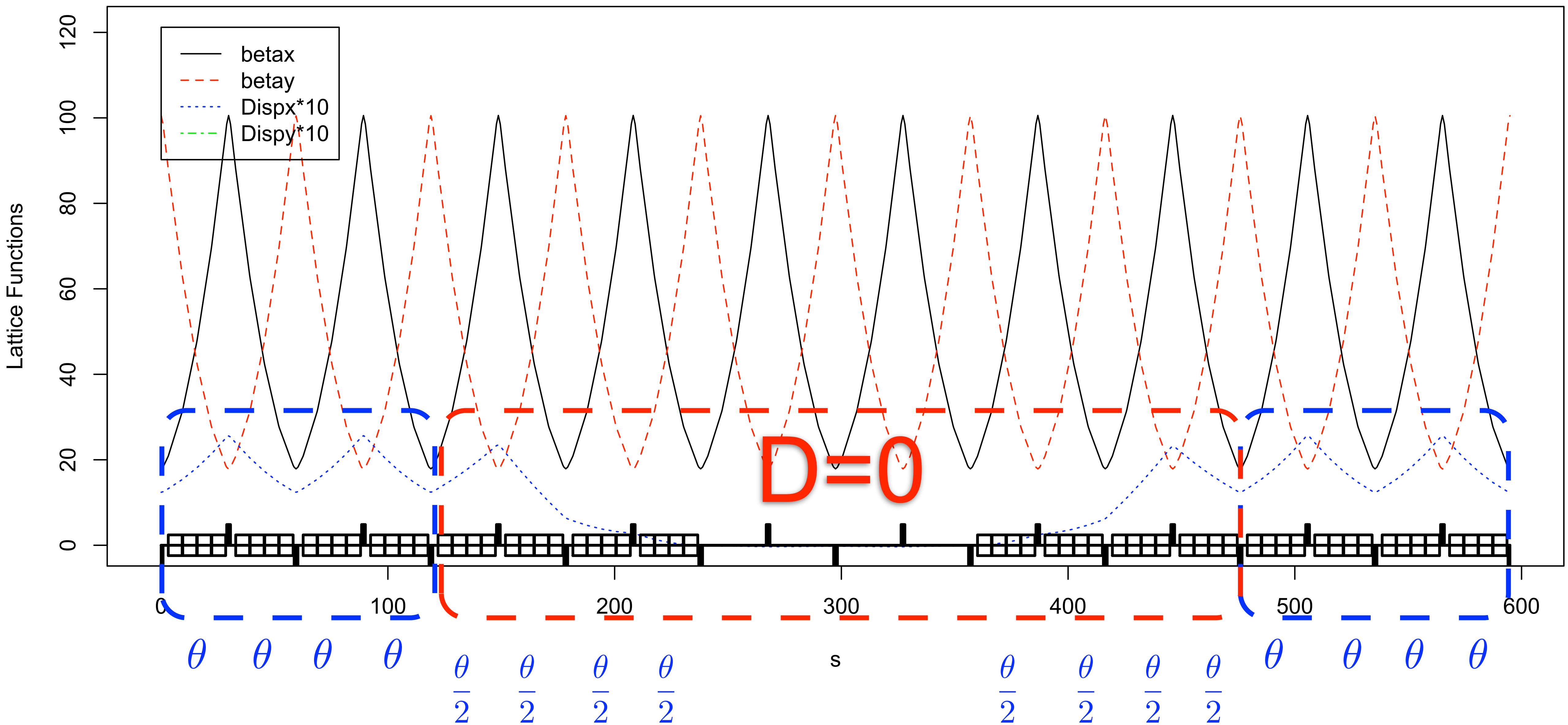
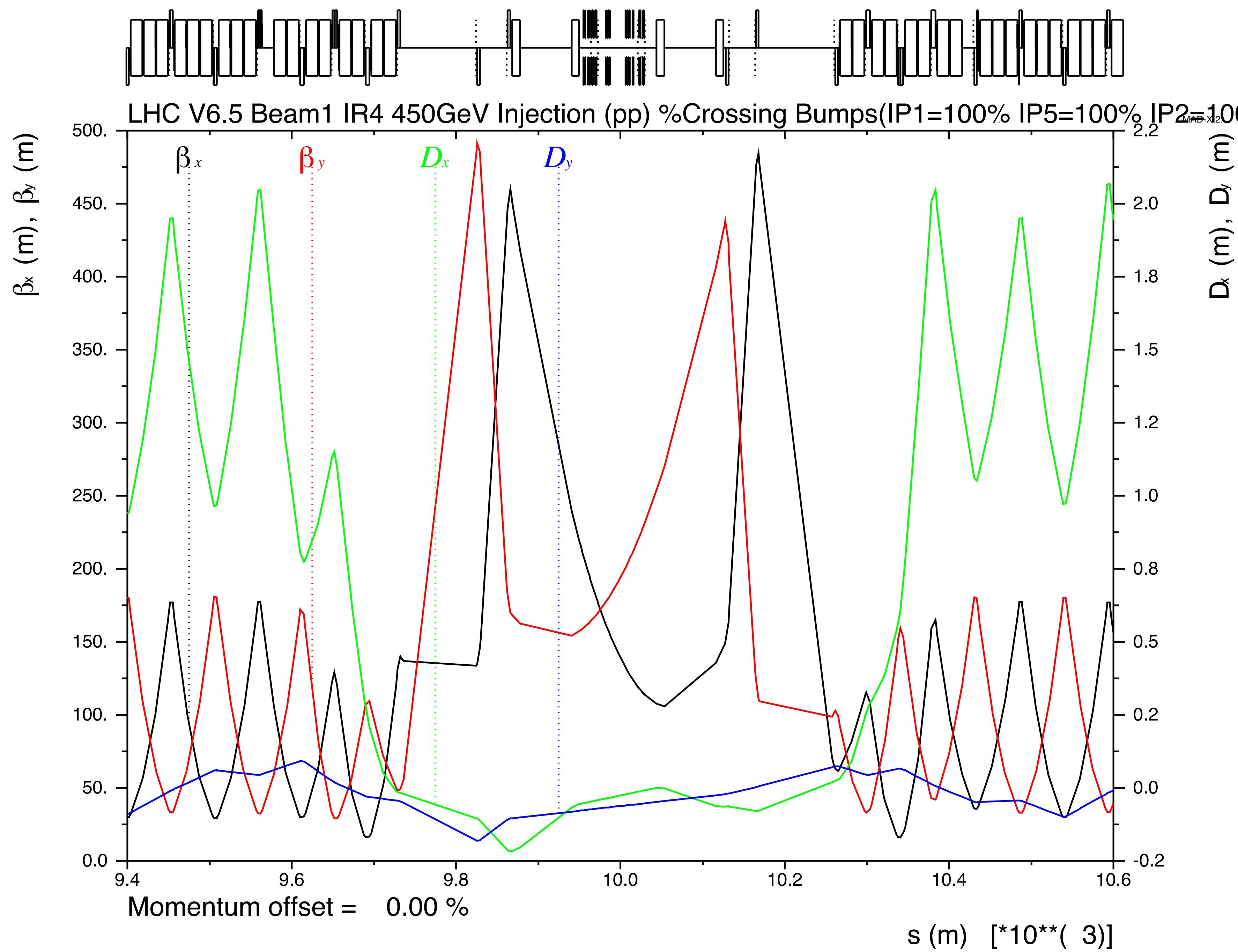


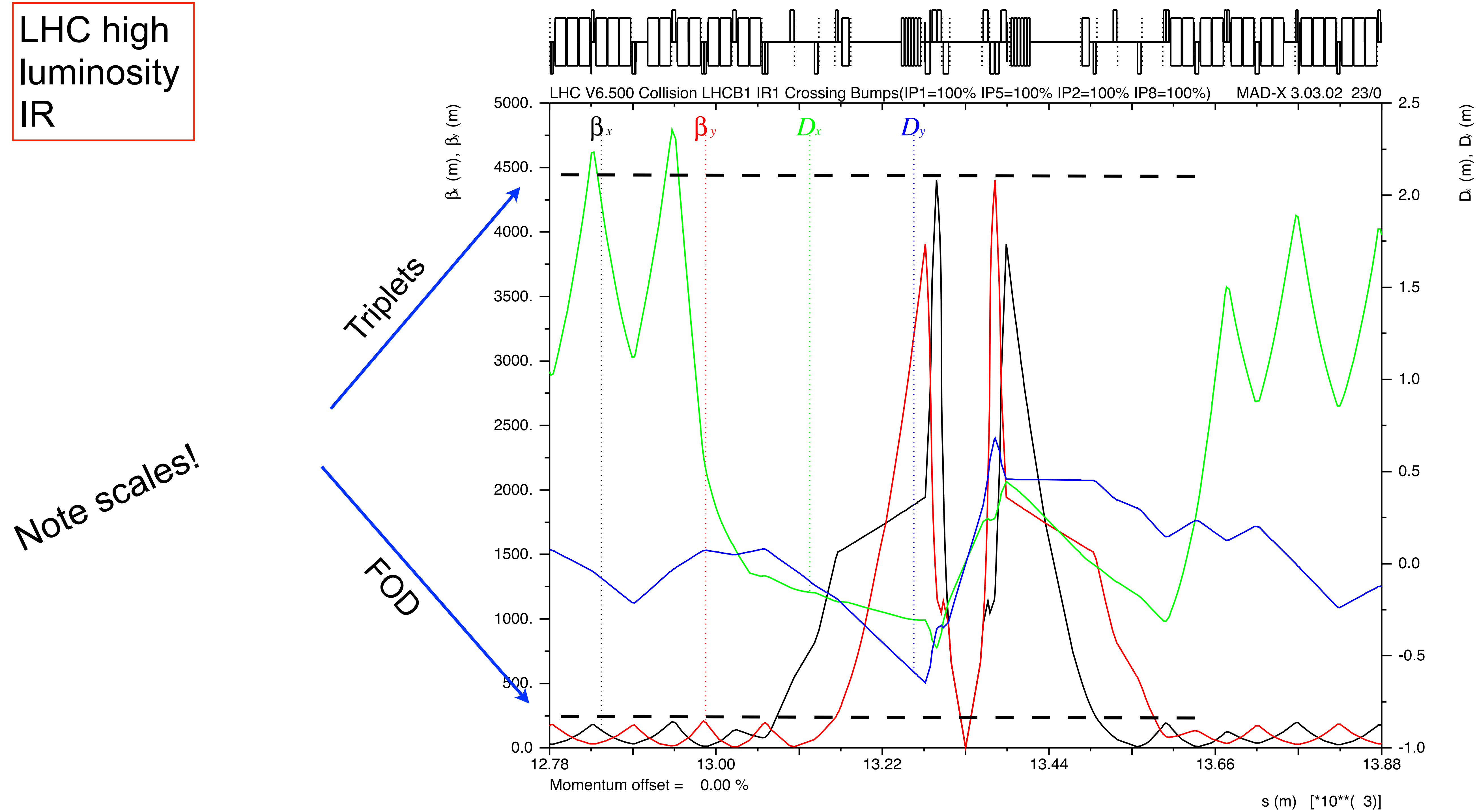
Long Straight Section

- a "matched insertion" that propagates the amplitude functions from their FODO values, through the new region, and reproduces them on the other side
- Here, we see an LHC section used for beam scraping



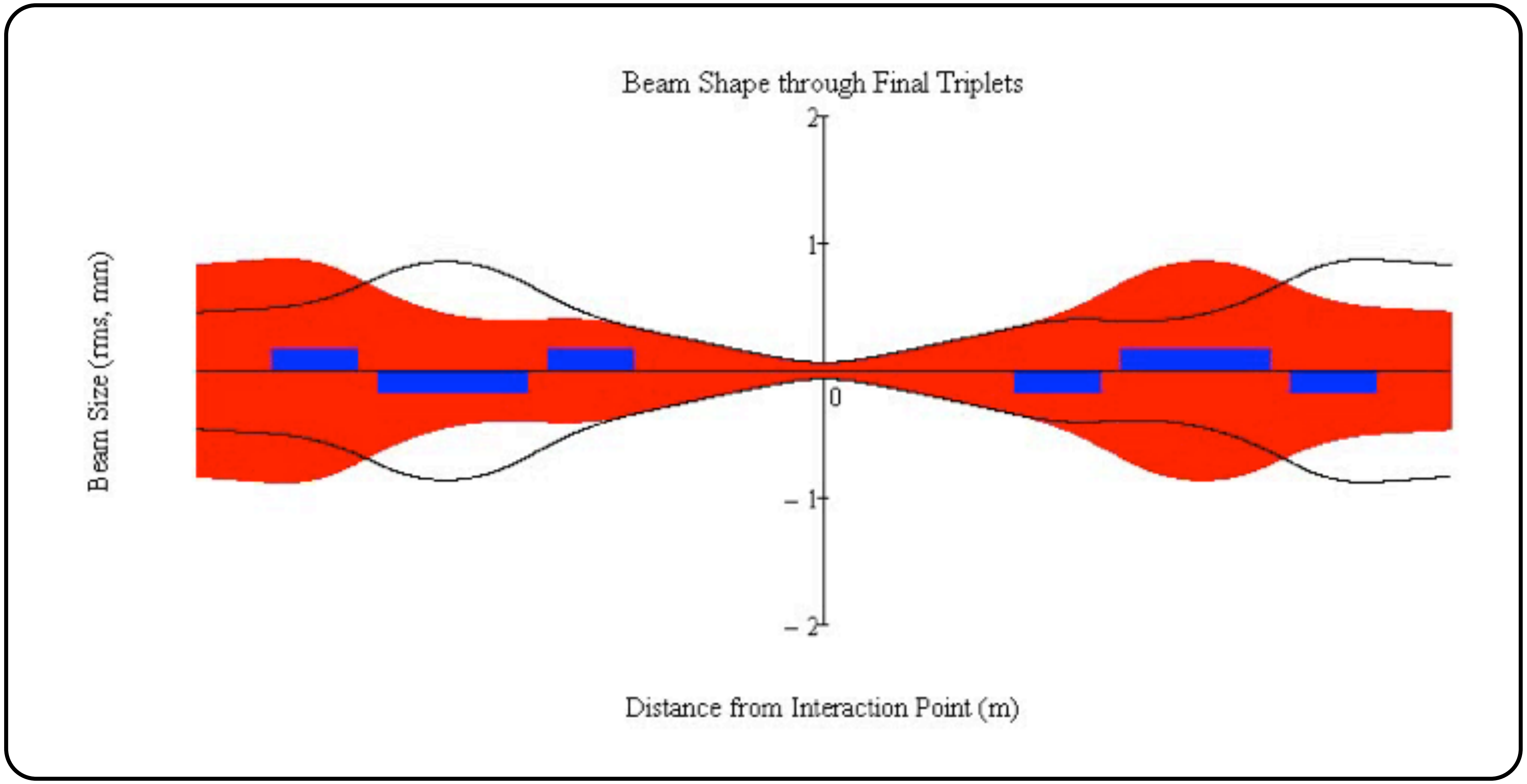
Interaction Region

LHC high luminosity IR



Low-Beta "Squeeze"

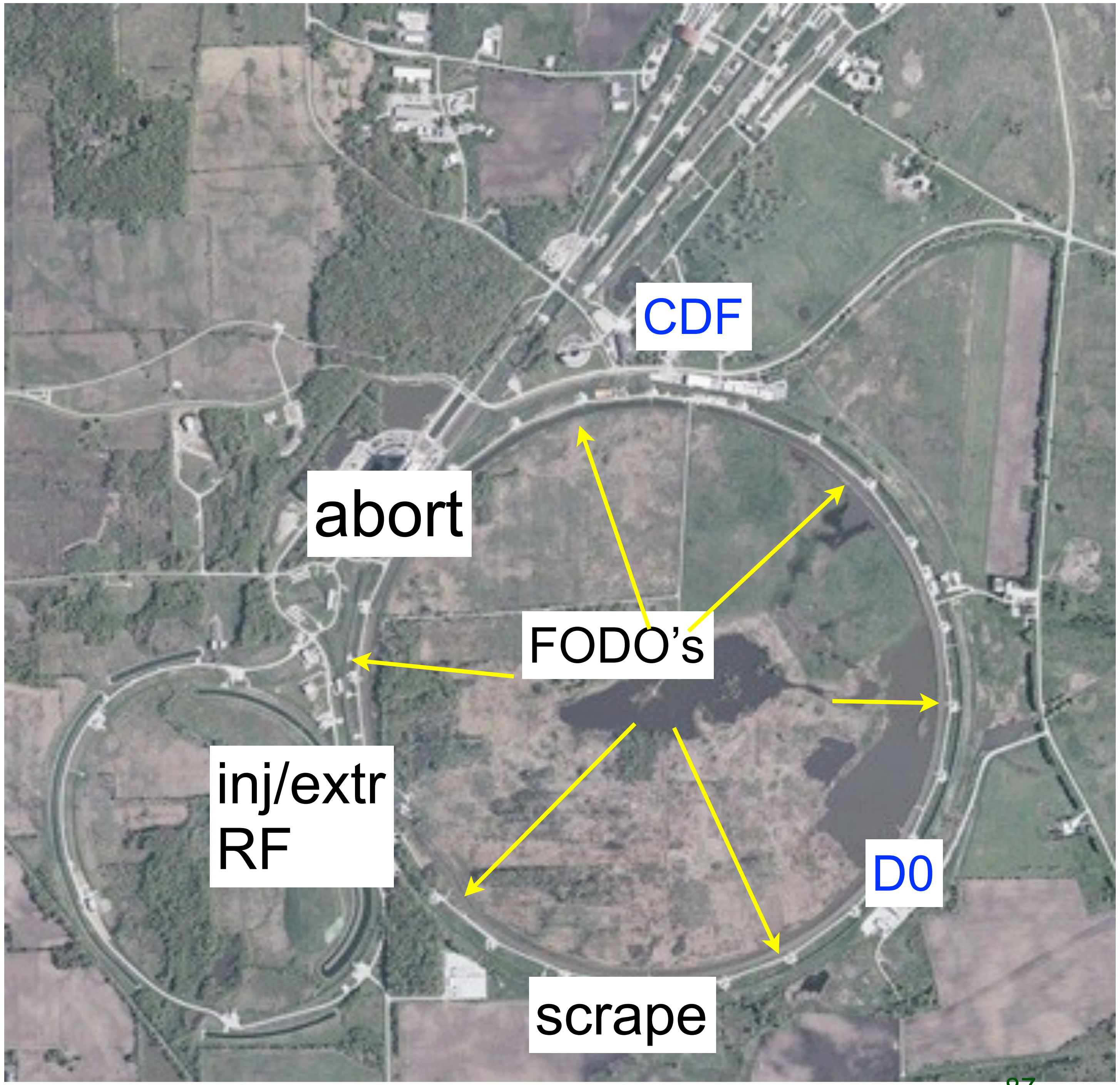
- As beam is larger at injection than it is at collision energy, do not want a "low-beta" condition during injection process
- Thus, the triplet and other nearby quadrupoles are tuned to adjust beam size at the focus; the beam is "squeezed" near the end of the sequence



Tevatron Example

Put it all Together

- make up a synchrotron out of FODO cells for bending, a few matched straight sections for special purposes...



Put it all Together

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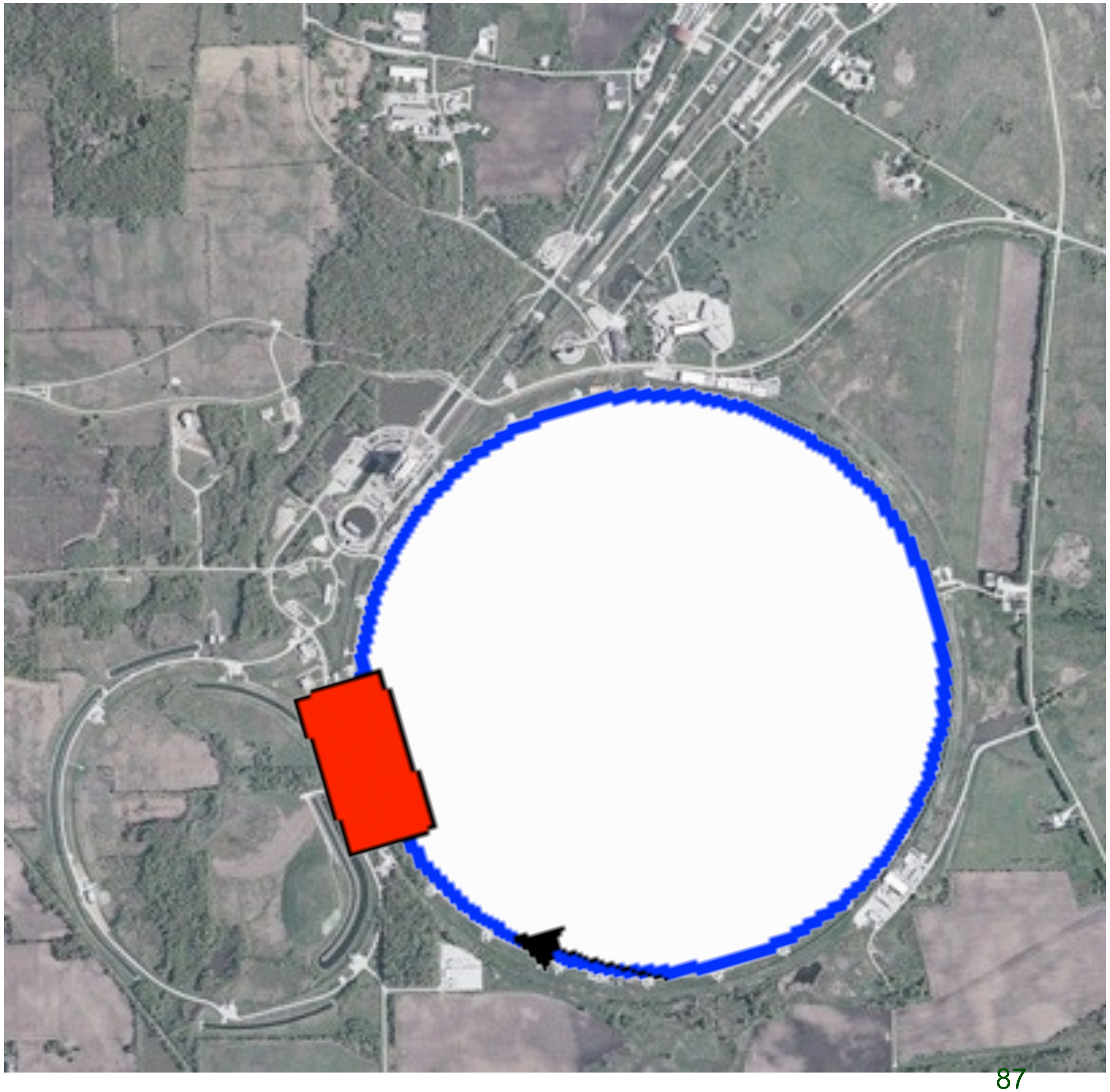
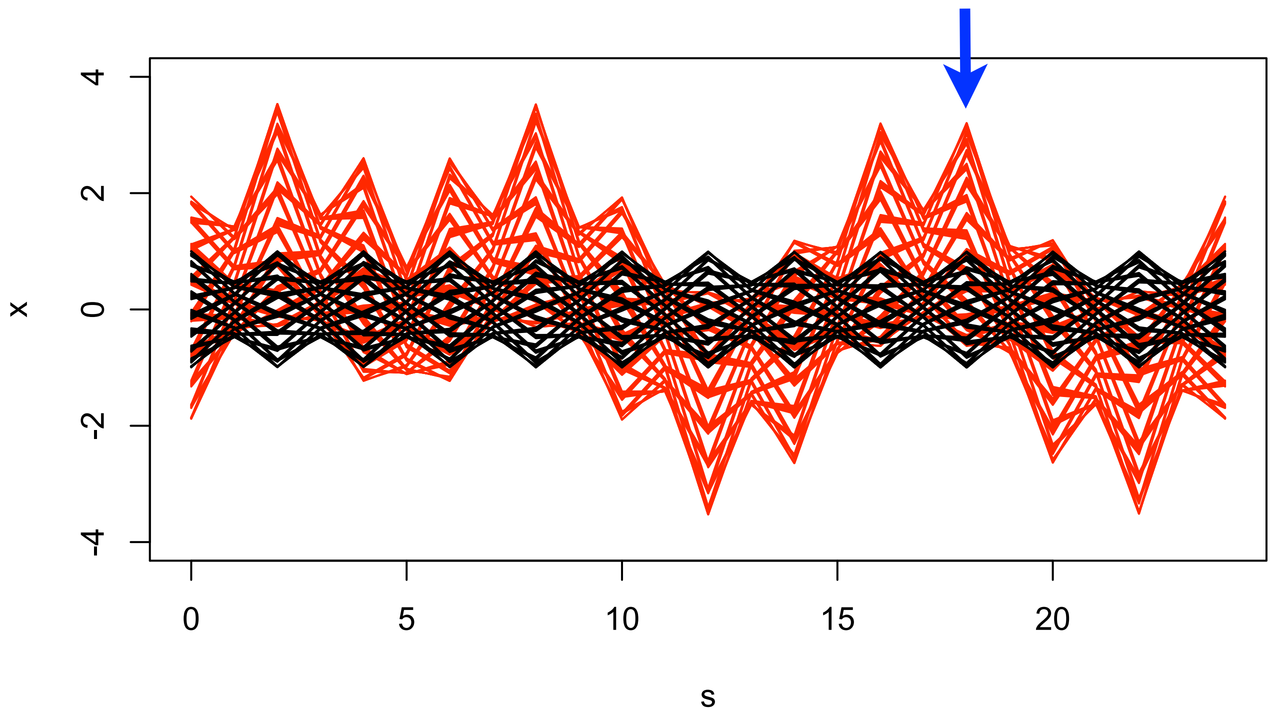


Corrections and Adjustments

- Correction/adjustment systems required for fine control of accelerator:
  - correct for misalignment, construction errors, drift, etc.
  - adjust operational conditions, tune up
- Use smaller magnetic elements for “fine tuning” of accelerator
  - dipole steering magnets for orbit/trajectory adjustment
  - quadrupole correctors for tune adjustment
  - sextupole magnets for chromaticity adjustment

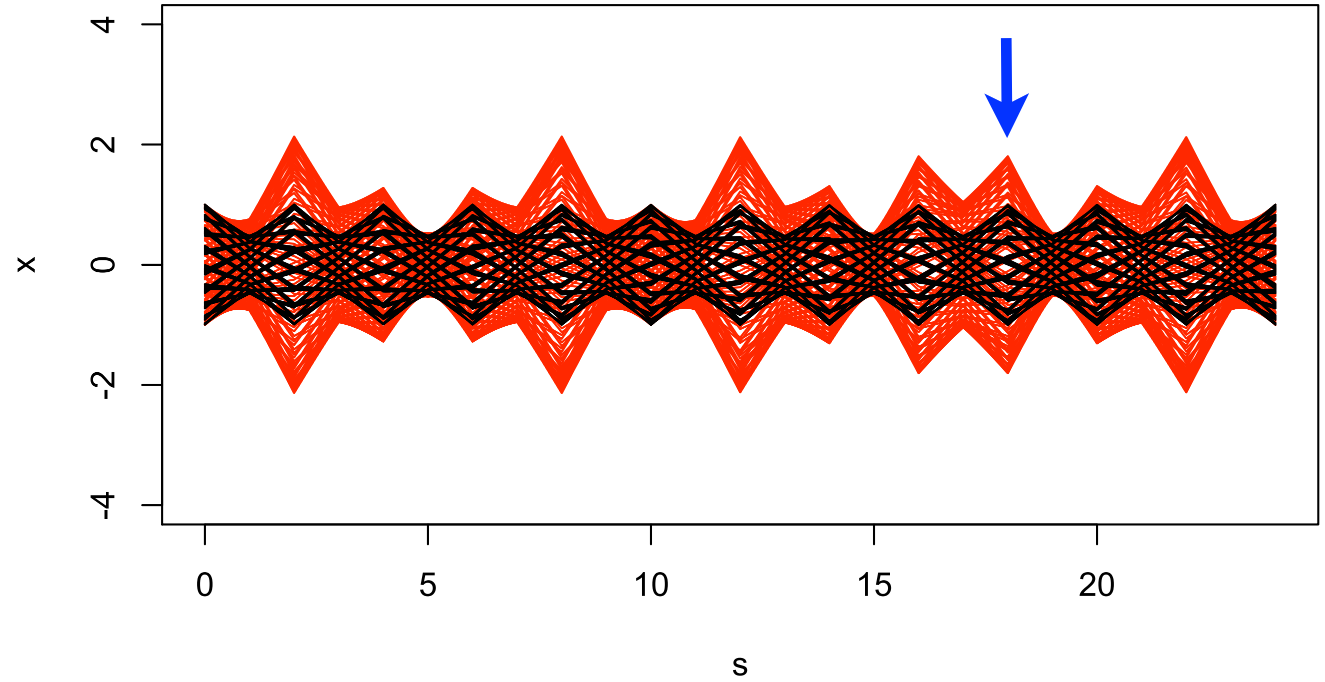
- Typically, place correctors and instrumentation near the major quadrupole magnets -- “corrector package”
  - control steering, tunes, chromaticity, etc.
  - monitor beam position (in particular), intensity, losses, etc.

Linear Distortions



Orbit distortion due to single dipole field error

Envelope Error (Beta-beat) due to gradient error



gradient error also generates a shift in the betatron tunes...

$$\Delta\nu = \frac{1}{4\pi}\beta_0 \Delta q$$

Resonances and Tune Space

- Error fields are encountered repeatedly each revolution -- can be resonant with tune

- repeated encounter with a steering (dipole) error produces an orbit distortion:
  - » thus, avoid integer tunes

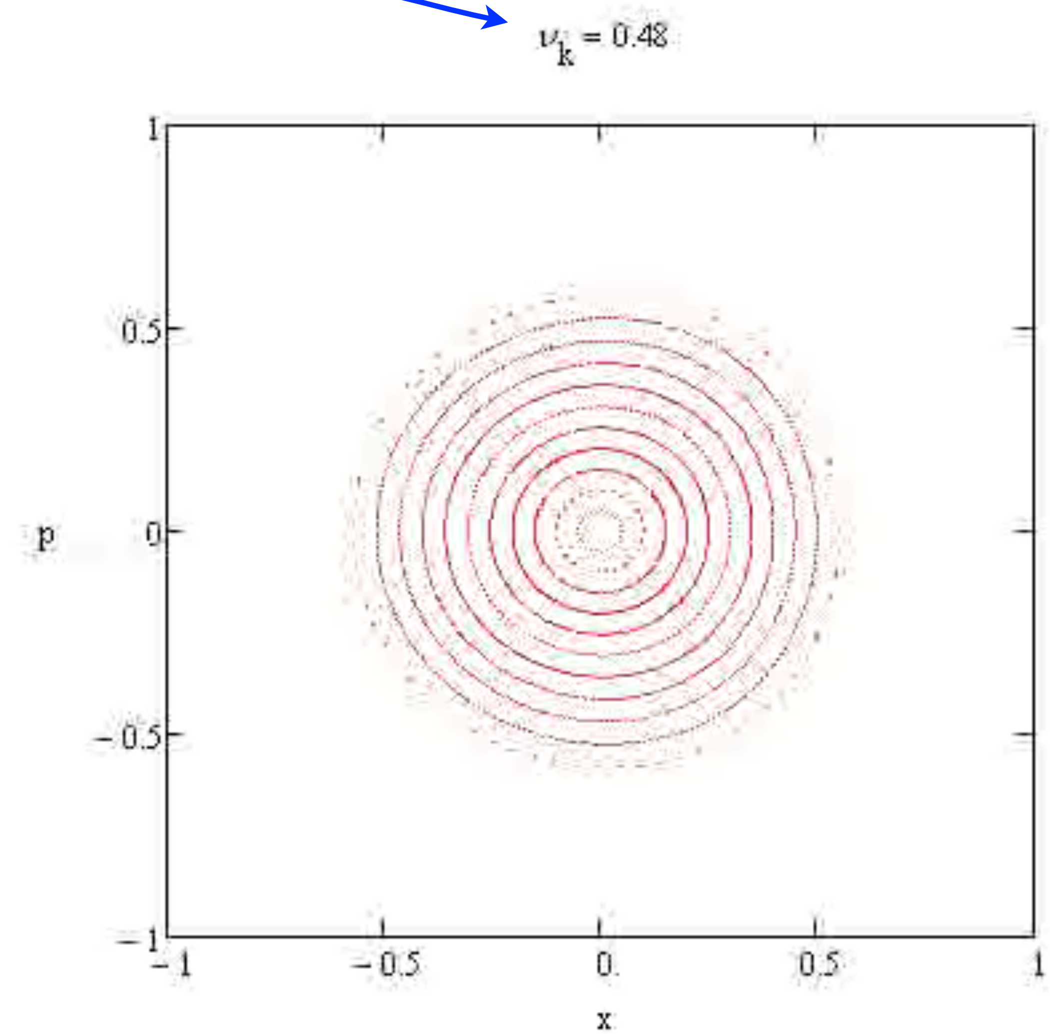
$$\boxed{\Delta x \sim \frac{1}{\sin \pi\nu}}$$

- repeated encounter with a focusing (quad) error produces distortion of amplitude fcn:
  - » thus, avoid half-integer tunes

$$\boxed{\Delta\beta/\beta \sim \frac{1}{\sin 2\pi\nu}}$$

Nonlinear Resonances

- Phase space w/ sextupole field present ($B_y \sim x^2$)
  - tune dependent:
  - "dynamic aperture"

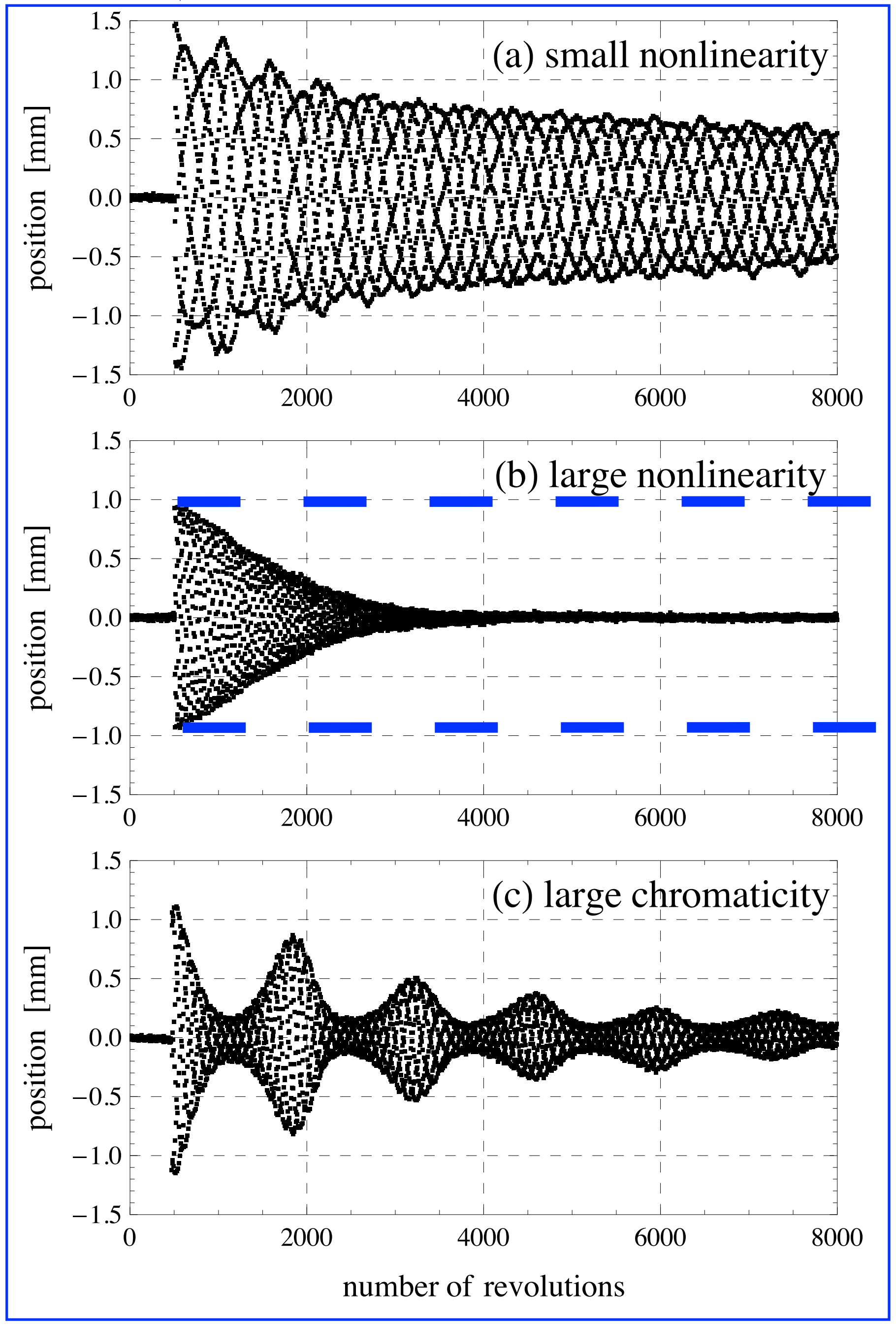


- Thus, avoid tune values:
  - k, $k/2$, $k/3$, ...

Tune Spread

- due to momentum -- chromaticity
  - "natural" chromaticity due to particle rigidity
  - also, field errors in magnets ~ x^2 in the presence of Dispersion
- due to nonlinear fields
  - field terms ~ x^2, x^3, etc.
- --> "decoherence" of beam position signal

"kick" the beam



Tune Diagram

- Always "error fields" in the real accelerator

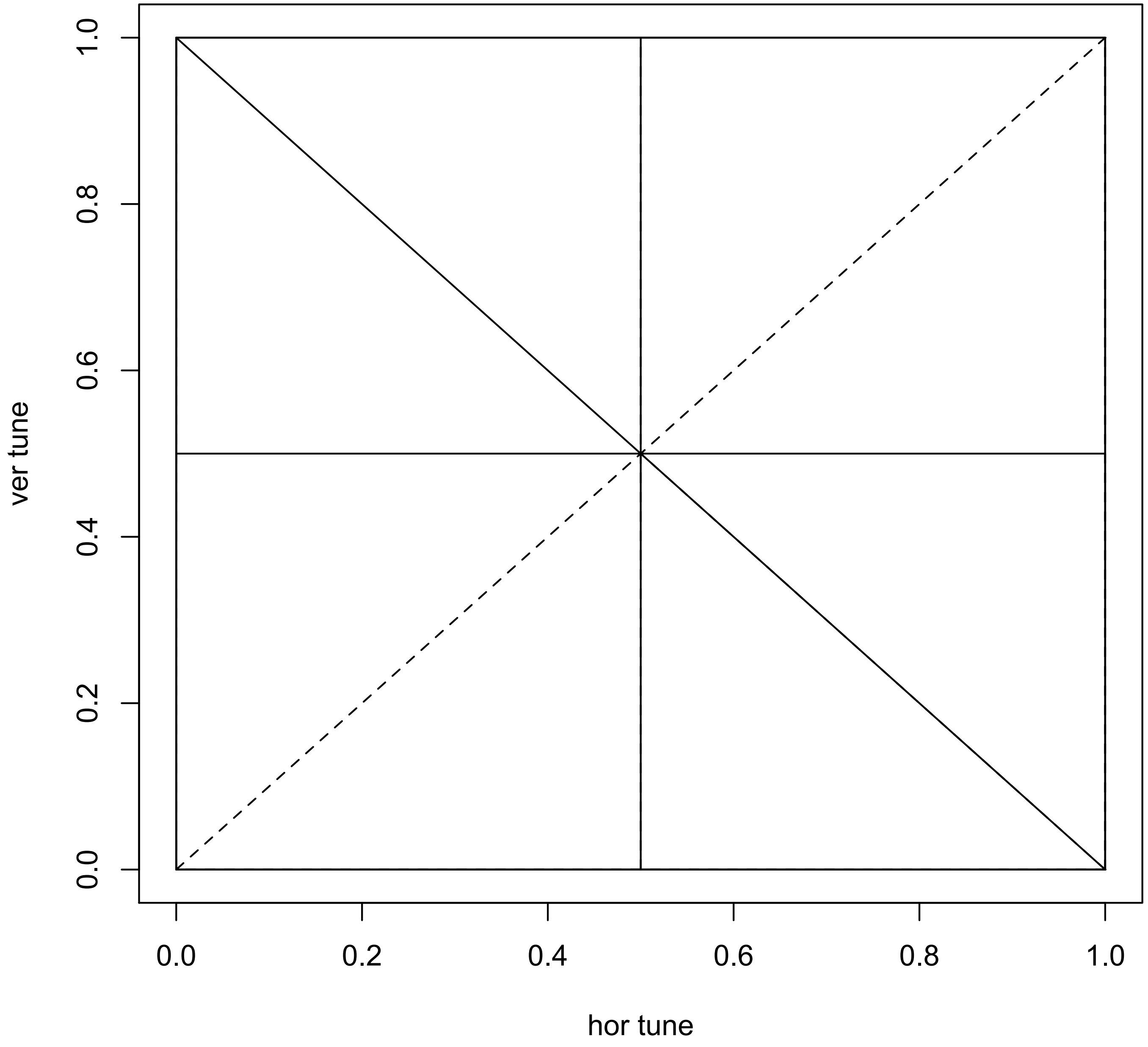
- Coupled motion also generates resonances (sum/difference resonances)

  - in general, should avoid: $$m\,\nu_x \pm n\,\nu_y = k$$

$\boxed{\text{avoid ALL rational tunes???}}$

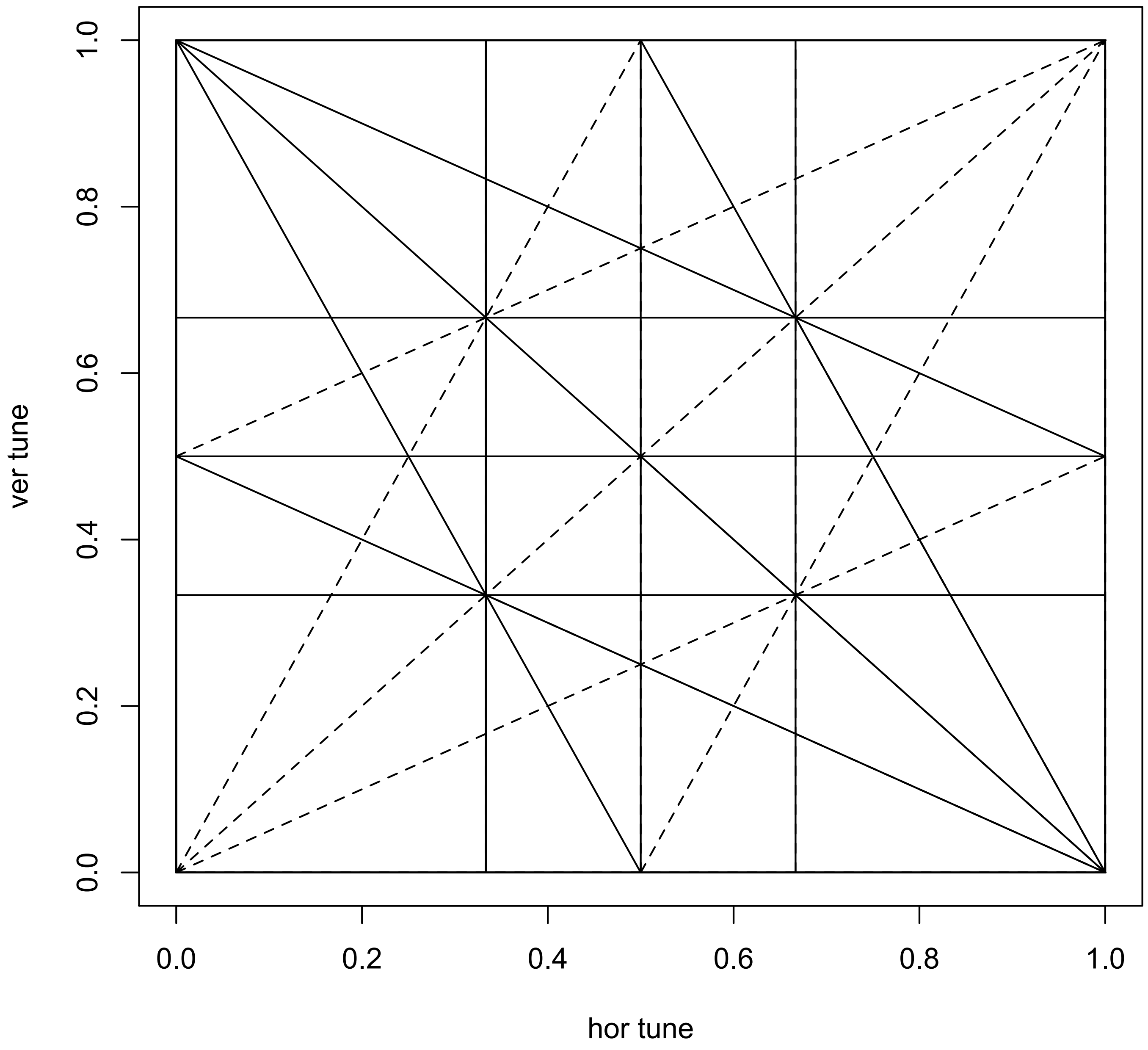
Tune Diagram

Through order
k =2



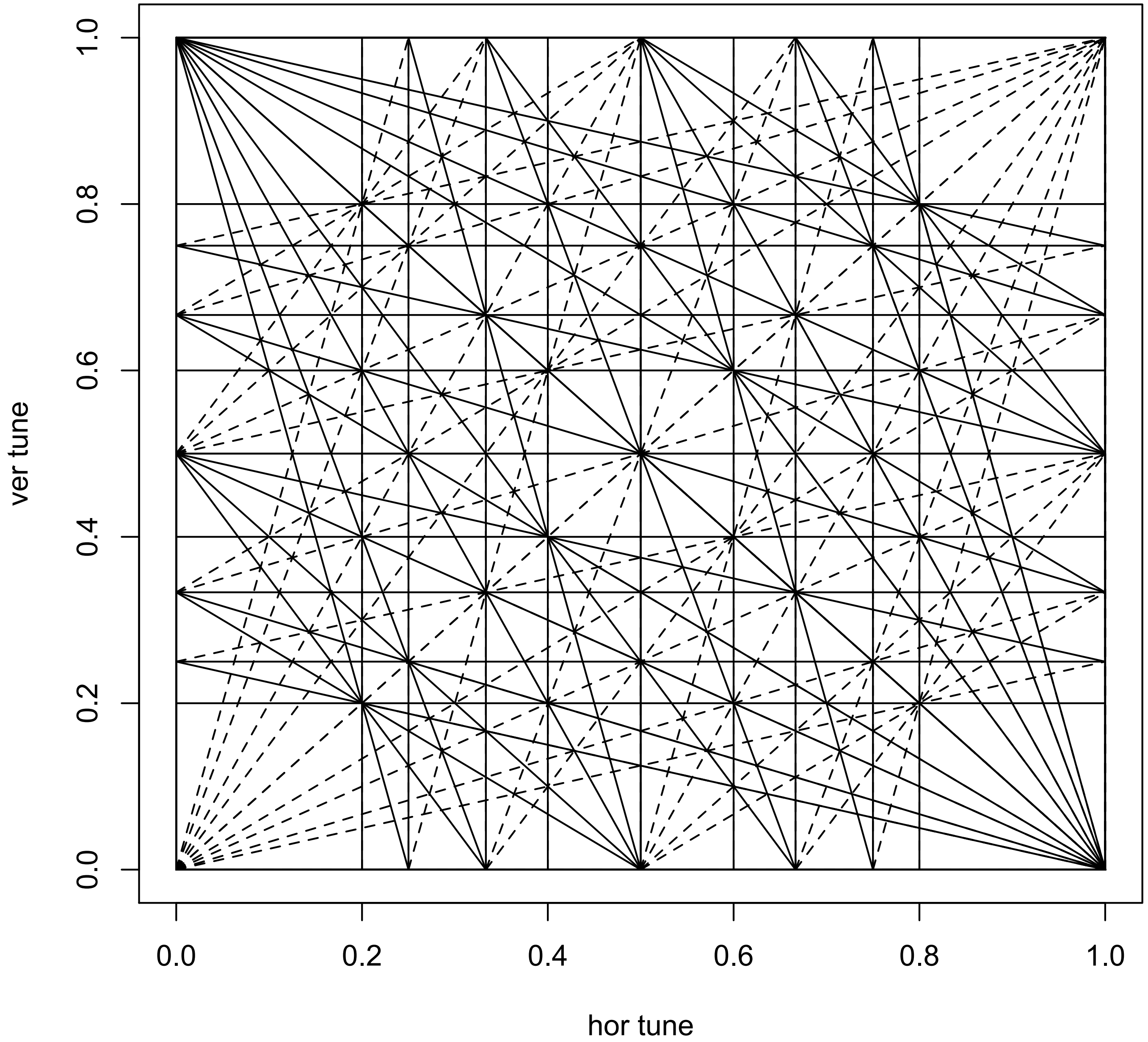
Tune Diagram

Through order
k =3

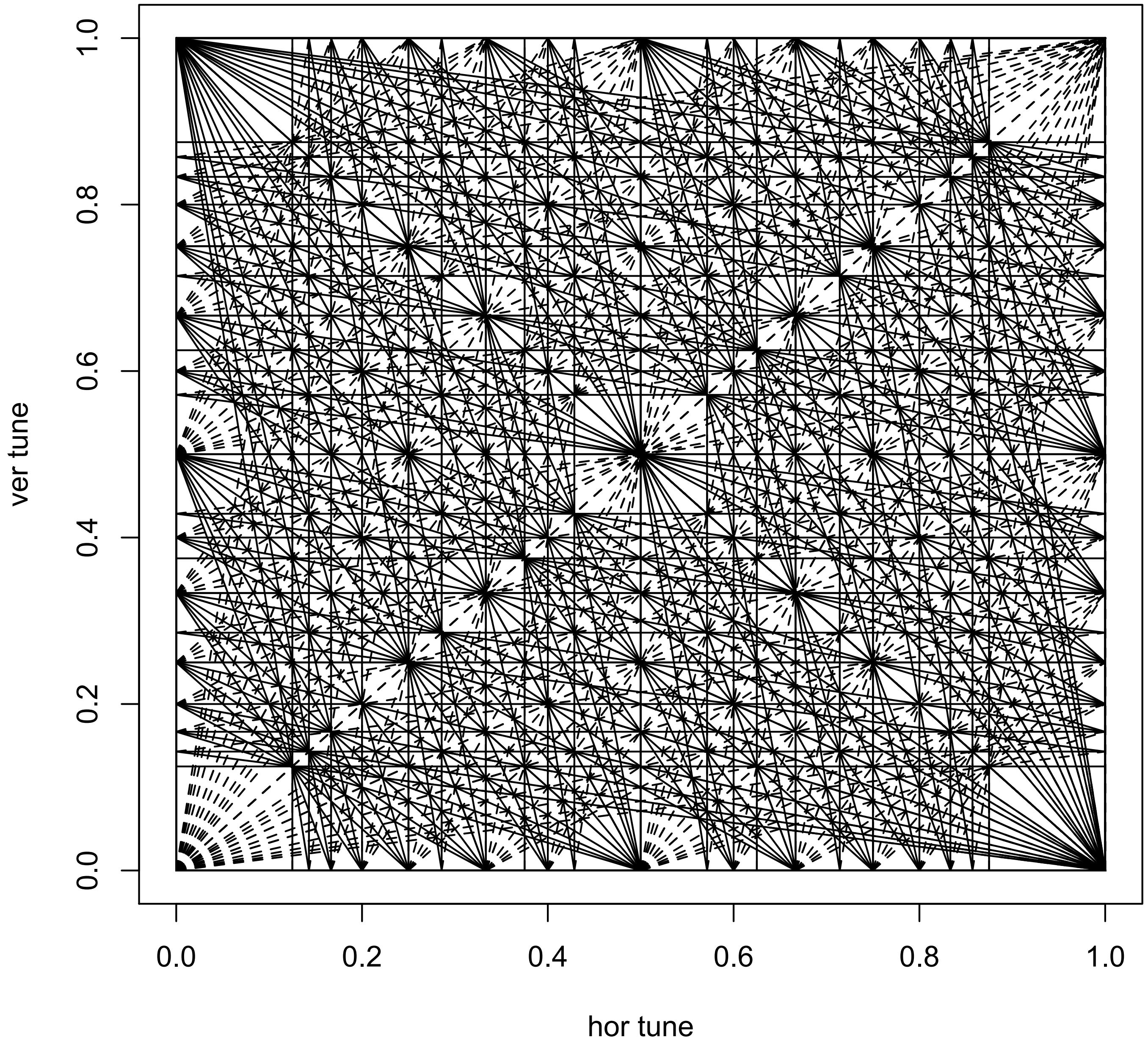


Tune Diagram

Through order
k =5

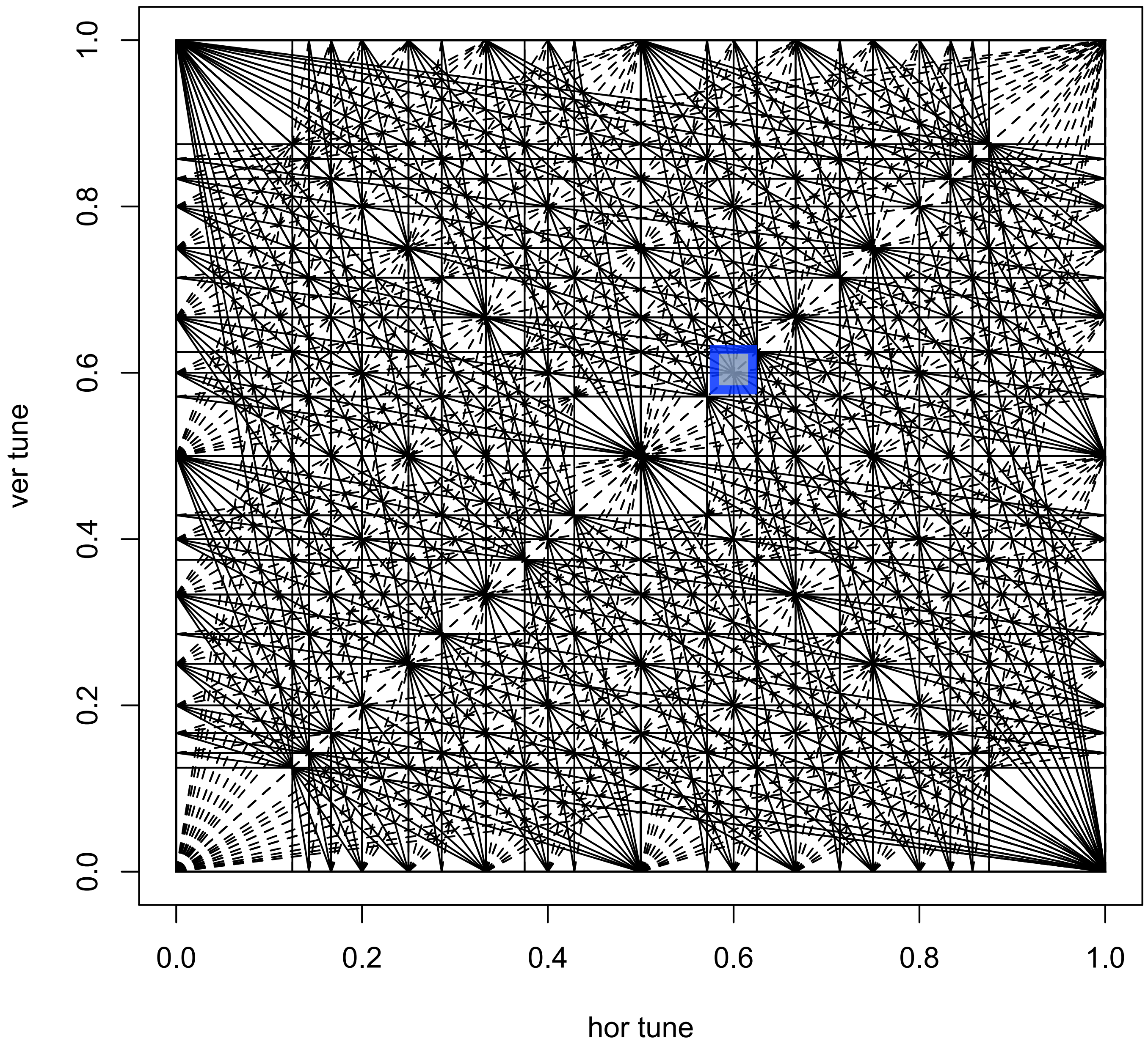


Tune Diagram

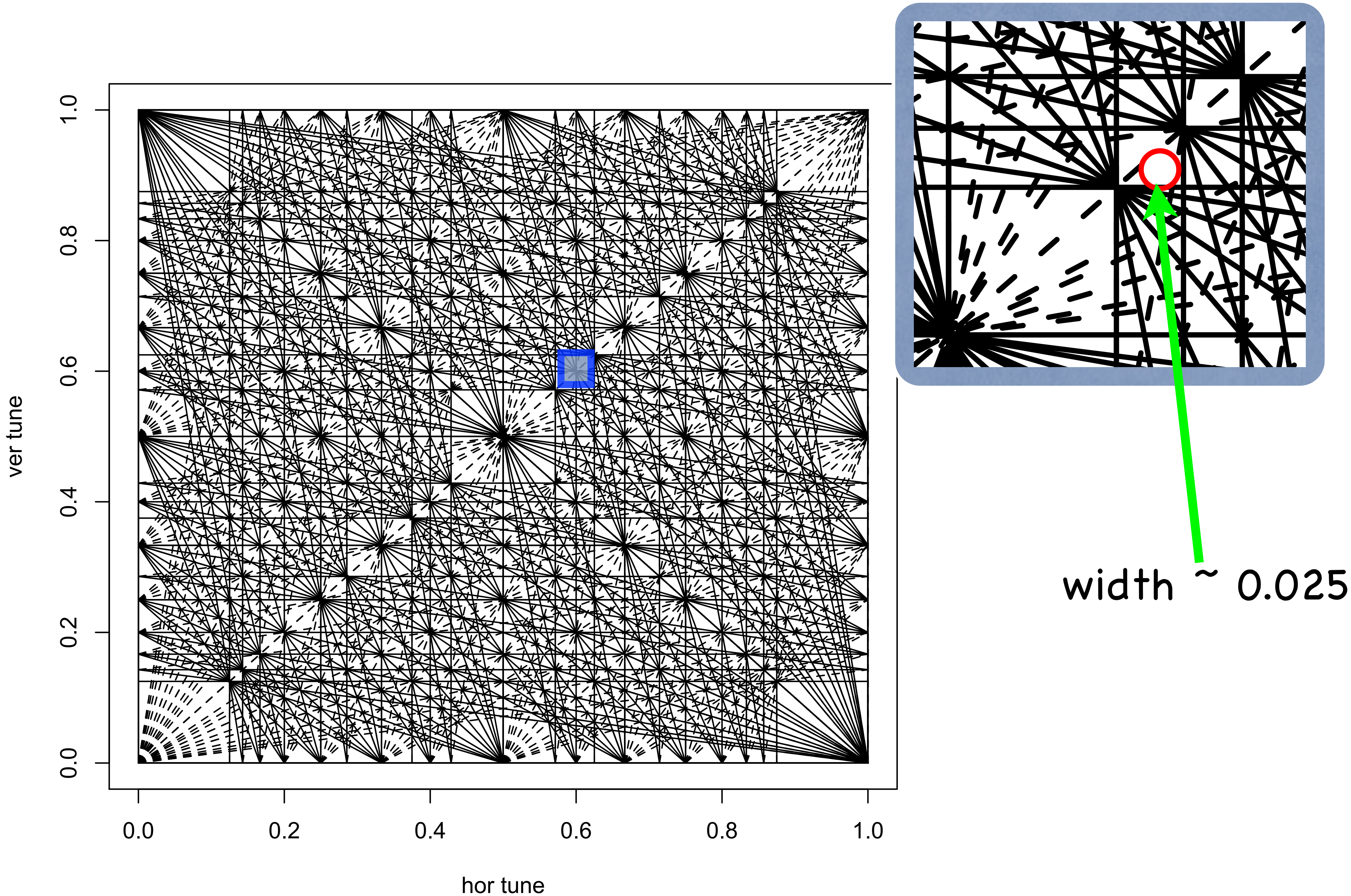


Through order
k =8

Tune Diagram



Tune Diagram



Break till Day Three...

- Tomorrow:
  - beam-beam interaction
  - energy deposition and synchrotron radiation
  - diffusion and emittance growth
  - hour glass and crossing angles
  - luminosity optimization
  - future directions